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Stability of discrete breathers in diatomic Fermi-Pasta-Ulam type lattices

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Introduction

Spatially localized excitations in nonlinear lattices have attracted great interest since the ground-breaking work by Takeno *et al.* [1]. The localized modes are called *discrete breathers* (DBs) or *intrinsic localized modes* (ILMs). The DB is expected to be a quite general object emerging in a variety of nonlinear space-discrete systems in nature. Indeed, experimental evidence for the existence of DB has been reported in various systems. Considerable progress has been achieved in understanding the nature of DB so far (e.g., [2]). From the theoretical point of view, the DB is a time-periodic and spatially localized solution of the equations of motion, which emerges due to nonlinearity and discreteness of the system. The existence of DBs has been proved rigorously with several mathematical methods in various nonlinear lattice models such as weakly coupled nonlinear oscillators [3] or the diatomic Fermi-Pasta-Ulam (FPU) lattice [4]. Another important issue, associated with DBs, is the stability of DB solutions. However, analytical methods for the linear stability analysis of DBs have not yet been fully developed. Rigorous results on the linear stability problem are still limited to a few lattice models, and no analytical result has been obtained for the diatomic FPU type lattices. Therefore, developing a new method to prove the existence of single/multi-site DBs of various configurations and evaluate their linear stability, we have obtained a stability criterion for DBs in the diatomic FPU type lattices in small mass ratio regime.

Model

We consider the one-dimensional diatomic FPU type lattice described by the Hamiltonian

$$H = \sum_{n=1}^{N-1} \frac{1}{2m_n} P_n^2 + \sum_{n=1}^{N} V(Q_n - Q_{n-1}), \tag{1}$$

where $Q_n \in \mathbb{R}$ and $P_n \in \mathbb{R}$ represent the position and momentum, respectively, and m_n represents the masses given by $m_{2j-1} = 1,\ m_{2j} = \bar{m} > 1,\ j \in \mathbb{N}$. For concreteness, we employ the fixed-end boundary conditions, i.e., $Q_0 = Q_N = 0$, and assume that N is even. Thus, the number of degrees of freedom is $N-1$ and the both ends at $n = 0, N$ are heavy particles. Let $W(X,\mu) : \mathbb{R} \times O \to \mathbb{R}$ be a C^2 function of X and μ such that $W(X,0) = 0$, where $\mu \in \mathbb{R}^l$ is a set of parameters and $O \subseteq \mathbb{R}^l$ is a neighbourhood of $\mu = 0$. We suppose the interaction potential V of the form

$$V(X) = W(X,\mu) + \frac{1}{k} X^k, \tag{2}$$

where $k \geq 4$ is an even integer.

Our results will hold for large $\bar{m} \gg 1$. The limit $\bar{m} \to \infty$ looks singular at first sight, however, it is known that the limit can be regularized by introducing the perturbation parameter [4]

$$\varepsilon = \frac{1}{\sqrt{\bar{m}}}. \tag{3}$$

Using this parameter, we define new coordinates q_n as

$$q_n = \begin{cases} Q_n & \text{if } n = 2j-1, \\ \varepsilon^{-1} Q_n & \text{if } n = 2j, \end{cases} \qquad j = 1, \dots, N/2. \tag{4}$$

In the rescaled coordinates, Hamiltonian (1) is rewritten as

$$H = \sum_{n=1}^{N-1} \frac{1}{2} p_n^2 + \sum_{j=1}^{N/2} \Big[V(\varepsilon q_{2j} - q_{2j-1}) + V(q_{2j-1} - \varepsilon q_{2j-2}) \Big], \tag{5}$$

where p_n is the momentum conjugate to q_n defined by $p_{2j-1} = P_{2j-1}$ and $p_{2j} = \varepsilon P_{2j}$. The boundary conditions are $q_0 = q_N = 0$. The equations of motion derived from Hamiltonian (5) are given by

$$\ddot{q}_{2j-1} = V'(\varepsilon q_{2j} - q_{2j-1}) - V'(q_{2j-1} - \varepsilon q_{2j-2}), \tag{6}$$

$$\ddot{q}_{2j} = \varepsilon V'(q_{2j+1} - \varepsilon q_{2j}) - \varepsilon V'(\varepsilon q_{2j} - q_{2j-1}). \tag{7}$$

These equations of motion decouple with each other in the limit $\varepsilon = 0$, which is called *the anti-continuous limit.* In what follows, we use the rescaled coordinates q_n and p_n, and describe our results for Hamiltonian (5).

Main result

Let us consider the case of $\varepsilon = 0$ and $\mu = 0$, i.e., the anti-continuous limit of the homogeneous potential lattice. In this case there are a number of periodic solutions to Eqs. (6) and (7) of the form

$$q_{2j-1} = 2^{-1/(k-2)}\sigma_{2j-1}\varphi(t), \quad q_{2j} = 0, \qquad j = 1, \ldots, N/2, \tag{8}$$

where $\sigma_{2j-1} \in \{-1, 0, 1\}$ and $\varphi(t)$ is a non-constant periodic solution of the differential equation

$$\ddot{\varphi} + \varphi^{k-1} = 0. \tag{9}$$

Equation (9) has the integral $\dot{\varphi}^2/2 + \varphi^k/k = h,$, where $h > 0$ is an integration constant. The period T of $\varphi(t)$ depends on h and it is given by

$$T = 2\sqrt{2}\, h^{-(1/2-1/k)} \int_0^{k^{1/k}} \frac{1}{\sqrt{1 - x^k/k}}\, dx. \tag{10}$$

This indicates that T continuously varies from $T = +\infty$ to 0 as h varies from $h = 0$ to $+\infty$ since the integral in Eq. (10) is independent of h. This implies that for any given $T > 0$, there exists a solution $\varphi(t)$ with the period T. Thus, for any code sequence $\sigma = (\sigma_1, \sigma_3, \ldots, \sigma_{2j-1}, \ldots, \sigma_{N-1}) \in \{-1, 0, 1\}^{N/2}$ and any $T > 0$, there exists the T-periodic solution of Eqs. (6) and (7) given by Eq. (8). We denote this periodic solution by $\Gamma(t; \sigma, T)$: i.e., $\Gamma(t; \sigma, T) = (q_1, \ldots, q_{N-1}, p_1, \ldots, p_{N-1})$, where q_n is given by Eq. (8) and $p_n = \dot{q}_n$.

When σ consists of a small number of nonzero components, the periodic solution of the form (8) with this σ represents a localized solution or a superposition of some localized solutions, i.e., discrete breather: for example, $\sigma = (\ldots, 0, 1, 0, \ldots)$ corresponds to a single-site DB and $\sigma = (\ldots, 0, 1, 0, 0, -1, 0, \ldots)$ corresponds to two single-site DBs located separately. In this study, more generally, we deal with an arbitrary code sequence σ. Let $\mathcal{A}$ be the set $\mathcal{A} = \{1, 2, \ldots, N/2\}$ and $\mathcal{A}_\sigma$ be its subset consisting of the indices for nonzero components of σ, i.e., $\mathcal{A}_\sigma = \{\, j\,;\, \sigma_{2j-1} \neq 0 \,\} \subseteq \mathcal{A}$. Suppose that σ contains m excited sites and $\mathcal{A}_\sigma = \{\, j_1, j_2, \ldots, j_m \,\}$, where $j_1 < j_2 < \cdots < j_m$. Consider two adjacent excited sites in $\Gamma(t; \sigma, T)$, which are represented by σ_{2j_i-1} and $\sigma_{2j_{i+1}-1}$. We say this pair of adjacent excited sites is *in-phase* if $\sigma_{2j_i-1} = \sigma_{2j_{i+1}-1}$ and *anti-phase* if $\sigma_{2j_i-1} = -\sigma_{2j_{i+1}-1}$. Let $N_{\text{in}}(\sigma)$ be the function of σ defined by

$$N_{\text{in}}(\sigma) = \sum_{i=1}^{m-1} \frac{1}{2} \left| \sigma_{2j_i-1} + \sigma_{2j_{i+1}-1} \right|. \tag{11}$$

If $m = 1$, we define $N_{\text{in}}(\sigma) = 0$. This function counts the number of in-phase pairs of adjacent excited sites. Therefore, $N_{\text{in}}(\sigma) = 0$ if and only if a single site is excited, $m = 1$, or all the adjacent excited sites are anti-phase for $m \geq 2$. The main theorem is stated as follows (for the proof, see [5]).

Theorem 1. *For any $\sigma \neq 0$ and $T > 0$, there exist a constant $\varepsilon_c > 0$ and, for $\varepsilon \in [0, \varepsilon_c)$, a family $\Gamma_\varepsilon(t; \sigma, T)$ of T-periodic solutions of lattice (5) with $\mu = 0$ such that $\Gamma_\varepsilon(t; \sigma, T)$ is analytic with respect to ε and t and $\Gamma_0(t; \sigma, T) = \Gamma(t; \sigma, T)$. For each $\varepsilon \in (0, \varepsilon_c)$, there exist a neighbourhood $U_\varepsilon(0) \subseteq \mathbb{R}^l$ of $\mu = 0$ and, for $\mu \in U_\varepsilon(0)$, a family $\Gamma_{\varepsilon,\mu}(t; \sigma, T)$ of periodic solutions of lattice (5) such that $\Gamma_{\varepsilon,\mu}(t; \sigma, T)$ is C^1 with respect to μ and t, $\Gamma_{\varepsilon,0}(t; \sigma, T) = \Gamma_\varepsilon(t; \sigma, T)$, and the period $T_\varepsilon(\mu)$ is a C^1 function of μ satisfying $T_\varepsilon(0) = T$. The continued periodic solution $\Gamma_{\varepsilon,\mu}(t; \sigma, T)$ is linearly stable if and only if $N_{\text{in}}(\sigma) = 0$, otherwise it is linearly unstable with exactly $N_{\text{in}}(\sigma)$ unstable characteristic multipliers.*

References

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