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β-expansion's Attractors Observed in A/D converters

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A new class of analog-to-digital (A/D) and digital-to-analog (D/A) converters using a flaky quantiser, called the β-encoder, as shown in Fig. 1 [1, 2, 3] has been shown to have exponential bit rate accuracy while possessing a self-correction property for fluctuations of the amplifier factor β and the quantiser threshold ν. Motivated by the close relationships [4, 5, 6] between β-transformations and β-expansion, we have recently observed[9, 10] that (1) such a flaky quantiser is exactly realized by the "*multi-valued Rényi-Parry map*", defined here so that probabilistic behavior in the "flaky region" is completely explained using dynamical systems theory; (2) a sample x is always confined to a subinterval of the contracted interval while the successive approximation of x is *stably* performed using β-expansion even if ν may vary at each iteration (*i.e.* a small real-valued quantity, approximately proportional to the quantisation error, does not necessarily converge to any fixed value, *e.g.,* 0 but may oscillate without diverging . Such a phenomenon is precisely the kind of "*chaos*"); (3) such a subinterval enables us to obtain the decoded sample easily, as it is equal to the midpoint of the subinterval and to prove two classic β-expansions, known as the *greedy* and *lazy* expansions [7, 8] are perfectly symmetrical in terms of their quantisation errors. The subinterval further suggests that ν should be set to around the midpoint of its associated greedy and lazy values. A switched-capacitor (SC) circuit technique [11, 12] has been proposed for implementing A/D convereter cicuit based on several types of β-encoders and SPICE simulations have been given to verify the validity of these circuits against deviations and mismatches of circuit parameters. Our review

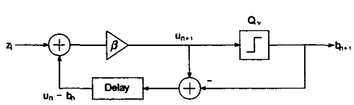

Figure 1: A typical β-encoder with its input $z_0 = y \in [0,1), z_i = 0, i > 1$ and output $(b_{i,\beta})_{i\geq 1}$.

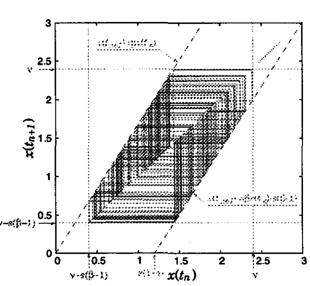

Figure 2: The scale-adjusted β-map: $S_{\beta,\nu,s}(x)$ and its eventually onto map in which an attractor can be observed.

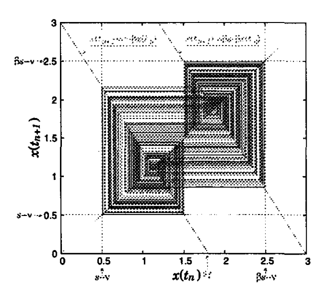

Figure 3: The scale-adjusted negative β-map: $R_{\beta,\nu,s}(x)$ and its eventually onto map in which an attractor can be observed.

is twofold. First, the β-encoder leads us to naturally define the "*multi-valued Rényi-Parry map*" [4, 5] with its eventually onto map, as it is identical to the Parry's $(\beta,\alpha)-$map [6]. Second, *chaos*, called "*β-expansion's attractors*" can be observed on the onto-map. Two types of β-expansion's attractors are as follows:

1. **Scale-Adjusted β-Map**[9, 11]: Daubechies et al. [1, 2] introduced a "flaky" version of an imperfect quantiser, defined as

$$Q^f_{\Delta_\beta}(z) = \begin{cases} 0, & \text{if } z \leq \nu_0, \\ 1, & \text{if } z \geq \nu_1, \\ 0 \text{ or } 1, & \text{if } z \in \Delta_\beta = [\nu_0, \nu_1], \nu_0 < \nu_1, \end{cases} \quad (1)$$

which is a ν-varying model of a quantiser $Q_\nu(z) = \begin{cases} 0, & \text{if } z \leq \nu, \\ 1, & \text{if } z \geq \nu, \end{cases}$ $\nu \in [\nu_0, \nu_1], \nu_0 < \nu_1$. We obtain:

Lemma 1[9]: *Let $S_{\beta,\nu,s}(x)$ be the scale-adjuted map with a scale s, defined by*

$$S_{\beta,\nu,s}(x) = \beta x - s(\beta-1)Q_{\gamma\nu}(x) = \begin{cases} \beta x, & x \in [0, \gamma\nu), \\ \beta x - s(\beta - 1), & x \in [\gamma\nu, s), \end{cases} \quad \nu \in [s(\beta-1), s),\ s > 0 \quad (2)$$

which is refferred to as the "multi-valued Rényi-Parry map" on the flaky region $\Delta_\beta = [s(\beta-1), s]$ and has its eventually onto Parry's $(\beta,\alpha)-$map [6] with the subinterval $[\nu - s(\beta-1), \nu)$ as shown in Fig.2. This map realises the flaky quantiser $Q^f_{[s(1-\gamma),s\gamma]}(\cdot)$. Let $b_{i,S^i_{\beta,\nu^i_1,s}}$ be its associated bit sequence for the threshold sequence $\nu^L_1 = \nu_1\nu_2\cdots\nu_L$, defined by

$$b_{i,S^i_{\beta,\nu^i_1,s}} = Q_{\gamma\nu}(S^{i-1}_{\beta,\nu^{i-1}_1,s}(x)) = \begin{cases} 0, & S^{i-1}_{\beta,\nu^{i-1}_1,s}(x) \in [0,\gamma\nu_i), \\ 1, & S^{i-1}_{\beta,\nu^{i-1}_1,s}(x) \in [\gamma\nu_i, s). \end{cases} \tag{3}$$

Then we get $x = s(\beta-1)\sum_{i=1}^L b_{i,S^i_{\beta,\nu^i_1,s}}\gamma^i + \gamma^L S^L_{\beta,\nu^L_1,s}(x)$ and its decoded value $\widehat{x}_{L,S^L_{\beta,\nu^L_1,s}} = s(\beta-1)\sum_{i=1}^L b_{i,S^i_{\beta,\nu^i_1,s}}\gamma^i + \frac{s\gamma^L}{2}$.

2. **Negative β-Map**[10, 12]:We get

Lemma 2[10]: Let $R_{\beta,\nu,s}(x) : [0,s) \to [0,s)$, $s > 0$ be the (scale-adjusted) negative β-map, defined by

$$R_{\beta,\nu,s}(x) = -\beta x + s[1+(\beta-1)Q_{\gamma\nu}(x)] = \begin{cases} s-\beta x, & x \in [0,\gamma\nu), \\ \beta s - \beta x, & x \in [\gamma\nu, s), \end{cases} \quad \nu \in [s(\beta-1), s] \tag{4}$$

which is another "*multi-valued Rényi-Parry map*" on the flaky region $\Delta_\beta = [s(\beta-1), s]$ realising $Q^f_{[s(1-\gamma),s\gamma]}(\cdot)$ and has its eventually onto Parry's $(\beta,\alpha)-$map [6] with the subinterval $[s-\nu, \beta s - \nu)$ as shown in Fig. 3. Let $b_{i,R^i_{\beta,\nu^i_1,s}}$ be the associated bit sequence for the threshold sequence ν^L_1, defined by

$$b_{i,R^i_{\beta,\nu^i_1,s}} = Q_{\gamma\nu}(R^{i-1}_{\beta,\nu^{i-1}_1,s}(x)) = \begin{cases} 0, & R^{i-1}_{\beta,\nu^{i-1}_1,s}(x) \in [0,\gamma\nu_i), \\ 1, & R^{i-1}_{\beta,\nu^{i-1}_1,s}(x) \in [\gamma\nu_i, s). \end{cases} \tag{5}$$

Then we get $x = (-\gamma)^L R^L_{\beta,\nu^L_1,s}(x) - s\sum_{i=1}^L f_{i,R^i_{\beta,\nu^i_1,s}}(-\gamma)^i$ and its decoded value $\widehat{x}_{L,R^L_{\beta,\nu^L_1,s}} = s\{(-\gamma)^L/2 - \sum_{i=1}^L f_{i,R^i_{\beta,\nu^i_1,s}}(-\gamma)^i\}$. where $f_{i,R^i_{\beta,\nu^i_1,s}} = 1 + b_{i,R^i_{\beta,\nu^i_1,s}}(\beta-1)$. Such a negative β-expansion defines a new A/D converter called a *negative β-encoder* which facilitates the implementation of stable analog circuits. Figures 2 [11] and 3 [12] show a typical β-expansion's attractor of Eqs.(2) and (4), respectively.

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References

[1] I. Daubechies, R. DeVore, C. Güntürk, and V. Vaishampayan, "Beta expansions: a new approach to digitally corrected A/D conversion," *IEEE ISCAS 2002*, **2**, 784-787, May. 2002.
[2] I. Daubechies, R. DeVore, C. Güntürk, and V. Vaishampayan, "A/D conversion with imperfect quantizers," *IEEE Transcations on Information Theory*, **52**-3, 874-885, Mar. 2006.
[3] I. 1Daubechies, and Ö. Yilmaz, "Robust and practical analog-to-digital conversion with exponential precision," *IEEE Transactions on Information Theory*, **52**-8, 3533-3545, Aug. 2006.
[4] A. Rényi, "Representations for real numbers and their ergodic properties," *Acta Mathematica Hungarica*, **8**, no.3-4, 477-493, Sep. 1957.
[5] W. Parry, "On the β-expansions of real numbers," *Acta Math. Acad. Sci. Hung.*, **11**, 401–416, 1960.
[6] W. Parry, "Representations for real numbers," *Acta Math.Acad. Sci.Hung.*, **15**, 95–105,1964.
[7] P. Erdös, and I. Joó, "On the expansion $1 = \sum_{i=1}^\infty q^{-n_i}$," *Periodica Mathematica Hungarica*, **23**-1, 25-28, Aug. 1991.
[8] P. Erdös, I. Joó and V. Komornik, "Characterization of the unique expansions $1 = \sum_{i=1}^\infty q^{-n_i}$ and related problems", *Bull. Soc. Math. France*, **118**, 377-390, 1990.
[9] Hironaka, S., Kohda, T. and Aihara, K., "Markov chain of binary sequences generated by A/D conversion using β-encoder", *Proc. of 15th IEEE International Workshop on Nonlinear Dynamics of Electronic Systems*, 261-264, 2007.
[10] T. Kohda, S. Hironaka, and K. Aihara, "Negative β-encoder,"arXiv:0808.2548v1 [cs.IT] 19 Aug 2008(arXiv:0808.2548v2 [cs.IT] 28 Jul 2009).
[11] Y. Horio, K. Jin'no, T. Kohda and K. Aihara. "Circuit Implementation of an A/D Converter Based on the Scale- Adjusted β-Map Using a Discrete-Time Integrator," *18th IEEE Workshop on Nonlinear Dynamics of Electronic Systems (NDES2010)*,110-113, 2010.
[12] Y. Horio, K. Jin'no, T. Kohda and K. Aihara. "Circuit Implementation of an A/D Converter Based on the Negative β-Map with a Discrete-Time Integrator," *2010 International Symposium on Nonlinear Theory and its Applications (Nolta2010)*, 265-268, 2010.