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Kyoto University
\(\beta\)-expansion’s Attractors Observed in A/D converters

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A new class of analog-to-digital (A/D) and digital-to-analog (D/A) converters using a flaky quantiser, called the \(\beta\)-encoder, as shown in Fig. 1 \([1, 2, 3]\) has been shown to have exponential bit rate accuracy while possessing a self-correction property for fluctuations of the amplifier factor \(\beta\) and the quantiser threshold \(\nu\). Motivated by the close relationships \([4, 5, 6]\) between \(\beta\)-transformations and \(\beta\)-expansion, we have recently observed\([9, 10]\) that (1) such a flaky quantiser is exactly realized by the “multi-valued Rényi-Parry map”, defined here so that probabilistic behavior in the “flaky region” is completely explained using dynamical systems theory; (2) a sample \(x\) is always confined to a subinterval of the contracted interval while the successive approximation of \(x\) is stably performed using \(\beta\)-expansion even if \(\nu\) may vary at each iteration (i.e. a small real-valued quantity, approximately proportional to the quantisation error, does not necessarily converge to any fixed value, e.g., 0 but may oscillate without diverging. Such a phenomenon is precisely the kind of “chaos”); (3) such a subinterval enables us to obtain the decoded sample easily, as it is equal to the midpoint of the subinterval and to prove two classic \(\beta\)-expansions, known as the greedy and lazy expansions \([7, 8]\) are perfectly symmetrical in terms of their quantisation errors. The subinterval further suggests that \(\nu\) should be set to around the midpoint of its associated greedy and lazy values. A switched-capacitor (SC) circuit technique \([11, 12]\) has been proposed for implementing A/D convereter circuit based on several types of \(\beta\)-encoders and SPICE simulations have been given to verify the validity of these circuits against deviations and mismatches of circuit parameters. Our review

Figure 1: A typical \(\beta\)-encoder with its input \(z_0 = y \in [0, 1)_x\), \(z_i = 0\), \(i > 1\) and output \((b_i, \beta)\)\(\geq 1\).

Figure 2: The scale-adjusted \(\beta\)-map: \(S_{\beta, v, s}(x)\) and its eventually onto map in which an attractor can be observed.

Figure 3: The scale-adjusted negative \(\beta\)-map: \(R_{\beta, v, s}(x)\) and its eventually onto map in which an attractor can be observed.

is twofold. First, the \(\beta\)-encoder leads us to naturally define the “multi-valued Rényi-Parry map” \([4, 5]\) with its eventually onto map, as it is identical to the Parry’s \((\beta, \alpha)\)-map \([6]\). Second, chaos, called \(\beta\)-expansion’s attractors” can be seen on the onto-map. Two types of \(\beta\)-expansion’s attractors are as follows:

1. **Scale-Adjusted \(\beta\)-Map\([9, 11]\):** Daubechies et al. \([1, 2]\) introduced a “flaky” version of an imperfect quantiser, defined as

\[
Q_{\Delta_\beta}^\nu(z) = \begin{cases} 
0, & \text{if } z \leq \nu_0 \\
1, & \text{if } z \geq \nu_1 \\
0 \text{ or } 1, & \text{if } z \in \Delta_\beta = [\nu_0, \nu_1], \nu_0 < \nu_1,
\end{cases}
\]

which is a \(\nu\)-varying model of a quantiser \(Q_v(z) = \begin{cases} 
0, & \text{if } z \leq \nu \\
1, & \text{if } z \geq \nu
\end{cases}, \nu \in [\nu_0, \nu_1], \nu_0 < \nu_1\). We obtain:

**Lemma 1\([9]\):** Let \(S_{\beta, v, s}(x)\) be the scale-adjusted map with a scale \(s\), defined by

\[
S_{\beta, v, s}(x) = \beta x - s(\beta - 1)Q_\nu(x) = \begin{cases} 
\beta x, & x \in [0, \gamma_\nu), \\
\beta x - s(\beta - 1), & x \in [\gamma_\nu, s), \nu \in [s(\beta - 1), s], s > 0
\end{cases}
\]
which is referred to as the “multi-valued Rényi-Parry map” on the flaky region $\Delta_\beta = [s(\beta - 1), s]$ and has its eventually onto Parry’s $(\beta, \alpha)$-map [6] with the subinterval $[\nu - s(\beta - 1), \nu]$ as shown in Fig. 2. This map realises the flaky quantiser $Q_s^{1}_{[s(1-\gamma), \gamma]}(\cdot)$. Let $b_i, s_{R, \nu, i}$ be its associated bit sequence for the threshold sequence $\nu_i = \nu_1 \nu_2 \cdots \nu_L$, defined by

$$b_i, s_{R, \nu, i} = Q_{\gamma \nu} (s_{R, \nu, i-1}, s) (x) = \begin{cases} 0, & s_{R, \nu, i-1}, s (x) \in [0, \gamma \nu_i), \\ 1, & s_{R, \nu, i-1}, s (x) \in [\gamma \nu_i, s). \end{cases}$$

(3)

Then we get $x = s(\beta - 1) \sum_{i=1}^L b_i, s_{R, \nu, i} \gamma_i + \gamma \nu L s_{R, \nu, i} (x)$ and its decoded value

$$\bar{x}_{L, s_{R, \nu, i}} = s(\beta - 1) \sum_{i=1}^L b_i, s_{R, \nu, i} \gamma_i + \frac{\nu L s_{R, \nu, i}}{2}.$$ 

2. Negative $\beta$-Map [10]: We get

**Lemma 2** [10]: Let $R_{\beta, \nu, \beta}(x) : [0, s) \to [0, s)$, $s > 0$ be the (scale-adjusted) negative $\beta$-map, defined by

$$R_{\beta, \nu, \beta}(x) = -\beta x + s[1 + (\beta - 1)Q_{\nu}(x)] = \begin{cases} s - \beta x, & x \in [0, \gamma \nu), \\ \beta s - \beta x, & x \in [\gamma \nu, s). \end{cases}$$

(4)

which is another “multi-valued Rényi-Parry map” on the flaky region $\Delta_\beta = [s(\beta - 1), s]$ realising $Q_s^{1}_{[s(1-\gamma), \gamma]}(\cdot)$ and has its eventually onto Parry’s $(\beta, \alpha)$-map [6] with the subinterval $[s - \nu, \beta s - \nu]$ as shown in Fig. 3. Let $b_i, R_{\beta, \nu, i}$ be the associated bit sequence for the threshold sequence $\nu_i$, defined by

$$b_i, R_{\beta, \nu, i} = Q_{\gamma \nu} (R_{\beta, \nu, i-1}, s) (x) = \begin{cases} 0, & R_{\beta, \nu, i-1}, s (x) \in [0, \gamma \nu_i), \\ 1, & R_{\beta, \nu, i-1}, s (x) \in [\gamma \nu_i, s). \end{cases}$$

(5)

Then we get $x = (-\gamma) L R_{\beta, \nu, i} (x) - s \sum_{i=1}^L f_i, R_{\beta, \nu, i} (x) - (\gamma)$ and its decoded value

$$\bar{x}_{L, R_{\beta, \nu, i}} = s(-\gamma) L R_{\beta, \nu, i} - 2 \sum_{i=1}^L f_i, R_{\beta, \nu, i} (x) - (\gamma) L,$$

where $f_i, R_{\beta, \nu, i} = 1 + b_i, R_{\beta, \nu, i} (\beta - 1)$. Such a negative $\beta$-expansion defines a new $A/D$ converter called a negative $\beta$-encoder which facilitates the implementation of stable analog circuits. Figures 2 [11] and 3 [12] show a typical $\beta$-expansion's attractor of Eqs.(2) and (4), respectively.

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References


