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Ergodic Coordinates for a Perturbed Hill's Spherical Vortex Flow

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Ergodic sets are the finest dynamically distinguishable sets in the state space of a dynamical system. By imposing topology on the collection of all ergodic sets, the ergodic partition, we can quantify degree of similarity of system's behaviors in different parts of the state space. To this end, we propose coordinates for the ergodic partition, in which euclidean distance is equivalent to the, dynamically meaningful, empirical distance on the space of invariant measures. This work describes a computational algorithm for computing the ergodic coordinates, and applies it to a non-autonomous area-preserving flow. We present a novel structural instability, uncovered through ergodic partition analysis, that exists in the high-perturbation regime for the flow.

Ergodic Coordinates through Time Averaging

The computational study of the ergodic partition is detailed in [1–3]. The main idea of identification of ergodic partition is the observation that, given two *distinct* ergodic sets, there exists *some* observable, i.e., function on the state space, whose average along a trajectory of the system will change depending on whether the initial condition was in one or the other ergodic set. The converse holds as well, i.e., points over which trajectory average of *any* observable changes cannot lie in the same ergodic set. Level sets of the time-averaged observable always partition the state space in invariant sets. By considering a function basis for observables, we can approximate the ergodic partition by joint level-sets of time-averages of observable basis [4]. Formally, contracting ergodic sets to points, by mapping them using time-averaged observable basis, amounts to taking a quotient the state space. We refer to the resultant structure as the ergodic quotient.

To endow the ergodic quotient with a dynamically meaningful distance, we chose Fourier functions as the function basis. It was demonstrated in [5] that an appropriately weighted l^2 norm, i.e., a negative-index Sobolev norm, over the space of time-averages of Fourier basis induces a distance that corresponds to empirical distance — a measure of the difference in average set residence times for two trajectories.

The dimension of the ambient space is determined by the number of averaged basis functions. The ergodic quotient itself, however, is of a much lower dimension – in particular it is at most of dimension of the state space, but potentially lower, e.g., for integrable systems. To recover a low-dimensional parametrization, we use the Diffusion Maps algorithm [6], which recovers the modes of diffusion on the low-dimensional objects, in our case the ergodic quotient. Using modes of diffusion as coordinates generates an embedding in which the euclidean distance approximates the empirical distance with exponential convergence in number of retained modes.

Hill's Spherical Vortex Flow Model

The dynamical model of perturbed Hill spherical vortex was presented in [7, 8]. The perturbed Hill spherical vortex is a three-dimensional system, described by canonical equations

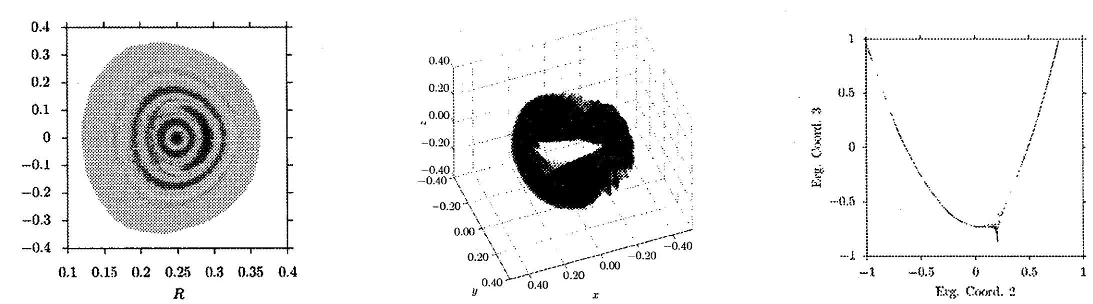
$$\begin{aligned}\dot{R} &= 2Rz + \varepsilon\sqrt{2R}\sin\theta\sin 2\pi t,\\ \dot{z} &= 1 - 4R - z^2 + \varepsilon\frac{z}{\sqrt{2R}}\sin\theta\sin 2\pi t\\ \dot{\theta} &= \frac{\varepsilon}{2R} + 2\varepsilon\cos\theta\sin 2\pi t,\end{aligned} \tag{1}$$

evolving on $X = \mathbb{R}^+ \times \mathbb{R} \times S^1$ domain. The system is volume-preserving in cylindrical coordinates[1]. When unperturbed ($\varepsilon = 0$), there are two sets of hyperbolic fixed points at $(R, z) = (0, \pm 1)$ and one set of elliptic fixed points at $(R, z) = (1/4, 0)$. All three equilibria in (R, z) exist for all values of angle θ, which has no dynamics for the unperturbed case. For small values of ε, the unperturbed dynamics in (R, z) is destroyed gradually in a KAM fashion, as described in [8].

Numerical Implementation and Results

The results presented were computed using an algorithm which simulates a uniform sample of initial conditions over the portion of state space we are interested in analyzing. By selecting a wavenumber cutoff for the Fourier basis, we specify the resolution of the algorithm. Ergodic coordinates are computed by solving an eigenproblem for a real symmetric matrix, which is a discretization of the kernel of a Laplace-Beltrami operator on the ergodic quotient. The index of the ergodic coordinate, as used in the remainder of the paper, is the order of the associated eigenvalue, ordered from the smallest to the largest.

[1] To convert to cylindrical coordinates, use $r = \sqrt{2R}$ transformation, with z and θ unchanged.



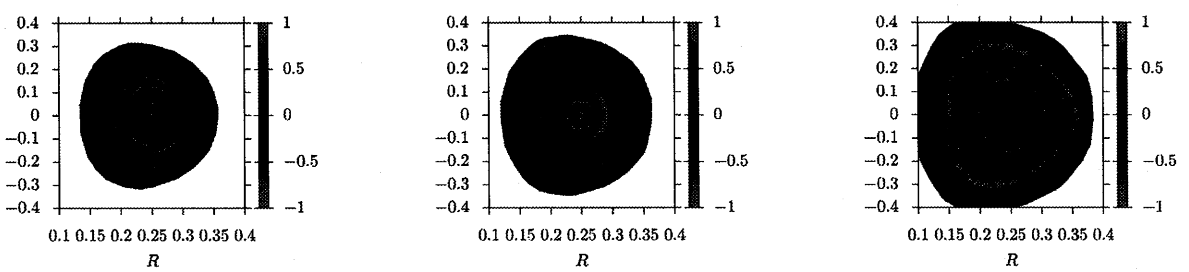
(a) Poincaré cut at $\theta = 0.0$ colored using a linear combination of ergodic coordinates 2–7

(b) Sickle-shaped feature (red) twisting around the central torus (blue) in Cartesian system; feature selection based on coord. 2 and 6

(c) Ergodic quotient embedding into euclidean space using ergodic coordinates; projection on coordinates 2 and 3

Figure 1: Visualizations based on ergodic coordinates. System (1), with $\varepsilon = 0.345$).

In Figure 1 we present a sampling of visualization possibilities based on ergodic trajectories. To colorize a trajectory, we take a pseudocolor vector with the same number of elements as there are trajectories, and assign a color to each trajectory depending on that vector. Ergodic coordinates used for coloring provide such vectors (Fig. 2), as do their linear combinations (Fig. 1(a)). Alternatively, we can place ergodic coordinates directly on axes to visualize a projection of the ergodic quotient on a subset of coordinates (Fig. 1(c)).



(a) $\varepsilon = 0.329$

(b) $\varepsilon = 0.345$

(c) $\varepsilon = 0.353$

Figure 2: Pseudocolorings of the Poincaré cut at $\theta = 0.0$ using ergodic coordinate 6

Using ergodic coordinates to visualize the model (1) uncovered a structural change for large perturbation values, between $\varepsilon = 0.32$ and $\varepsilon = 0.36$, which was not previously described. The central elliptic cycle, which is born out of the circle of elliptic fixed-points in the unperturbed system, is replaced by another elliptic cycle, which pierces the center of the sickle-shaped region in Figure 2(b). A similar behavior, although for hamiltonian, not volume-preserving systems, was described in [9, §3.4], where a series of pitchfork bifurcations ending in a saddle-node bifurcation accounts for the structural change.

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