IUTAM Symposium on 50 Years of Chaos : Applied and Theoretical
Kyoto, Japan, November 28 - December 2, 2011

Phase description of periodic solutions in reaction-diffusion systems

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Phase reduction theory for stable limit-cycle solutions of one-dimensional reaction-diffusion systems is developed. By locally approximating the isochrons of the limit-cycle orbit, we derive the phase sensitivity function, which is a key quantity in the phase description of limit cycles. As an example, synchronization of traveling pulses in a pair of mutually interacting reaction-diffusion systems is analyzed. It is shown that the traveling pulses can exhibit multimodal phase locking.

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Introduction

Spontaneous rhythms naturally arise in nonlinear dissipative systems. Interactions of rhythms can lead to various complex collective dynamics, which may play crucial roles for their functions. The phase reduction method is a powerful technique for analyzing models of interacting regular rhythmic elements, i.e., coupled limit-cycle oscillators. It has been well established for low-dimensional nonlinear oscillators [1]. Non-trivial collective dynamics of coupled oscillator systems have been revealed with this technique, including macroscopic synchronization transition as the prominent example.

The purpose of this work is to extend the applicability of the phase reduction method to limit-cycle solutions of reaction-diffusion systems with infinite-dimensional phase space. The isochrons of the system, which give asymptotic phase values to given spatial patterns in the basin of the limit-cycle orbit, can be locally approximated near the unperturbed limit-cycle solution. The phase sensitivity function of the limit-cycle solution, which quantifies linear response of the phase to weak spatial perturbations, is then derived from the approximated isochrons. Based on this formulation, we analyze a pair of interacting traveling pulses in coupled reaction-diffusion systems and reveal their multi-modal phase locking phenomena.

Phase sensitivity function

Here we briefly summarize the main points of the theory. Detailed formulation will be given elsewhere [3]. For low-dimensional limit cycles described by $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X})$ with $\mathbf{X}(t)$ being a state vector of the oscillator, the isochron $\phi(\mathbf{X}) \in [0, T]$ can be introduced for the stable limit-cycle solution, $\mathbf{X}_0(t)$, which constantly satisfies a simple phase equation $\dot{\phi}(t) = \nabla_{\mathbf{X}}\phi(\mathbf{X}) \cdot \dot{\mathbf{X}}(t) = \nabla_{\mathbf{X}}\phi(\mathbf{X}) \cdot \mathbf{F}(\mathbf{X}) = 1$. With this definition, the time and the phase are equivalent and the location on the limit cycle can be specified by its phase ϕ as $\mathbf{X}_0(\phi)$. When this oscillator is weakly perturbed as $\dot{\mathbf{X}}(t) = \mathbf{F}(\mathbf{X}) + \epsilon\mathbf{p}(t)$, the corresponding phase equation at the lowest order is given by $\dot{\phi}(t) = 1 + \epsilon\mathbf{Z}(\phi) \cdot \mathbf{p}(t)$, where the *phase sensitivity function* $\mathbf{Z}(\phi) = \nabla_{\mathbf{X}=\mathbf{X}_0(\phi)}\phi(\mathbf{X})$, a gradient of the isochron on the limit cycle, is introduced. This function $\mathbf{Z}(\phi)$ encapsulates dynamical properties of the weakly perturbed oscillator and plays a central role in the phase reduction theory. Let us now consider a limit-cycle solution $\mathbf{X}_0(x, t)$ of a reaction-diffusion system on a ring of length L described by

$$\frac{\partial}{\partial t}\mathbf{X}(x,t) = \mathbf{F}(\mathbf{X}) + \mathrm{D}\frac{\partial^2}{\partial x^2}\mathbf{X}.$$

To extend the phase reduction theory, we need to define the *isochron functional* $\phi\{\mathbf{X}(x,t)\}$ for this partial differential equation, which assigns a scalar value $\phi \in [0, T]$ to a given spatial profile $\mathbf{X}(x, t)$ of the system that eventually converges to the limit-cycle solution. This is difficult in general, but, in the vicinity of the limit-cycle solution, $\mathbf{X}(x,t) \simeq \mathbf{X}_0(x; \phi_0)$, its linear approximation can be given as

$$\phi\{\mathbf{X}(x,t)\} = \phi_0 + [\mathbf{Q}(x,\phi_0), \mathbf{X}(x,t) - \mathbf{X}_0(x,\phi_0)],$$

where the function $\mathbf{Q}(x,t)$ is a time-periodic solution of the following adjoint equation:

$$\frac{\partial}{\partial t}\mathbf{Q}(x,t) = -\hat{L}^{\dagger}(x,t)\mathbf{Q}(x,t)$$

with the initial condition $\mathbf{Q}(x,0) = \mathbf{u}^{\dagger}(x,0)$. Here, the inner product is defined by $[\mathbf{A}(x), \mathbf{B}(x)] = \int_0^L \mathbf{A}(x) \cdot \mathbf{B}(x)dx$, $\hat{L}^{\dagger}(x,t)$ is an adjoint operator to a linearized operator $\hat{L}(x,t)$ of the above reaction-diffusion equation near the periodic solution, $\mathbf{X}_0(x,\phi)$, and $\mathbf{u}^{\dagger}(x,0)$ is a zero eigenvector of $\hat{L}^{\dagger}(x,0)$. This $\mathbf{Q}(x,t)$ is the phase sensitivity function of the present reaction-diffusion system, and, if the system is weakly perturbed as

$$\frac{\partial}{\partial t}\mathbf{X}(x,t) = \mathbf{F}(\mathbf{X}) + \mathrm{D}\frac{\partial^2}{\partial x^2}\mathbf{X} + \epsilon\mathbf{h}(x,t),$$

the phase dynamics of its approximate solution $\mathbf{X}(x,t) \simeq \mathbf{X}_0(x,\phi(t))$ is given at the lowest order as

$$\dot{\phi}(t) = 1 + \epsilon[\mathbf{Q}(x,\phi(t)), \mathbf{h}(x,t)].$$

Thus, the above weakly perturbed reaction-diffusion system can be approximately reduced to a single phase dynamics in the vicinity of the unperturbed limit-cycle solution, just as in the case of low-dimensional dynamical systems. The above result for the function $\mathbf{Q}(x,t)$ is actually a straightforward generalization of the Malkin-Izhikevich-Hoppensteadt theorem [2] for ordinary low-dimensional limit cycles.

Synchronization of traveling pulses between a pair of reaction-diffusion systems

Let us consider a mutually interacting pair of reaction-diffusion systems (the FitzHugh-Nagumo model is used for numerical simulations), both of which exhibit stable traveling pulse solutions, $\mathbf{X}_0^{A,B}(x,\phi(t))$:

$$\frac{\partial}{\partial t}\mathbf{X}^A(x,t) = \mathbf{F}(\mathbf{X}^A) + \mathrm{D}\frac{\partial^2}{\partial x^2}\mathbf{X}^A + \epsilon(\mathbf{X}^B - \mathbf{X}^A), \quad \frac{\partial}{\partial t}\mathbf{X}^B(x,t) = \mathbf{F}(\mathbf{X}^B) + \mathrm{D}\frac{\partial^2}{\partial x^2}\mathbf{X}^B + \epsilon(\mathbf{X}^A - \mathbf{X}^B).$$

Using our formulation, we can reduce these equations to coupled phase equations,

$$\dot{\phi}^A(t) = 1 + \epsilon\Gamma(\phi^B - \phi^A), \quad \dot{\phi}^B(t) = 1 + \epsilon\Gamma(\phi^A - \phi^B),$$

where $\phi^A(t)$ and $\phi^B(t)$ represent the phases of the traveling pulses, which are simply their locations in the present case, and the function $\Gamma(\phi^B - \phi^A) = [\mathbf{Q}(x,\phi^A), \mathbf{X}_0(x,\phi^B)]$ is the phase coupling function that determines effective interaction between the traveling pulses (due to translational symmetry in the present case, Γ depends only on the phase difference $\phi^B - \phi^A$ without performing the averaging step [1, 2]). It is predicted that the pulses can mutually phase lock at multiple locations, reflecting complex functional shapes of the phase sensitivity function. This prediction is confirmed by numerical simulations as shown in Fig. 1.

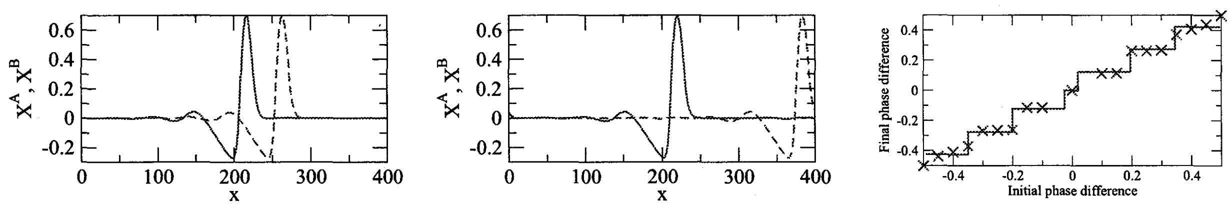

Figure 1: (Left, Middle) Multimodal phase locking between two pulses of coupled FitzHugh-Nagumo equations. (Right) Predicted steady phase shifts compared with the results of direct numerical simulations.

Summary

We have briefly explained our framework of phase reduction approach to limit-cycle solutions in infinite-dimensional reaction-diffusion systems. Interacting traveling pulses have been analyzed within the interface dynamics approach in the past. Our approach may be advantageous to the conventional approach in that it can be applicable to systems without spatial translational symmetry. For example, we can analyze synchronization of coupled breathing solutions in coupled reaction-diffusion systems as well. Further details will be reported in Ref. [3]. This work is supported by MEXT (Grant No. 22684020) and by the JST-CREST program, Japan.

References

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