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Exponential Transient Rotating Waves and Their Bifurcations in a Ring of Unidirectionally Coupled Bistable Lorenz Systems

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Introduction

We study rotating waves in a ring of unidirectionally coupled Lorenz systems. A model is given by

$$\begin{aligned} dx_n/dt &= \sigma(y_n - x_n) \\ dy_n/dt &= \rho x_{n-1} - y_n - x_n z_n \\ dz_n/dt &= -\beta z_n + x_n y_n \qquad (1 \le n \le N, \quad x_0 = x_N) \end{aligned} \tag{1}$$

where x_n in the first term of the equation for y_n of the nth system is replaced by x_{n-1} of the n-1st system. This coupling is equivalent to the linear convection of x_n in the equations for y_n with strength equal to ρ. A total N systems make a closed loop with $x_0 = x_N$. This ring of coupled Lorenz systems has been studied in [1] and it has then been shown that periodic and hyperchaotic rotating waves are generated from synchronized spatially uniform chaotic states through the Hopf bifurcations. These rotating waves have been observed in a range of parameters in which a single Lorenz system is chaotic. In this study, we consider Eq. (1) consisting of bistable Lorenz systems and show (i) unstable transient rotating waves the duration of which increases exponentially with the number N of systems (exponential transients); (ii) the stabilization of the rotating waves through pitchfork bifurcations and the generation of chaotic rotating waves.

Exponential Duration of Transient Rotating Waves

The origin ($x_n = y_n = z_n = 0$ ($1 \le n \le N$)) is a steady state of Eq. (1). The eigenvalues of the Jacobian matrix of Eq. (1) evaluated at the origin are given by

$$\lambda = -\beta, \quad -\left(1 + \sigma \pm \sqrt{1 - 2\sigma + 4\exp(i2n\pi/N)\rho\sigma + \sigma^2}\right)/2 \qquad (0 \le n < N) \tag{2}$$

We set $\sigma = 10$, $\beta = 8/3$ and use ρ for a bifurcation parameter. The origin is stable when $0 \le \rho < 1$, and a pair of stable spatially uniform steady states ($x_n = y_n = \pm\sqrt{\beta(\rho-1)}$, $z_n = \rho - 1$ ($1 \le n \le N$)) is generated through the pitchfork bifurcation from the origin at $\rho = 1$ so that Eq. (1) becomes bistable. An unstable symmetric rotating wave solution is then generated through the Hopf bifurcation as ρ increases, in which each system oscillates with phase difference $2\pi/N$ between the adjacent systems (Fig. 1(a)). It is shown that the largest eigenvalue of the the Poincare map of the unstable rotating wave decreases to zero double exponentially with N so that the relaxation time of it to converge to one of the steady states increases exponentially with N. As a result, the mean duration of transient rotating waves generated from random initial states increases exponentially with N. Figure 2 shows the results of computer simulation of 10^4 runs under Gaussian random initial conditions: $x_n, y_n, z_n \sim N(0, 1)$ for each N when $\rho = 1.5$. Such exponential transient states are common in static kink and pulse patterns in symmetric bistable reaction-diffusion systems [2] and have been recently found in dynamic rotating waves in a spatially discrete coupled system (a ring neural network) [3].

Bifurcations of Rotating Waves

Bifurcation diagrams of rotating waves for $N = 9$ are shown in Fig. 3, in which the sum $S_x = \sum_{n=1}^{N} x_n$ of x_n is plotted. In Fig. 3(a), an unstable symmetric rotating wave ($S_x = 0$) (RW1) is generated through the Hopf bifurcation from the origin at $\rho \approx 1.24$ (H1). An unstable symmetric rotating wave of second harmonics (RW2), which has two spatial periods, is generated through the second Hopf bifurcation from the origin at $\rho \approx 2.62$ (H2) successively. In Fig. 3(b), the stability of the RW1 changes alternately through the successive pitchfork bifurcations at $\rho \approx 1.92$, 2.53 and 3.88 (PFs), and three pairs of asymmetric rotating waves (RW11-13) are generated, the second one (RW12) of which is stable (Fig. 1(b)). Such pitchfork bifurcations of rotating waves have been shown in a ring neural network with inertia [4]. More complicated bifurcations of interest occur in Eq. (1) as follows. A stable quasiperiodic rotating wave, in which S_x changes periodically, is

generated from the RW1 through the Neimark-Sacker bifurcation at $\rho \approx 10.58$ (NS) (Fig. 1(c)), which is destabilized at $\rho \approx 10.64$. In Fig. 3(c), the second symmetric rotating wave (RW2) generated at the origin causes the pitchfork bifurcation at $\rho \approx 5.78$ (PF) and a pair of asymmetric rotating waves (RW21) is generated. The generated RW21 causes the Neimark-Sacker bifurcation at $\rho \approx 5.93$ (NS) and the period doubling bifurcation at $\rho \approx 6.07$ (PD) successively. The rotating wave (RW22) of period two generated at PD disappears through the saddle-node bifurcation with the RW12 at $\rho \approx 8.62$ (SN1). The symmetric RW2 is stabilized through the Neimark-Sacker bifurcation at $\rho \approx 10.34$ (NS).

A chaotic rotating wave, in which S_x changes intermittently, coexists with the stable RW1 in $8.62 < \rho < 10.34$ (Fig. 1(d)). A sequence $\{y_1(t_k)\}$ at t_k when the sign of x_1 changes from a negative to a positive at kth time are shown in Fig. 4, and a return map of the successive maxima $y_1^{min}{}_m$ in $\{y_1(t_k)\}$ is shown in Fig. 5, in which we can see type-II intermittency. The largest Liapunov exponent is estimated to be 0.11 at $\rho = 10.0$. It is worth noting that this chaotic rotating wave appears in a range of parameters in which a single Lorenz system is bistable and nonchaotic ($\rho < 24.06$), which is in contrast to the results in [1].

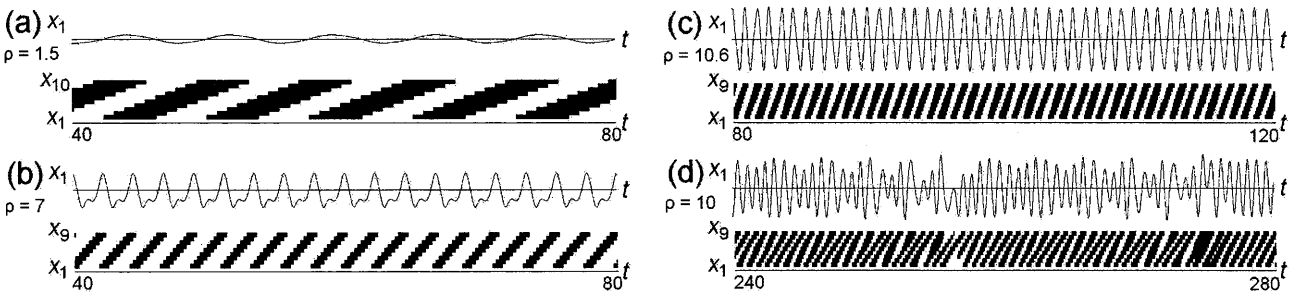

Figure 1: Rotating Waves in a Ring of Lorenz Systems (positive: black, negative: white)

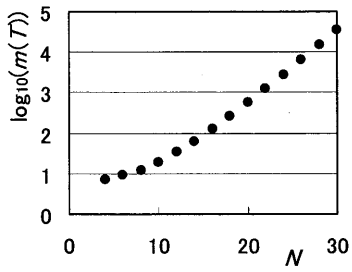

Figure 2: Mean Duration $m(T)$ vs N

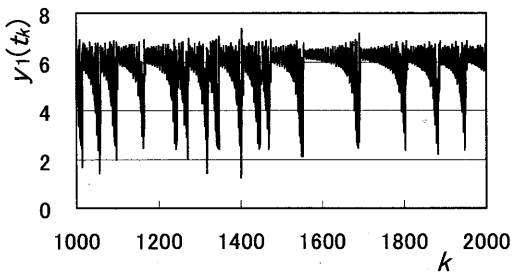

Figure 4: Sequence $\{y_1(t_k)\}$

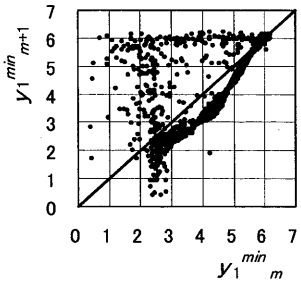

Figure 5: Return map of $y_1^{min}{}_m$

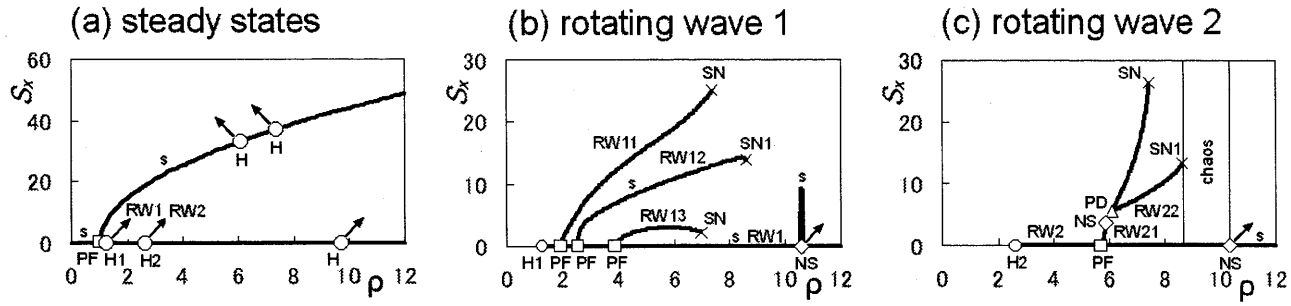

Figure 3: Bifurcation diagrams of steady states and rotating waves (s: stable branch)

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