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\( \beta \)-expansion's Attractors Observed in A/D converters

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A new class of analog-to-digital (A/D) and digital-to-analog (D/A) converters using a flaky quantiser, called the \( \beta \)-encoder, as shown in Fig. 1 [1, 2, 3] has been shown to have exponential bit rate accuracy while possessing a self-correction property for fluctuations of the amplifier factor \( \beta \) and the quantiser threshold \( \nu \). Motivated by the close relationships [4, 5, 6] between \( \beta \)-transformations and \( \beta \)-expansion, we have recently observed [9, 10] that (1) such a flaky quantiser is exactly realized by the "multi-valued Rényi-Parry map", defined here so that probabilistic behavior in the "flaky region" is completely explained using dynamical systems theory; (2) a sample \( x \) is always confined to a subinterval of the contracted interval while the successive approximation of \( x \) is stably performed using \( \beta \)-expansion even if \( \nu \) may vary at each iteration (i.e. a small real-valued quantity, approximately proportional to the quantisation error, does not necessarily converge to any fixed value, e.g., 0 but may oscillate without diverging. Such a phenomenon is precisely the kind of "chaos"; (3) such a subinterval enables us to obtain the decoded sample easily, as it is equal to the midpoint of the subinterval and to prove two classic \( \beta \)-expansions, known as the greedy and lazy expansions [7, 8] are perfectly symmetrical in terms of their quantisation errors. The subinterval further suggests that \( \nu \) should be set to around the midpoint of its associated greedy and lazy values. A switched-capacitor (SC) circuit technique [11, 12] has been proposed for implementing A/D convereter circuit based on several types of \( \beta \)-encoders and SPICE simulations have been given to verify the validity of these circuits against deviations and mismatches of circuit parameters. Our review

is twofold. First, the \( \beta \)-encoder leads us to naturally define the "multi-valued Rényi-Parry map" [4, 5] with its eventually onto map, as it is identical to the Parry's \((\beta, \alpha)\)-map [6]. Second, chaos, called "\( \beta \)-expansion's attractors" can be observed on the onto-map. Two types of \( \beta \)-expansion's attractors are as follows:

1. Scale-Adjusted \( \beta \)-Map [9, 11]: Daubechies et al. [1, 2] introduced a "flaky" version of an imperfect quantiser, defined as

\[
Q^\nu_{\Delta \beta}(z) = \begin{cases} 
0, & \text{if } z \leq \nu_0, \\
1, & \text{if } z \geq \nu_1, \\
0 \text{ or } 1, & \text{if } z \in \Delta \beta = [\nu_0, \nu_1], \nu_0 < \nu_1,
\end{cases}
\]

(1)

which is a \( \nu \)-varying model of a quantiser \( Q_\nu(z) = \begin{cases} 
0, & \text{if } z \leq \nu, \\
1, & \text{if } z \geq \nu,
\end{cases} \nu \in [\nu_0, \nu_1], \nu_0 < \nu_1 \). We obtain:

Lemma 1 [9]: Let \( S_{\beta, \nu, s}(x) \) be the scale-adjusted map with a scale \( s \), defined by

\[
S_{\beta, \nu, s}(x) = \beta x - s(\beta - 1)Q^\nu(x) = \begin{cases} 
\beta x, & x \in [0, \nu], \\
\beta x - s(\beta - 1), & x \in [\nu, s], \nu \in [s(\beta - 1), s], s > 0
\end{cases}
\]

(2)
which is referred to as the “multi-valued Rényi-Parry map” on the flaky region $\Delta_\beta = [s(\beta - 1), s]$ and has its eventually onto Parry’s $(\beta, \alpha)$–map [6] with the subinterval $[\nu - s(\beta - 1), \nu]$ as shown in Fig. 2. This map realises the flaky quantiser $Q^f_{s(1-\gamma), \gamma}(\cdot)$. Let $b_{i,S_{\beta,\gamma_1}, \gamma_1}$ be its associated bit sequence for the threshold sequence $\nu^f_1 = \nu_1 \nu_2 \cdots \nu_L$, defined by

$$b_{i,S_{\beta,\gamma_1}, \gamma_1} = Q_{\gamma_\nu}(S_{\beta,\gamma_1}^{-1}, s)(x)$$

(3)

Then we get $x = s(\beta - 1) \sum_{i=1}^{L} b_{i,S_{\beta,\gamma_1}, \gamma_1} \gamma_1 + \gamma_1 S_{\beta,\gamma_1}^{L} s(x)$ and its decoded value

$$\hat{x}_{L,S_{\beta,\gamma_1}, \gamma_1} = s(\beta - 1) \sum_{i=1}^{L} b_{i,S_{\beta,\gamma_1}, \gamma_1} \gamma_1 + \gamma_1 S_{\beta,\gamma_1}^{L}.$$  

2. Negative $\beta$-Map[10, 12]; We get

Lemma 2[10]: Let $R_{\beta,\nu,s}(x): [0, s) \to [0, s)$, $s > 0$ be the (scale-adjusted) negative $\beta$-map, defined by

$$R_{\beta,\nu,s}(x) = -s x + s[1 + (\beta - 1)Q_{\gamma_\nu}(x)] \begin{cases} s - \beta x, & x \in [0, \gamma_\nu), \\ \beta s - \beta x, & x \in [\gamma_\nu, s), \end{cases}$$

which is another “multi-valued Rényi-Parry map” on the flaky region $\Delta_\beta = [s(\beta - 1), s]$ realising $Q^f_{s(1-\gamma), \gamma}(\cdot)$ and has its eventually onto Parry’s $(\beta, \alpha)$–map [6] with the subinterval $[\nu - s, \beta s - \nu]$ as shown in Fig. 3. Let $b_{i,R_{\beta,\nu,s}, \nu}$ be the associated bit sequence for the threshold sequence $\nu^f_1$, defined by

$$b_{i,R_{\beta,\nu,s}, \nu} = Q_{\gamma_\nu}(R_{\beta,\nu,s}^{-1}, \nu)(x)$$

(5)

Then we get $x = (-\gamma)^L R_{\beta,\nu,s}^{-1}(\nu)(x) - s \sum_{i=1}^{L} f_i R_{\beta,\nu,s}^{-1}(-\gamma)^i$ and its decoded value

$$\hat{x}_{L,R_{\beta,\nu,s}, \nu} = s(-\gamma)^L/2 - \sum_{i=1}^{L} f_i R_{\beta,\nu,s}^{-1}(-\gamma)^i,$$  

where $f_i R_{\beta,\nu,s}^{-1} = 1 + b_{i,R_{\beta,\nu,s}, \nu} (\beta - 1)$. Such a negative $\beta$-expansion defines a new A/D converter called a negative $\beta$-encoder which facilitates the implementation of stable analog circuits. Figures 2 [11] and 3 [12] show a typical $\beta$-expansion’s attractor of Eqs.(2) and (4), respectively.

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References