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From Random to Data Driven Dynamical Systems: Application to the Lorenz-96 Atmospheric Model

Nicolas Perkowski[1], Hoong Chieh Yeong[2], Peter Imkeller[1], and N. Sri Namachchivaya[2]

[1]*Department of Mathematics, Humboldt-Universitaet zu Berlin*
Unter den Linden 6, 10099 Berlin, Germany, mathematik.hu-berlin.de
[2]*Department of Aerospace Engineering, University of Illinois at Urbana-Champaign,*
104 South Wright Street, Urbana, IL 61801, USA, navam@illinois.edu

An important aspect in the study of *random dynamical systems* is the estimation of state variables that are often hidden, based on observational data. Sensor data are sparse and usually contain noise, and mathematical models are limited in accuracy due to model uncertainties. But, when used together, the resulting prediction of the state of large-scale dynamical systems is superior to using either the models or the data alone. The optimal estimate is given by the conditional expectation and can be generated by a recursive equation, called the filter, *driven by the observation process.* Multi-scale properties are ubiquitous in science and engineering and complex behavior can occur in a wide range of dynamical systems. Nonlinearities of the governing physical processes allow energy transfer between different scales, and many aspects of this complex behavior can be represented by stochastic models. Multi-scale problems are commonly recognized for their complexity, yet the main challenge in multi-scale modeling is to recognize their simplicity, and make use of it to see how information interacts with these complex structures and scales. This paper outlines efficient methods that combine the information from the observations with the dynamics of coupled ocean - atmosphere models.

We present a reduced-order particle filtering algorithm, the Homogenized Hybrid Particle Filter (HHPF), for state estimation in nonlinear multi-scale dynamical systems. This method combines stochastic homogenization with nonlinear filtering theory to construct an efficient multi-scale particle filtering algorithm for the approximation of a reduced-order nonlinear filter for multi-scale systems. In this work, we show that the HHPF gives a good approximation of the conditional law of the coarse-grained dynamics in the limit as the scaling parameter $\epsilon \to 0$, and the number of particles $n \to \infty$.

Problem Formulation and Theoretical Results

Consider the signal process $(X_t^\varepsilon, Z_t^\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m$ governed by:

$$\begin{aligned} dX_t^\varepsilon &= \left[b\left(X_t^\varepsilon, Z_t^\varepsilon\right) + \varepsilon^{-1} b_1\left(X_t^\varepsilon, Z_t^\varepsilon\right)\right] dt + \sigma\left(X_t^\varepsilon, Z_t^\varepsilon\right) dW_t, \quad X_0^\varepsilon = \xi; \\ dZ_t^\varepsilon &= \varepsilon^{-1} f\left(X_t^\varepsilon, Z_t^\varepsilon\right) dt + \varepsilon^{-1/2} g\left(X_t^\varepsilon, Z_t^\varepsilon\right) dV_t, \quad Z_0^\varepsilon = \eta \end{aligned} \tag{1}$$

with associated probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and observation process $Y_t^\varepsilon \in \mathbb{R}^d$,

$$dY_t^\varepsilon = h\left(X_t^\varepsilon, Z_t^\varepsilon\right) dt + dB_t, \quad Y_0^\varepsilon = 0.$$

W_t, V_t and B_t are independent Wiener processes, ξ and η are random initial conditions, independent of W_t, V_t and B_t, and the small parameter ε measures the ratio of the timescale separation between the slow, X_t^ε, and fast, Z_t^ε, processes.

We assume that (Doeblin condition) for any test function $f \in C_b(\mathbb{R}^m)$ and Z_t^x denoting the fast process with fixed x, $\mathbb{E}_z\left[f(Z_t^x)\right] \to \int f(z)\mu_x(dz)$ as $t \to \infty$ uniformly in x and z in any compact set, where $\mu_x(dx)$ is an invariant measure attained by Z_t^ε. Then, averaging theory results tell us that, as $\varepsilon \to 0$, the generator $\mathscr{L}^\varepsilon$ of the system (1) converges to a generator $\bar{\mathscr{L}}$ with the associated homogenized process governed by

$$dX_t^0 = \bar{b}\left(X_t^0\right) dt + \bar{\sigma}\left(X_t^0\right) dW_t, \quad X_0^0 = \xi, \tag{2}$$

where $X^\varepsilon \xrightarrow{d} X^0$.

For the filtering problem, we are interested in obtaining the best estimate of X^ε based on observations Y^ε, i.e. obtaining $\pi_t^\varepsilon(\varphi) = \mathbb{E}\left[\varphi(X_t^\varepsilon)|\mathscr{Y}_t^\varepsilon\right] = \rho_t^\varepsilon(\varphi)/\rho_t^\varepsilon(1)$, for π_t^ε a finite measure on $\mathbb{R}^m$ and $\varphi \in C_b(\mathbb{R}^m, \mathbb{R})$, where $\mathscr{Y}_t^\varepsilon$ is the σ-algebra of the observation process and ρ_t^ε is the unique solution of the Zakai equation:

$$d\rho_t^\varepsilon(\varphi) \;=\; \rho_t^\varepsilon(\mathscr{L}^\varepsilon\varphi)dt + \rho_t^\varepsilon(h\varphi)dY_t^\varepsilon, \quad \rho_0^\varepsilon(\varphi) \;=\; \mathbb{E}(\varphi(X_0^\varepsilon)). \tag{3}$$

As long as we are only interested in the slow component, we want to take advantage of the fact that $X^\varepsilon \xrightarrow{d} X^0$, by showing that, for small ε, ρ_t^ε is close to the solution of the filter equation for the homogenized process X_t^0:

$$d\rho_t^0(\varphi) \;=\; \rho_t^0(\bar{\mathscr{L}}\varphi)dt + \rho_t^0(\bar{h}\varphi)dY_t^\varepsilon, \quad \rho_0^0(\varphi) \;=\; \mathbb{E}(\varphi(X_0^0)), \tag{4}$$

Note that the homogenized filter of (4) is still defined in terms of the real observation Y_t^ε instead of the fictitious "homogenized observation" Y_t^0 ($dY_t^0 = \bar{h}\left(X_t^0\right) dt + dB_t$, $Y_0^0 = 0$).

Our goal is to obtain a strong result by showing L^p-convergence of the filters:

$$\lim_{\varepsilon\to 0,\; n\to\infty} \mathbb{E}\left[\sup_{t\le T} d(\pi_t^\varepsilon, \pi_t^{0,n})^p\right] = 0, \; \forall\, T > 0,$$

where d denotes the Prokhorov distance on the space of probability measures and $\pi_t^{0,n}$ represents the homogenized particle filter with n particles. The desired convergence result will be achieved by invoking the dual representations of the filters and applying backward stochastic differential equations techniques and existing results on homogenization and particle filter convergence. On this premise, the HHPF was developed to numerically approximate the nonlinear filter for the coarse-grained dynamics of a multiscale system from the reduced filtering equation (4).

Homogenized Hybrid Particle Filter

The HHPF incorporates the homogenization technique of the HMM developed in [2] with the nonlinear filtering method of the BPF of [1] to numerically approximate the solution to (4) using a system of branching particles with varying weights. Particle weights are updated based on actual observation $Y^\varepsilon_{[0,t]}$ using a continuous version of the Kallianpur-Streibel formula: $w^{\bar{x}^p_{(i+1)\Delta}} = \exp\left(\int_{i\Delta}^{(i+1)\Delta} \tilde{h}(\bar{x}^p_t)\, dY^\varepsilon_t - \frac{1}{2}\int_{i\Delta}^{(i+1)\Delta} |\tilde{h}(\bar{x}^p_t)|^2 dt\right)$ where $\tilde{h}$ is the averaged sensor function, approximated using the HMM, similar to $\tilde{b}$. Particles $\bar{x}^p_t$ are branched into offspring or eliminated based on the corresponding weights. Using the algorithm in [1], the random variables $o^p_{(i+1)\Delta}$ for the numbers of offspring generated are correlated with minimal variance. Through this correlation, sample size kept constant at N_s, thus avoiding sample size blow up or particle degeneracy.

Numerical Example

We apply the HHPF to Lorenz [3] atmospheric model to illustrate its potential for high-dimensional complex problems.

$$\dot{X}_k = -X_{k-1}(X_{k-2} - X_{k+1}) - X_k + F_x + \frac{h_x}{J}\sum_{j=1}^{J} Z_{j,k}, \qquad \dot{Z}_{j,k} = \frac{1}{\epsilon}(-Z_{j+1,k}(Z_{j+2,k} - Z_{j-1,k}) - Z_{j,k} + h_z X_k)$$

The Lorenz-96 model is used to describe the dynamics of an atmospheric quantity, X_j, such as temperature (macroscopic variable) at N-equally distributed grid points on a circle of constant latitude defined over a latitude circle in the mid-latitude region [3]. The latitude circle is divided into K sectors, where each X_k is coupled to its neighbors, and generalized to all values of k by letting $X_{k+K} = X_{k-K} = X_k$. The macroscopic dynamical equation in the above Lorenz model comprises of an advective-like nonlinear term that conserves the total energy, a linear dissipation, an external forcing F, and an average contribution from the local "convective-like" microscopic processes, $B_k = \frac{h_x}{J}\sum_{j=1}^{J} Z_{j,k}$, where h_x is the associated coupling strength.

Conclusions

The HHPF is a numerical scheme that incorporates homogenization techniques with particle filtering to approximate the solution to a reduced nonlinear filtering equation. The resulting scheme overcomes the dimensionality issues associated with particle filtering for high-dimensional systems and is more computationally efficient, applicable to complex, multi-scale systems. Works in progress on the development of the HHPF are application to high-dimensional multi-scale systems and the development of a rigorous convergence proof for the numerical scheme, which will be presented in the symposium.

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