Non-hyperbolic Equilibria of SD Oscillator

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Introduction

An oscillator with strong geometrical nonlinearities, or a smooth and discontinuous (SD) oscillator was proposed and investigated by Cao *et al.* [1, 2, 3], previously, which allows one to study the transition from smooth to discontinuous dynamics depending on the value of the smoothness parameter α. In this paper, we investigate the non-hyperbolic equilibria of SD Oscillator under the external harmonic excitation and visco-damping. Averaging technique is introduced to this irrational system with strongly nonlinearities. The results derived in this paper are valid for both smooth and discontinuous regines of the system.

Governing Equation

The earlier studies on the SD oscillator [1, 2, 3] showed that this system behaves a rich nonlinear behaviour of chaos [6, 7] and co-dimension bifurcations [4, 5] depending on the value of parameter α. In this paper, we will focuss our attention on the periodic motions of the system.

Let consider the dimensionless governing equation of the geometry nonlinear oscillator in the following form, given in [1].

$$\ddot{x} + \omega_0^2 x(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}) = 0, \tag{1}$$

which is a strongly nonlinear system with an irrational type of nonlinearity due to the geometry configuration of the mechanism. This system is smooth for $\alpha > 0$ and discontinuous while $\alpha = 0$. The equilibria of this system can be derived by letting $y = \dot{x}$ and written as $(x_0, y_0) = (0,0)$ and $(x_{2,3}, y_{2,3}) = (\pm\sqrt{1-\alpha^2}, 0)$. It is worth noticing that the trivial solution $(x_0, y_0) = (0,0)$ is unique and stable for $\alpha > 1$. It bifurcates into a pair of stable solutions when α decreases crossing the bifurcation point $\alpha = 1$ into the interval of $0 \leq \alpha < 1$ and itself becomes unstable, the details seen in [4, 5]. The dimensionless form of the system under external harmonic excitation and visco-damping can be written as the following form.

$$\ddot{x} + \delta\dot{x} + \omega_0^2 x(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}) = f\cos\omega t, \tag{2}$$

which admits multiple co-existences of periodic solutions which are investigated in the following sections near both $(0,0)$ and $(\pm\sqrt{1-\alpha^2}, 0)$.

Periodic solutions for hyperbolic equilibrium

By letting $x = a\cos(\tau + \theta), \dot{x} = -a\sin(\tau + \theta)$, the response equation can be obtained and written as

$$\left(G(a,\alpha) + (\omega^2 - \omega_0^2)\right)^2 + \delta^2\omega^2 = \frac{f^2}{a^2}, \tag{3}$$

where

$$G(a,\alpha) = \frac{4\omega_0^2}{\pi a^2}\left[g_1(a,\alpha) - g_2(a,\alpha)\right], \; g_1(a,\alpha) = \sqrt{a^2+\alpha^2}E\left(\sqrt{\frac{a^2}{a^2+\alpha^2}}\right), \; g_2(a,\alpha) = \frac{\alpha^2}{\sqrt{a^2+\alpha^2}}K\left(\sqrt{\frac{a^2}{a^2+\alpha^2}}\right),$$

$K(k)$ and $E(k)$ are the complete elliptic integral of the first and second kind respectively, and $0 < k < 1$.

The response curves for both $\alpha = 0.5$ and $\alpha = 0.0$ at $\delta = 0.05$, $\omega = 0.6$ and $\omega_0 = 1$ near $(0,0)$ are plotted in Fig. 1(a) and (b) marked c_3 and c_1, respectively, the solid lines represents the stable and the dashed line marks the unstable periodic solutions.

Periodic solutions near non-hyperbolic equilibria

In the same way, the periodic solutions near the equilibria $(\sqrt{1-\alpha^2}, 0)$ can also be obtained by letting $x = a\cos(\tau + \theta) + c, \dot{x} = -a\sin(\tau + \theta) + \dot{c}$, and consider the conditions $\ddot{c}, \dot{c} = 0, \dot{a} = 0, \dot{\theta} = 0$, we have

$$\left[\omega^2 - \omega_0^2 + \frac{\omega_0^2}{a}Q_2(\tau,\alpha)\right]^2 + \delta^2\omega^2 = \frac{f^2}{a^2}, \tag{4}$$

where $Q_2(\tau,\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{[a\cos(\tau+\theta)+c]\cos(\tau+\theta)}{\sqrt{(a\cos(\tau+\theta)+c)^2+\alpha^2}} d(\tau+\theta)$ and $c = \frac{1}{2\pi}\int_0^{2\pi} \frac{a\cos(\tau+\theta)+c}{\sqrt{(a\cos(\tau+\theta)+c)^2+\alpha^2}} d(\tau+\theta)$ is an implicit function.

The response curves for both $\alpha = 0.5$ and $\alpha = 0.0$ at $\delta = 0.05$, $\omega = 0.6$ and $\omega_0 = 1$ near $(\pm\sqrt{1-\alpha^2}, 0)$ are plotted in Figs. 1(a) and (b) and they are marked c_4 and c_2, respectively. The solid lines represents the stable and the dashed line marks the unstable periodic solutions.

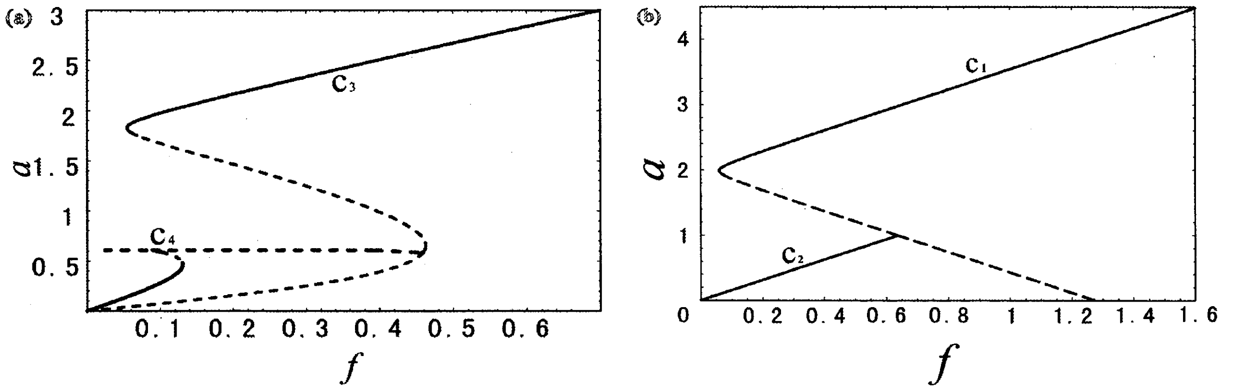

Figure 1: Response curves for $\delta = 0.05$, $\omega = 0.6$ and $\omega_0 = 1$ near $(0,0)$ and $(\pm\sqrt{1-\alpha^2}, 0)$ in (3) and (4), respectively, the solid lines represents the stable and the dashed line marks the unstable periodic solutions: (a) response curve c_3 and c_4 for $\alpha = 0.5$ and (b) response curve c_1 and c_2 for $\alpha = 0.0$, respectively.

It can be easily seen that the response curves, as shown in Fig. 1, there appears to have "hysteresis" and "jumps". There is also an indication of multiple co-existing solutions for the perturbed system near near both $(0,0)$ and $(\pm\sqrt{1-\alpha^2}, 0)$.

Conclusion

In this paper, the periodic motions of the SD oscillator have been investigated and the response curves for both smooth and discontinuous stages, near both the hyperbolic equilibrium $(0,0)$ and the non-hyperbolic equilibria $(\pm\sqrt{1-\alpha^2}, 0)$ have been calculated. Stability analysis has been carried out to determine the stable and unstable periodic solution branches in the corresponding response curves. An averaging procedure is successfully applied to this strongly irrational nonlinear system without truncation or approximation. Further investigations of supper and sub-harmonic resonances and the analytical chaotic behaviour are being actively carried out by the authors.

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References

[1] Q. Cao, M. Wiercigroch, E. E. Pavlovskaia, C. Grebogi, J.M.T. Thompson, Archetypal oscillator for smooth and discontinuous dynamics, *Physical Review E* 74(4), 046218, 2006.

[2] Q. Cao, M. Wiercigroch, E. E. Pavlovskaia, C. Grebogi, J.M.T. Thompson, Piecewise linear approach to an archetypal oscillator for smooth and discontinuous dynamics, *Phi. Tran. Roy. Soc. A*, 366(1865), 635-652, 2008.

[3] Q. Cao, M. Wiercigroch, E. E. Pavlovskaia, C. Grebogi, J.M.T. Thompson, The limit case response of the archetypal oscillator for smooth and discontinuous dynamics, *Inter. J. Non. Mech.* 43, pp. 462–473, 2008.

[4] R. Tian. Q. Cao, S. Yang, The codimension-two bifurcation for the recent proposed SD oscillator *Nonlinear Dynamics* 59, 1-2, 19-27, 2009.

[5] R. Tian, Q. Cao, Z. Li, Hopf bifurcations for the recently proposed smooth-and-discontinuous oscillator, *Chin. Phys. Lett.* 27, No. 7, 074701, 2010.

[6] E. N. Lorenz, Deterministic non-periodic flow, *J. Atoms. Sci.*, vol. 20, pp. 130–141, 1963.

[7] Y. Ueda, *The Road to Chaos–II*, (Aerial Press, Inc., 2001).