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A combinatorial framework for nonlinear dynamics

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In building a robust computational framework for studying global dynamics we must acknowledge some inherent obstacles in the classical approach to the analysis of nonlinear systems which arise from the fact that global dynamical structures can vary on all scales in both phase space and parameter space. For example, in a chaotic system arbitrarily small perturbations/errors rapidly lead to order one differences in the behavior of trajectories. Thus, the behavior of an individual orbit obtained through a numerical simulation may or may not represent a true orbit of the system. Dynamical systems theory deals with this by focusing on the existence and structure of invariant sets as opposed to the behavior of any single orbit. A similar more profound phenomenon occurs in the context of bifurcation theory; for parameterized families of non-uniformly hyperbolic systems there can exist regions in parameter space for which bifurcations occur on Cantor sets of positive measure. This implies that even if one has a perfect model and can perform perfect computations, then the existence, stability, and structures of the invariant sets for the computed dynamics may or may not match that of the system being studied. In the context of applications the possibility of mismatch is even more likely as models are never exact and computations typically contain errors of various magnitudes. This suggests the value of extracting coarse, robust, but mathematically rigorous, descriptions of the dynamics. With this in mind we believe it is essential to de-emphasize the direct focus on invariant sets and their associated structures.

The computational methods by which we propose to study the global dynamics of nonlinear multiparameter systems are based on ideas of C. Conley [2]. The core objects that define the theory are robust with respect to perturbations, and while not invariant, are able to characterize underlying invariant structures both globally and locally. Thus, this theoretical foundation directly addresses the issues of identifying robust, coarse structures described above. Moreover, Conley theory appears to be inherently combinatorial and hence can be formulated in an algorithmic manner [4]. In its simplest setting Conley theory consists of three core components: isolation, decomposition, and reconstruction via the Conley index. In the following sections, we describe these components in terms of discrete systems determined by iterating a continuous map. This restriction is for simplicity of exposition; the theory applies equally well to dynamics generated by differential equations, and in principle can be applied in systems lacking closed-form mathematical formulations.

Isolation

Consider a multiparameter family of dynamical systems given by a continuous function $f\colon X \times \Lambda \to X$ (not necessarily a family of homeomorphisms) where the phase space X is a locally compact metric space and the parameter space Λ is a compact, locally contractible, connected metric space. For each parameter value $\lambda \in \Lambda$ we have a dynamical system $f_\lambda\colon X \to X$. A compact set $N \subset X$ is an *isolating neighborhood* for f_{λ_0} if $\mathrm{Inv}\,(N, f_{\lambda_0})$, the maximal invariant set in N under f_{λ_0}, is contained in the interior of N. We focus on identifying isolating neighborhoods for the following two reasons: (1) if N is an isolating neighborhood for f_{λ_0}, then N is an isolating neighborhood for f_λ for all λ sufficiently close to λ_0; and (2) there exist efficient algorithms for finding isolating neighborhoods.

From our perspective it is important to have explicit knowledge about the set of parameter values over which isolation occurs. Thus we consider the associated system which includes the parameters as state variables, i.e.

$$\begin{array}{rcl} F\colon X \times \Lambda & \to & X \times \Lambda \\ (x, \lambda) & \mapsto & (f_\lambda(x), \lambda). \end{array} \tag{1}$$

and perform our computations over sets of parameter values. Thus, given $\Lambda_0 \subset \Lambda$, we consider $F_{\Lambda_0}\colon X \times \Lambda_0 \to X \times \Lambda_0$, the restriction of F defined by (1).

To perform the computations we discretize the phase space x via a finite grid $\mathcal{X}$[1]. A *combinatorial multivalued map* $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ takes grid elements to sets of grid elements. It is an *outer approximation* of f_λ if $f(G) \subset \mathrm{int}\,(|\mathcal{F}(G)|)$ for each grid element G, where $|\mathcal{F}(G)|$ denotes the union of the grid elements in $\mathcal{F}(G)$. To compute over sets of parameter values, consider a finite grid $\mathcal{Q}$ for Λ. For fixed $Q \in \mathcal{Q}$ construct $\mathcal{F}_Q$ such that $f(G, Q) \subset \mathrm{int}\,(|\mathcal{F}_Q(G)|)$ for all $G \in \mathcal{X}$. Since f is a continuous map, one can obtain bounds on f, including numerical errors, to construct a rigorous outer approximation computationally via interval arithmetic. Note that there is a minimal multivalued map defined by $\mathcal{F}(G) = \{H \in \mathcal{X} \mid H \cap F(G) \neq \emptyset\}$ which is useful for theoretical purposes but cannot be computed in practice.

[1] In practice this is done in a multiscale manner

Decomposition

Conley introduced the notion of a *Morse decomposition* of an isolated invariant set which provides a language for describing the decomposition of dynamics into gradient-like, i.e. strictly non-recurrent, and recurrent-like dynamics. For the isolated invariant set S_Q under F_Q, this consists of a finite collection of disjoint isolated invariant sets $M_Q(p) \subset S_Q$ called *Morse sets* which are indexed by a finite poset $(\mathsf{P}_Q, >_Q)$, such that for every $(x, \lambda) \in S_Q \setminus \bigcup_{p \in \mathsf{P}} M_Q(p)$ and any complete orbit γ of f_λ through x in S_λ there exist indices $p >_Q q$ such that under f_λ, $\omega(\gamma) \subset M_\lambda(q)$ and $\alpha(\gamma) \subset M_\lambda(p)$. Thus S_Q is composed of the Morse sets and connecting orbits between Morse sets. Due to the strict partial order, the dynamics is gradient-like on the complement of the Morse sets.

As in the case of isolating neighborhoods there are two important facts. Since Morse sets are isolated invariant sets, they can be characterized in terms of isolating neighborhoods, and the partial order can be determined by approximating orbits between isolating neighborhoods. Given $\mathcal{F}$ and outer approximation for F_Q there are linear time algorithms for determining isolating neighborhoods for Morse decompositions [3].

Reconstruction via the Conley index

The final ingredient to this approach to dynamics is the Conley index. To define this one needs an index pair $P = (P_Q^1, P_Q^0)$ for S_Q where $P_Q^i \subset X \times Q$. The shift equivalence class of the induced map on homology $F_{Q,P*}: H_*(P_Q^1/P_Q^0, [P_Q^0]) \to H_*(P_Q^1/P_Q^0, [P_Q^0])$ represents the Conley index of the isolated invariant set S_Q. The continuation of the Conley index follows from the fact that for any $\lambda \in Q$ we have $f_{\lambda,P*} : H_*(P_\lambda^1/P_\lambda^0, [P_\lambda^0]) \to H_*(P_\lambda^1/P_\lambda^0, [P_\lambda^0])$ is shift equivalent to $F_{Q,P*}$.

It should be noted that given an outer approximation $\mathcal{F}$ there are fast algorithms for determining index pairs of Morse sets in terms of the grid $\mathcal{X}$. Furthermore, if $\mathcal{F}$ takes acyclic values then there exists algorithms to compute $F_{Q,P*}$.

The Conley index provides information about the existence, structure, and stability of the maximal invariant set in $(P_Q^1 \setminus P_Q^0)$. The most basic result is that if F_{QP*} is not nilpotent, then $\mathrm{Inv}\,(P_Q^1 \setminus P_Q^0, f_\lambda) \neq \emptyset$, for all $\lambda \in Q$. This robust description is by necessity coarse, but it provides a systematic method to overcome the problem of bifurcations occurring on all scales of parameter space.

For each grid element Q in parameter space the information obtained from the computation of the Morse decomposition and the Conley indices of the Morse sets is codified via a *Conley-Morse graph*. This is an acyclic directed graph obtained from the poset $(\mathsf{P}_Q, >_Q)$ where the nodes represent the Morse sets and carries the Conley index information for the Morse set. Since there are only a finite number of grid elements that cover parameter space this provides a finite combinatorial encoding of the dynamics.

Applications

The above mentioned techniques have been applied to a variety of systems including the simple nonlinear Leslie model $f: \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}^2$

$$(x, \lambda) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} \theta_1 \\ \theta_2 \\ p \end{bmatrix} \right) \mapsto f(x, \lambda) = \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-0.1(x_1+x_2)} \\ p x_1 \end{bmatrix} \tag{2}$$

which was studied in [1]. The resulting Conley-Morse graphs can be found at `http://chomp.rutgers.edu/databas`

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