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Covariant Lyapunov Analysis of Chaotic Kolmogorov Flows and Time-correlation Function

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Introduction

Fluid turbulence is a typical and important example of chaotic dynamical systems, and some aspects of turbulence have been studied from this point of view by several researchers. However, it is not well understood how and what sort of properties of dynamical systems are related to physically important properties of turbulence including the Kolmogorov scaling laws and the intermittency.

Hyperbolicity is one of the fundamental properties of dynamical systems. A dynamical system is called to be hyperbolic if the tangent space of the phase space can be decomposed into the stable and unstable directions (Oseledec decomposition), i.e. the stable and unstable manifolds intersect at nonzero angles.

It is an important problem in the dynamical system theory whether the system is hyperbolic or not, and so several studies have revealed the hyperbolic parameter regions of some dynamical systems. Kuptsov et al. (2009) performed a parameter survey of the one-dimensional coupled Ginzburg-Landau equations [2] using the covariant Lyapunov analysis (mentioned below), and argued that the system becomes non-hyperbolic at a certain parameter value, demonstrating an extensive spatiotemporal chaos after the hyperbolic-nonhyperbolic transition. However, little is known about the hyperbolicity of fluid systems governed by the Navier-Stokes equations. Here, we investigate numerically the hyperbolicity of a fluid system (Kolmogorov flows) and its relations to physical properties. We here focus our attention on behaviors of the time-correlation function of vorticity as one of the fundamental physical properties.

Methods of analysis

Kolmogorov flows are fluid flows governed by the two-dimensional incompressible Navier-Stokes equation on the two-dimensional torus $T^2((x,y) \in [0,2\pi] \times [0,2\pi])$ and the vorticity equation which we solved numerically is

$$\partial_t \zeta + \boldsymbol{u} \cdot \nabla \zeta = \frac{1}{R}\Big(\Delta\zeta - n^3 \cos ny\Big), \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x,y,t) = (u,v)$ is the velocity, $\zeta = \zeta(x,y,t) = \partial_x v - \partial_y u$ the vorticity, R the Reynolds number, and n the wavenumber of external forcing ($n = 2$ in this paper). The governing equation (1) possesses a steady solution $\zeta = -n \cos ny$ which we call the trivial solution and we denote by $R_{cr}(= n\sqrt{2})$ the critical Reynolds number beyond which the trivial solution becomes linearly unstable.

Direct numerical simulations of the vorticity equation (1) were performed by means of the standard 2/3 dealiased spectral method on the periodic domain $T^2 = [0,2\pi] \times [0,2\pi]$ and the 4th order Runge-Kutta method. For the covariant Lyapunov analysis, we used the data set of the Fourier coefficients in the period from $t = 1.00 \times 10^4$ to $t = 17.0 \times 10^4$ where the solution is well within the attractor.

Degree of hyperbolicity is estimated quantitatively along the solution orbit by measuring the angle between the local stable and unstable manifolds by the use of the covariant Lyapunov analysis recentry proposed by Ginelli et al. (2007) [1], which gives Lyapunov vectors tangent to the local stable and unstable manifolds.

Results

Kolmogorov flow becomes chaotic at the Reynolds number $R/R_{cr} \simeq 18.2$, at which a positive Lyapunov exponent arises. We are interested in the degree of hyperbolicity of the chaotic Kolmogorov flows and its relations with physical properties, and employ the covariant Lyapunov analysis in the range of $20.0 \leq R/R_{cr} \leq 24.0$ where positive Lyapunov exponents are observed.

Degree of hyperbolicity

We numerically obtain the probability density function (PDF, $P(\theta)$) of the angles between the local stable and unstable manifolds along the solution orbit. Figure 1 shows close-up ($0 \leq \theta \leq 0.1$[rad]) of the PDF $P(\theta)$ at $\mathrm{R/R_{cr}} = 20.0, 21.0, 22.0, 23.0, 24.0$ from top to bottom (linear-log plot). At the small Reynolds number ($\mathrm{R/R_{cr}} \simeq 20.0$) the angle θ is bounded from below by a certain small angle, which indicates that the attractor is hyperbolic. However, as the Reynolds number is increased, smaller angles appear and probability density of θ increases near $\theta = 0$. And at a certain Reynolds number ($\mathrm{R/R_{cr}} \simeq 23.0$) the distribution is observed to reach the zero angle, which implies that the attractor is non-hyperbolic.

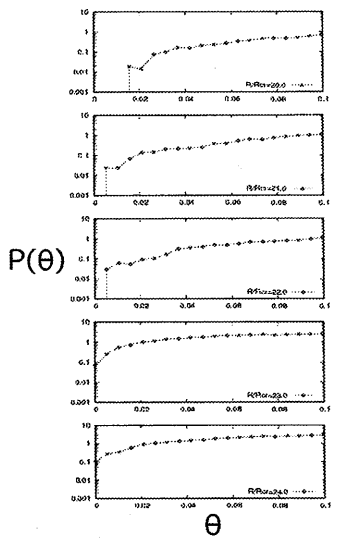

Figure 1: Close-up ($0 \leq \theta \leq 0.1$[rad]) of the PDF $P(\theta)$ of the angles between the local stable and unstable manifolds at $\mathrm{R/R_{cr}} = 20.0, 21.0, 22.0, 23.0, 24.0$ from top to bottom (linear-log plot).

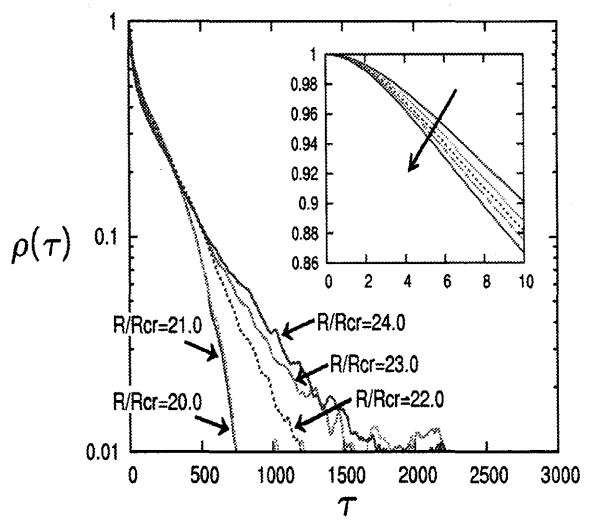

Figure 2: The time-correlation function $\rho(\tau)$ (linear-log plot). Inset is an close-up of the time-correlation function in $0 \leq \tau \leq 10$ (the arrow indicates increase of the Reynolds number).

Time-correlation function

We compute the time-correlation function of vorticity for several Reynolds numbers across the hyperbolic/non-hyperbolic transition point. Figure 2 shows the ensemble averaged time-correlation function $\rho(\tau)$ of vorticity in the range of $20.0 \leq R/R_{cr} \leq 24.0$ (inset is an close-up of the time-correlation function in $0 \leq \tau \leq 10$). The time-correlation function $\rho(\tau)$ for $\tau \lesssim 10$ is almost the same in the whole of this range of Reynolds number. However, the long-time asymptotic form of the correlation function ($\rho(\tau)$, $\tau \gtrsim 100$) changes at $R/R_{cr} = 22.0 \sim 23.0$.

In the range of $23.0 \lesssim R/R_{cr} \lesssim 24.0$ the correlation function has an exponential tail $\rho(\tau) \simeq e^{-\tau/T}$, while in the range of $20.0 \lesssim R/R_{cr} \lesssim 22.0$ the correlation changes its sign. We employ the least-square method to fit the correlation function with $\rho(\tau) = ae^{-\tau/T}\cos\omega\tau$ via three fitting parameters (a, T, ω) in long-time region $100 \leq \tau \leq 700$. While the fitting parameters a and T are found to be almost independent of the Reynolds number, the fitting parameter ω depends strongly on the Reynolds number, as shown in Figure 3. Apparently, the value of the fitting parameter ω shows a clear transition from finite ($\omega \simeq 0.002$) to 0 at $R/R_{cr} \simeq 22.0$. The qualitative change of the long-time correlation of vorticity occurs at $R/R_{cr} \simeq 22.0$ close to that of the hyperbolic/non-hyperbolic transition, which suggests that the time-correlation function reflects the transition to non-hyperbolicity.

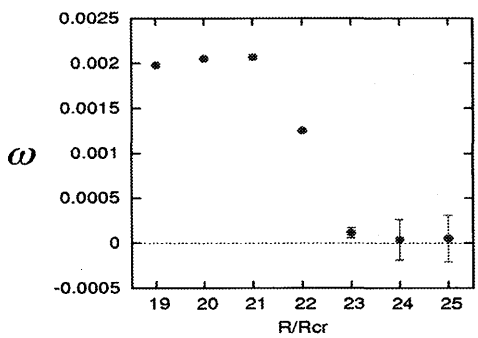

Figure 3: Dependence of the fitting parameter ω on the Reynolds number ($19.0 \leq R/R_{cr} \leq 25.0$). Error bars indicate the standard errors in least-squares method.

References

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[2] P. V. Kuptsov, and S. P. Kuznetsov, Violation of hyperbolicity in a diffusive medium with local hyperbolic attractor, phys. rev. E 80, 016205(11), (2009).