Lattice Polyhedra and Submodular Flows

Satoru Fujishige · **Britta Peis**

Abstract Lattice polyhedra, as introduced by Gröflin and Hoffman, form a common framework for various discrete optimization problems. They are specified by a lattice structure on the underlying matrix satisfying certain sub- and supermodularity constraints. Lattice polyhedra provide one of the most general frameworks of total dual integral systems. So far no combinatorial algorithm has been found for the corresponding linear optimization problem. We show that the important class of lattice polyhedra in which the underlying lattice is of modular characteristic can be reduced to the Edmonds-Giles polyhedra. Thus, submodular flow algorithms can be applied to this class of lattice polyhedra. In contrast to a previous result of Schrijver, we do not explicitly require that the lattice is distributive. Moreover, our reduction is very simple in that it only uses an arbitrary maximal chain in the lattice.

1 Introduction

Discrete optimization problems often allow, or even ask for, a formulation as an integer linear program, i.e., as a problem to find optimal integral solutions of a linear program of type

$$\max\{w^T x \mid Ax \le f\},$$

where $A \in \{-1, 0, 1\}^{m \times n}$, $w \in \mathbb{R}^n$ and $f \in \mathbb{R}^m$. By duality theory, the value of the linear problem is equal to the value of its dual linear program $\min\{y^T f \mid y^T A = w^T,\, y \ge 0\}$.

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Being interested in integral solutions, a fundamental problem in polyhedral combinatorics is to decide which linear programs have guaranteed integer optimal solutions. The probably most important concept here is that of total dual integral inequality systems [22], where system $Ax \le f$ is called *total dual integral (TDI)* if the minimum in $\min\{y^T f \mid y^T A = w^T,\ y \ge 0\}$ can be achieved by an integral vector y for each integral w for which the optima exist. Various classes of linear programs with specially structured right-hand sides (often submodular) can be proven to have integral optimal solutions. However, the development of polynomial algorithms to solve TDI problems has lagged behind the theoretical results, although there have been some recent successes (see, e.g., [16], [17], [18].)

When asking for combinatorial algorithms for as general TDI-systems as possible, one promising model to look at is that of *lattice polyhedra* which were introduced by Hoffman and Schwartz [12] for $\{0,1\}$-matrices, and generalized by Gröflin and Hoffman [11] to $\{-1,0,1\}$-matrices.

1.1 Lattice polyhedra

The name "lattice polyhedron" comes from a certain, very general, lattice structure on the underlying matrix on which the right-hand side is submodular. We first need some notation: A poset $(L, \preceq)$ is a *lattice* if for any two elements $i, j \in L$ there exist a "meet" $i \wedge j = \sup\{k \in L \mid k \preceq i, j\}$ and a "join" $i \vee j = \inf\{k \in L \mid k \succeq i, j\}$. We denote by m and M the (unique) minimal and maximal element of lattice L, respectively. A function $f : L \to \mathbb{R}$ is *submodular* if it satisfies for all $i, j \in L$ the inequality $f(i) + f(j) \ge f(i \wedge j) + f(i \vee j)$.

Definition 1 [Lattice polyhedron [11]] Let $(L, \preceq, \wedge, \vee)$ be a lattice on which $f \in \mathbb{R}^{|L|}$ is submodular. Given a matrix $A \in \{-1,0,1\}^{|L| \times |E|}$ with entries $\chi(i,e)$ for $i \in L$ and $e \in E$, lower and upper bounds $c, d \in \mathbb{R}^{|E|}$ the polyhedron

$$\mathbb{P}(A,f) = \{x \in \mathbb{R}^{|E|} \mid Ax \le f,\ c \le x \le d\}$$

is called a *lattice polyhedron* if for all $i, j, k \in L$ and all $e \in E$ the following three hold:

(C1) if $i \prec j \prec k$ and $\chi(i,e) = \chi(k,e) = t \ne 0$, then $\chi(j,e) = t$,
(C2) if $i \prec j$, then $\chi(i,e) \cdot \chi(j,e) \ge 0$, and
(C3) $\chi(i,e) + \chi(j,e) \le \chi(i \vee j,e) + \chi(i \wedge j,e)$.

Analogously, if, in the above definition, function f is supermodular (i.e., if $-f$ is submodular) and (C3) is replaced by (C3') $\chi(i,e) + \chi(j,e) \ge \chi(i \vee j,e) + \chi(i \wedge j,e)$, the polyhedron $\mathbb{P}'(A,f) = \{x \in \mathbb{R}^{|E|} \mid Ax \ge f,\ c \le x \le d\}$ is also called a lattice polyhedron. The pair of the matrix A and the lattice L is called *consecutive* if properties (C1) and (C2) are satisfied. If (A,L) satisfies (C3) (or (C3'), or both), we say that (A,L) has supermodular (or submodular, or modular) characteristic. For technical reasons, we assume throughout that for each $e \in E$ there exists at least one $i \in L$ such that $\chi(i,e) \ne 0$.

Lattice polyhedra form a common framework for various combinatorial structures. For example, lattice polyhedra with $\{0,1\}$-matrices include polyhedra types

such as polymatroids, the intersection of polymatroids, and submodular systems. The lattice polyhedra with $\{-1,0,1\}$-matrix include Edmonds-Giles polyhedra [3], so that the optimization problem for lattice polyhedra also covers the well-studied submodular flow problem. (See Section 2 below for the definition of Edmonds-Giles polyhedra and the submodular flow problem). Several min-max results for combinatorial structures can be derived from the following theorem:

Theorem 1 ([11]) *If either (A,L) is of modular characteristic, or $c \geq 0$, then the system of linear inequalities for the lattice polyhedron is totally dual integral. If f, c and d are integral, then all vertices of lattice polyhedra are integral.*

However, Gröflin and Hoffman's integrality result for lattice polyhedra is only a structural existence theorem without algorithmic foundation. In the last decades, combinatorial algorithms have been developed only for various special instances of lattice polyhedra. For example, there exist several greedy-type algorithms for special classes of $\{0,1\}$-lattice polyhedra (e.g., [7], [6], [2], [4], [5], [13], [15]), as well as a primal-dual algorithm for $\{0,1\}$-lattice polyhedra with supermodular right-hand side [18].

Lattice polyhedra on $\{-1,0,1\}$-matrices seem to be a lot harder to investigate under algorithmic aspects. For $\{-1,0,1\}$-lattice polyhedra, algorithms have mainly been developed for the special case of Edmonds-Giles polyhedra. (See e.g. [20], [14], [19] for the currently fastest submodular flow algorithms, or the survey paper [10].) The reason why polyhedra on $\{-1,0,1\}$-matrices and their underlying structure are not so tractable might be that it is much more natural to play around with subset families $L \subseteq 2^E$ than with families of disjoint ordered tuples $L \subseteq 3^E = \{(X,Y) \mid X, Y \subseteq E,\ X \cap Y = \emptyset\}$ in which the elements may occur "positive" or "negative". Thus, in order to design a combinatorial algorithm for $\{-1,0,1\}$-lattice polyhedra in general, we would need to better understand the underlying combinatorial structure, and maybe even give some graph-theoretic interpretation of the problem. Moreover, we would need to find an appropriate oracle to access the lattice.

Our contribution. This paper can be seen as a further step toward this goal. We will show that $\{-1,0,1\}$-lattice polyhedra in which (A,L) is of modular characteristic can be reduced to Edmonds-Giles polyhedra via a very simple reduction. For our reduction, we only need an oracle that gives us an arbitrary maximal chain in L. In contrast to a related result of Schrijver [21], we do not require explicitly that L is a distributive lattice, and we do not need an oracle to return minimal and maximal elements in certain sublattices. The differences between Schrijver's and our reduction will be described below.

2 Preliminaries and Definitions

Let $\mathbb{P}(A,f) = \{x \in \mathbb{R}^{|E|} \mid Ax \leq f,\ c \leq x \leq d\}$ be a lattice polyhedron on lattice $L = (L, \preceq, \wedge, \vee)$. Thus, by Definition 1, the pair (A,L) is consecutive and of supermodular characteristic, and f is submodular.

Note that, since A is a $\{-1,0,1\}$-matrix, we might as well identify each $i \in L$ with some element (X_i, Y_i) in 3^E, where 3^E denotes the family of all ordered pairs (X,Y) of disjoint subsets of a finite set E, i.e., A can be interpreted as the incidence matrix of some (multi-)set of elements in 3^E via

$$\chi(i,e) = \begin{cases} 1 & \text{if } e \in X_i \\ -1 & \text{if } e \in Y_i \\ 0 & \text{otherwise.} \end{cases}$$

With this interpretation, $\mathbb{P}(A,f)$ can equivalently be written as $\mathbb{P}(L,f) = \{x \in \mathbb{R}^E \mid \forall i \in L: \ x(X_i) - x(Y_i) \leq f_i, \ c \leq x \leq d\}$. To shorten notation, we call lattice L *consecutive, supermodular, or modular* if the pair (A,L) is consecutive, supermodular, or modular, respectively. For each $i \in L$ we denote by $\chi(i)$ the $\{-1,0,1\}$-vector $(\chi(i,e) \mid e \in E)$. For technical reasons, we assume that $\chi(m) = \chi(M) = \emptyset$ and $f(m) = f(M) = 0$ for the unique minimal and maximal element m and M in L, respectively.

As usual, we assume that the size of E and the height of the lattice (i.e., the size of a maximal chain in L) are rather small, whereas the size of the lattice L might be huge, say exponential in $|E|$. These assumptions are justified by all the known examples and applications of lattice polyhedra (see, e.g., [21]). The function values of f are assumed to be given implicitly via some oracle.

Chains, ideals, lower neighbours, join-irreducible elements. Given a poset $(P, \preceq)$, a subset $C \subseteq P$ is a *chain*, if for any two elements $i, j \in C$ either $i \preceq j$ or $j \preceq i$. A subset $I \subseteq P$ is called *ideal w.r.t.* $(P, \preceq)$ if $i \in I$, $j \preceq i$ implies $j \in I$. An element $i \in P$ is called *lower neighbour* of $j \in P$ if $i \prec j$ and there exists no $k \in P$ with $i \prec k \prec j$. Given a lattice $(L, \preceq, \wedge, \vee)$ an element $i \in L$ is called *join-irreducible* if for every j and k in L it holds that $i = j \vee k$ implies $i = j$ or $i = k$.

Distributive lattices. A lattice $(L, \preceq, \wedge, \vee)$ is called *distributive* if the binary operators $\wedge, \vee$ satisfy the distributive law $i \wedge (j \vee k) = (i \wedge j) \vee (i \wedge k)$ for all $i, j, k \in L$. (Note that this implies that also $i \vee (j \wedge k) = (i \vee j) \wedge (i \vee k)$ holds.) Alternatively, distributive lattices can be characterized by the exclusion of the sublattices N_5 and M_3 (see Figure 1).

Birkhoff's representation of distributive lattices. By a theorem of Birkhoff [1], a finite distributive lattice $L = (L, \preceq, \wedge, \vee)$ admits a canonical representation as a union- and intersection-closed system of sets in the following way: Let $P \subseteq L$ denote the set of all join-irreducible elements of L. Then the ordering $\preceq$ in L induces the poset $P = (P, \preceq)$. Let $\mathscr{D}(P)$ denote the collection of all ideals w.r.t. $(P, \preceq)$. Then there exists an order isomorphism $\phi : L \to \mathscr{D}(P)$ between the two lattices $(L, \preceq, \wedge, \vee)$ and $(\mathscr{D}(P), \subseteq, \cap, \cup)$ with $\phi(i) = \{p \in P \mid p \preceq i\}$ and $\phi^{-1}(I) = \bigvee\{p \in P \mid p \in I\}$. For further details on lattices, the reader is referred to [1].

Distributive lattice polyhedra. Let us call a lattice polyhedron *distributive* if (A,L) is of modular characteristic and lattice L is distributive. One important example of distributive lattice polyhedra are *Edmonds-Giles polyhedra*:
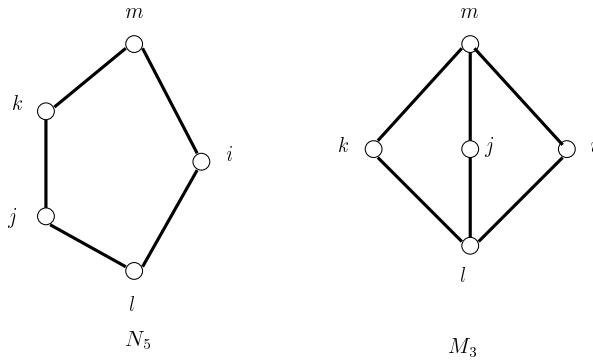
Fig. 1 Lattices N_5 and M_3.

Edmonds-Giles polyhedra and the submodular flow problem. Independent of the lattice polyhedron model, Edmonds and Giles introduced the *submodular flow problem* [3], i.e., the problem to optimize a linear function over an Edmonds-Giles polyhedron as a common generalization of the min-cost flow problem, the polymatroid intersection, and the minimum directed-cut covering problem:

Let $G = (V, E)$ be a connected directed graph and $\mathscr{F} \subseteq 2^V$ be a *ring family*, i.e., a set family that is closed with respect to the set-theoretic intersection and union. Note that $(\mathscr{F}, \subseteq, \cap, \cup)$ is a distributive lattice. Given a submodular function $f : \mathscr{F} \to \mathbb{R}$, an edge-assignment $x \in \mathbb{R}^{|E|}$ is called a *submodular flow* if

$$x(\Delta^+(S)) - x(\Delta^-(S)) \leq f(S) \ \forall S \in \mathscr{F}, \tag{1}$$

where $\Delta^+(S)$ and $\Delta^-(S)$ denote, respectively, the sets of arcs leaving S and entering S. Edmonds and Giles [3] proved that the linear inequality system in (1) is box-TDI, i.e., the system is TDI even if we add box-constraints $c \leq x \leq d$ for any integral lower and upper bound capacities $c, d \in \mathbb{Z}^E$. The polyhedron

$$\mathbb{P}_{\mathrm{EG}}(G, \mathscr{F}, f) = \{x \in \mathbb{R}^E \mid x(\Delta^+(S)) - x(\Delta^-(S)) \leq f(S) \ \forall S \in \mathscr{F}, c \leq x \leq d\},$$

is called *Edmonds-Giles polyhedron*, and the corresponding linear optimization problem is called *submodular flow problem (SF)* [1]. In the last decades, numerous algorithms have been developed for the submodular flow problem, which is said to be "one of the most important frameworks for efficiently solvable combinatorial optimization problems"[20].

It is not hard to see that an Edmonds-Giles polyhedron belongs to the class of distributive lattice polyhedra: given an Edmonds-Giles polyhedron $\mathbb{P}_{\mathrm{EG}}(G, \mathscr{F}, f)$ consider the collection of ordered pairs $L = \{(\Delta^+(S), \Delta^-(S)) \mid S \in \mathscr{F}\} \subseteq 3^E$ partially

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[1] Originally, the submodular flow problem was defined in terms of crossing submodular functions on crossing families. It is, however, shown in [8] that is suffices to consider submodular functions on ring families.

ordered by $(\Delta^+(S), \Delta^-(S)) \preceq (\Delta^+(T), \Delta^-(T))$ if and only if $S \subseteq T$. Note that $(L, \preceq)$ forms a distributive lattice with meet and join operations corresponding to the cuts induced by the intersection and union of the two vertex sets in question. Also for all $S \subseteq V$ and $e \in E$ define

$$\chi(S, e) = \begin{cases} 1 & \text{if } e \in \Delta^+(S) \\ -1 & \text{if } e \in \Delta^-(S) \\ 0 & \text{otherwise.} \end{cases}$$

Then $(L, \preceq, \wedge, \vee)$ together with the incidence matrix $A = (\chi(S, e))_{S \in \mathscr{F}, e \in E}$ is consecutive and of modular characteristic. Moreover, $\mathscr{F}$ (and thus L) is distributive. Thus, the integrality of Edmonds-Giles polyhedra follows as a consequence of Theorem 1.

Schrijver's reduction of distributive lattice polyhedra to Edmonds-Giles polyhedra.
Let $\mathbb{P}(L, f)$ be a distributive lattice polyhedron, i.e., L is a consecutive distributive lattice of modular characteristic on which f is submodular. Schrijver [21] showed that this model coincides with the Edmonds-Giles model via the following reduction: Consider the set V of all join-irreducible elements of lattice L, which will turn out to be the vertices in the corresponding Edmonds-Giles model. For each $i \in L$ define the vertex set $V_i := \{j \in V \mid j \preceq i\}$. Then $\mathscr{F} = \{V_i \mid i \in L\} \subseteq 2^V$ is a ring family on subsets of vertices. Now, pick any $e \in E$ (recall that we assumed that there exists at least one $i^* \in L$ with $\chi(i^*, e) \neq 0$, say $\chi(i^*, e) = 1$). Since L has modular characteristic, we know that there exist unique minimal and maximal elements i_e and M_e in the sublattice $L_e^+ = \{i \in L \mid \chi(i, e) = 1\}$. Schrijver showed that the set $\{j \in L \mid j \succ M_e\}$ contains a unique minimal element j_e. Furthermore, he proved that the two elements i_e (as defined above as unique minimal element in L_e^+) and j_e are both join-irreducible. Assigning the arc $a_e = (i_e, j_e)$ (in the case $\chi(i^*, e) = -1$ the arc gets opposite direction) one can show for each $i \in L$ that a_e enters the vertex set V_i iff $\chi(i, e) = -1$, and a_e leaves V_i iff $\chi(i, e) = 1$. Thus, the distributive lattice polyhedron model coincides with the Edmonds-Giles model.

Schrijver's reduction of distributive lattice polyhedra to Edmonds-Giles polyhedra is a big step into the direction of understanding the combinatorial structure of lattice polyhedra in general, and finding a combinatorial algorithm for them. However, if we really want to transform his reduction into an algorithm to solve the problem, we would need an efficient way to construct the auxiliary digraph, i.e., we would need to identify for each $e \in E$ the minimal element i_e and the maximal element M_e in L_e^+ (resp. $L_e^- = \{i \in L \mid \chi(i, e) = -1\}$ in the case $\chi(i^*, e) = -1$), as well as the minimal element j_e in the set $\{j \in L \mid j \succ M_e\}$ (note that such an element j_e always exists by our assumption that the unique maximal element M satisfies $\chi(M, e) = 0$ for all $e \in E$). Furthermore, Schrijver requires that lattice L is not only of modular characteristic, but also distributive whereas Gröflin and Hoffman's integrality result holds for all L of modular characteristic. (In fact, in [21], the requirement of L being distributive seems to be forgotten. However, the distributivity is necessary for the correctness of the proof in [21].) So one of the first questions to answer is whether we can find a combinatorial algorithm for lattice polyhedra where we only require that L has modular characteristic.

Our contribution. In this paper we will see that the modular characteristic of L is almost sufficient for the lattice polyhedron to be distributive (Theorem 2). It turns out that it suffices to require the additional property that $\chi(i) \neq \chi(j)$ whenever i is a lower neighbor of j in $(L, \preceq)$. We call this property (*). In Section 3.1, we give an alternative, constructive proof to show that the inequality system of a distributive lattice polyhedron corresponds to one in an Edmonds-Giles polyhedron, and vice versa. That is, given a lattice polyhedron in which L has modular characteristic and satisfies property (*), we give a simple way to construct an auxiliary digraph G so that the lattice polyhedron is equivalent to some Edmonds-Giles polyhedron on G. We will see that in order to construct G, we only need to consider the elements of an *arbitrary* maximal chain in L.

3 Distributive lattice polyhedra and Edmonds-Giles polyhedra

Birkhoff's representation of the lattice elements by ideals in $(\mathscr{D}(P), \subseteq)$ as described above turns out to be the key for the reduction of distributive lattice polyhedra to Edmonds-Giles polyhedra in the subsequent Section 3.1. Before going into its detail, let us observe that the modular characteristic of L (i.e., property (C3) is satisfied with equality) is almost sufficient for the lattice polyhedron to be distributive.

Theorem 2 *Let $\mathbb{P}(A, f)$ be a lattice polyhedron where the underlying lattice L has modular characteristic and satisfies*

$$\chi(i) \neq \chi(j) \text{ whenever } i \text{ is a lower neighbor of } j \text{ in } (L, \preceq). \qquad (*)$$

Then the underlying lattice $(L, \preceq, \wedge, \vee)$ is distributive.

Proof For the sake of contradiction, assume that L is not distributive, i.e., that it contains an N_5- or an M_3-sublattice (see Figure 1). Then there exist five distinct elements $i, j, k, l, m \in L$ such that $l = i \wedge j = i \wedge k$ and $m = i \vee j = i \vee k$. Note that we can assume that j is a lower neighbor of k in the lattice L if j and k are elements in the N_5 sublattice, and that j and k are lower neighbors of m in the lattice L if j, k, and m are elements in the M_3-sublattice. By (*), it follows immediately that $\chi(j) \neq \chi(k)$ must be true in the N_5-sublattice. However, $\chi(j) \neq \chi(k)$ must also be true in the M_3-sublattice as otherwise $\chi(j) = \chi(m)$ by the consecutivity and modular characteristic of L, in contradiction to (*). Thus, we can choose some element $e \in E$ with $\chi(j, e) \neq \chi(k, e)$. Since L is of modular characteristic, it follows that $\chi(l, e) + \chi(m, e) = \chi(i, e) + \chi(j, e) = \chi(i, e) + \chi(k, e)$, which implies $\chi(j, e) = \chi(k, e)$, a contradiction.

3.1 Reduction to Edmonds-Giles polyhedra

Let $\mathbb{P}(A, f)$ (respectively $\mathbb{P}(L, f)$) be a distributive lattice polyhedron, and P denote the set of join-irreducible elements of the underlying distributive lattice $(L, \preceq, \wedge, \vee)$. By Birkhoff's Theorem, we may identify each element $i \in L$ (resp. the corresponding ordered pair $(X_i, Y_i) \in 3^E$ with $X_i := \{e \in E \mid \chi(i, e) = 1\}$ and $Y_i := \{e \in E \mid \chi(i, e) = $

$-1\}$) with some ideal $I_i \in \mathscr{D}(P)$. Thus, our inequality system $\{x \in \mathbb{R}^E \mid Ax \le f\}$ can be rewritten as $\{x \in \mathbb{R}^E \mid x(X_I) - x(Y_I) \le f_I, \ \forall I \in \mathscr{D}(P)\}$. In the following, we will construct an auxiliary digraph $G = (P, E')$ whose vertices correspond to the join-irreducibles of L, and whose edges are copies of E, i.e., $E' = \{e' \mid e \in E\}$, such that for all $I \in \mathscr{D}(P)$

$$e \in X_I \quad \Longleftrightarrow \quad e' \in \Delta^+(I), \tag{2}$$

$$e \in Y_I \quad \Longleftrightarrow \quad e' \in \Delta^-(I). \tag{3}$$

This directly implies that the vectors in $\mathbb{P}(A, f)$ are exactly the vectors in the Edmonds-Giles polyhedron $\mathbb{P}(G, \mathscr{D}(P), f)$.

Construction of auxiliary digraph. To construct our digraph corresponding to $\mathbb{P}(A, f)$, we arbitrarily choose a maximal chain $(\mathscr{C}, \preceq) = \{(X_1, Y_1) \prec (X_2, Y_2) \prec \cdots \prec (X_n, Y_n)\}$ in $(L, \preceq)$.

Observation 1 *Each $e \in E$ occurs either in some set $\{X_i\}_{i \in P}$ or in some set $\{Y_i\}_{i \in P}$.*

Proof By (C2), an element $e \in E$ cannot occur both in some X_i and some Y_j for $i, j \in \{1, \dots, n\}$. Suppose for the sake of contradiction that e does not occur in any chain element $(X_i, Y_i) \in \mathscr{C}$ at all. Let (X, Y) be the minimal element in $(L, \preceq)$ with $e \in X \cup Y$, and let (X_i, Y_i) be the minimal element of chain $(\mathscr{C}, \preceq)$ with $(X, Y) \prec (X_i, Y_i)$. Then (X, Y) and (X_{i-1}, Y_{i-1}) must be incomparable. Since e occurs neither in (X_{i-1}, Y_{i-1}) nor in $(X_i, Y_i) = (X, Y) \vee (X_{i-1}, Y_{i-1})$, the modular characteristic of L implies that e occurs in $(X, Y) \wedge (X_{i-1}, Y_{i-1})$. But, since $(X, Y) \wedge (X_{i-1}, Y_{i-1}) \prec (X, Y)$, this is a contradiction to the choice of (X, Y).

Note that, by Birkhoff's Theorem, our chosen chain $(\mathscr{C}, \preceq)$ is isomorphic to a chain $(\bar{\mathscr{C}}, \subseteq) = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ in $(\mathscr{D}(P), \subseteq)$, where $P_n = \{p_1, \dots, p_n\}$ is a linear extension of $(P, \preceq)$ (i.e., $p_i \prec p_j$ implies $i < j$), and $P_{i-1} = P_i \setminus \{p_i\}$ for $i = 1, \dots, n$. We now define for all $e \in E$ the potential $\nu_e : P \to \{-1, 0, 1\}$ with $\nu_e(p_i) = \chi(P_i, e) - \chi(P_{i-1}, e)$ for all $i = 1, \dots, n$. By the consecutiveness of L we observe that each ν_e uniquely defines i_e, j_e with $\nu_e(i_e) = 1$ and $\nu_e(j_e) = -1$. These vertices i_e and j_e will now form the tail and head of arc e', i.e., we define the arc set $E' = \{e' = (i_e, j_e) \mid e \in E\}$. (See Figure 2.)

Theorem 3 *If $\mathbb{P}(A, f)$ is a distributive lattice polyhedron and $G = (P, E')$ is the auxiliary digraph described above, then $\mathbb{P}(A, f) = \mathbb{P}_{EG}(G, \mathscr{D}(P), f)$.*

Proof Note that the statement of the theorem follows immediately if we can show that for all ideals $I \in \mathscr{D}(P)$ with the corresponding lattice element $(X_I, Y_I) \in L$ and all $e \in E$ the equivalences (2) and (3) hold. Note that these are equivalent to

$$\chi(I, e) = 1 \Longleftrightarrow e' \in \Delta^+(I), \tag{4}$$

$$\chi(I, e) = -1 \Longleftrightarrow e' \in \Delta^-(I). \tag{5}$$

Thus, in order to show the statement of Theorem 3 it suffices to prove that

$$\chi(I, e) = \sum_{p \in I} \nu_e(p) \quad \forall I \in \mathscr{D}(P) \quad e \in E. \tag{6}$$
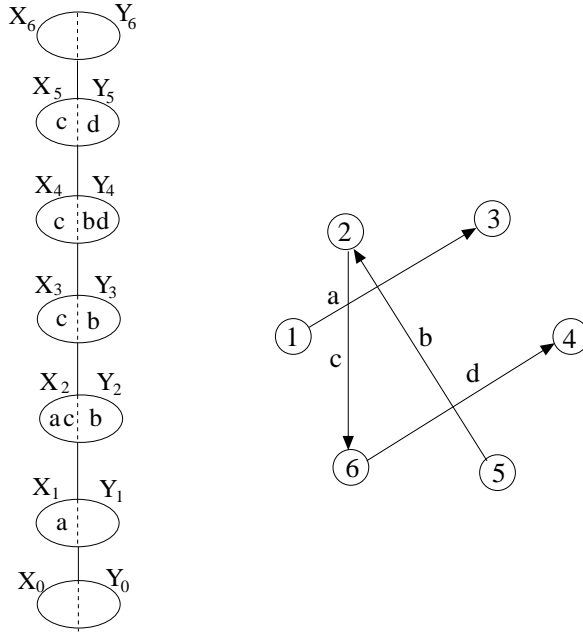
Fig. 2 A maximal chain in L and the corresponding auxiliary graph.

This follows as a consequence of Lemma 7.5 in [9]. For the sake of completeness, we add the proof of (6) here: For any integers $1 \le i_1 < \cdots < i_m \le n$ consider $I = \{p_{i_1}, \ldots, p_{i_m}\}$ with $\{p_{i_k}\} = P_{i_k} \setminus P_{i_k - 1}$ $(k = 1, \ldots, m)$. Since $\chi(\cdot, e)$ is a modular function, it follows that

$$\chi(I, e) + \sum_{k=1}^{m} \chi(P_{i_k - 1}, e)$$

$$= \chi(\underbrace{I \cup P_{i_m - 1}}_{P_{i_m}}, e) + \sum_{k=2}^{m} \chi(\underbrace{(I \cap P_{i_k - 1}) \cup P_{i_{k-1} - 1}}_{P_{i_{k-1}}}, e) + \chi(\underbrace{I \cap P_{i_1 - 1}}_{\emptyset}, e)$$

$$= \sum_{k=1}^{m} \chi(P_{i_k}, e),$$

which implies that

$$\chi(I, e) = \sum_{k=1}^{m} \chi(P_{i_k}, e) - \sum_{k=1}^{m} \chi(P_{i_k - 1}, e) = \sum_{p \in I} v_e(p).$$

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