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Kyoto University
On the universal $sl_2$ invariant of Brunnian bottom tangles

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Abstract

The universal $sl_2$ invariant is an invariant of bottom tangles from which one can recover the colored Jones polynomial of links. We are interested in the relationship between topological properties of bottom tangles and algebraic properties of the universal $sl_2$ invariant. A bottom tangle $T$ is called Brunnian if every proper subtangle of $T$ is trivial. In this paper, we prove that the universal $sl_2$ invariant of $n$-component Brunnian bottom tangles takes values in a small subalgebra of the $n$-fold completed tensor power of the quantized enveloping algebra $U_h(sl_2)$. As an application, we give a divisibility property of the colored Jones polynomial of Brunnian links.

1. Introduction

The universal invariant of tangles associated with a ribbon Hopf algebra [5, 6, 7, 9, 10, 12, 13] has the universality property for the colored link invariants which are defined by Reshetikhin and Turaev [13].

The universal $sl_2$ invariant $J_T$ of an $n$-component bottom tangle $T$ takes values in the $n$-fold completed tensor power $U_h(sl_2)^{\otimes n}$ of $U_h(sl_2)$, and we can obtain the colored Jones polynomial of the closure link $cl(T)$ from $J_T$ by taking the quantum traces. Here, a bottom tangle is a tangle in a cube consisting of only arc components such that each boundary point is on the bottom and the two boundary points of each arc are adjacent to each other, see Figure 1 (a) for example. The closure of a bottom tangle is defined as in Figure 1 (b).

Our interest is in the relationship between topological properties of tangles and links and algebraic properties of the universal $sl_2$ invariant and the colored Jones polynomial. Habiro [3] proved that the universal $sl_2$ invariant of $n$-component, algebraically-split,
0-framed bottom tangles takes values in a subalgebra \((\hat{U}_q^{\text{ev}})^{\otimes n}\) of \(U_h(sl_2)^{\otimes n}\) (Theorem 4.3). A bottom tangle is called ribbon if its closure is a ribbon link (cf. [3, 14]). A bottom tangle is called boundary if its components admit mutually disjoint Seifert surfaces of bottom tangles (cf. [3, 15]). The present author [14, 15] proved improvements of Habiro’s result in the special cases of ribbon bottom tangles and boundary bottom tangles, with a smaller subalgebra \((\hat{U}_q^{\text{ev}})^{\otimes n} \subset (\hat{U}_q^{\text{ev}})^{\otimes n}\) (Theorem 4.4). Here, the result for boundary bottom tangles is a refined version of Habiro’s conjecture [3].

A link \(L\) is called Brunnian if every proper sublink of \(L\) is trivial. Similarly, a bottom tangle \(T\) is called Brunnian if every proper subtangle of \(T\) is trivial, i.e., looks like \(\cap \cdots \cap\).

Habiro [4, Proposition 12] proved that for every Brunnian link \(L\), there is a Brunnian bottom tangle whose closure is isotopic to \(L\).

In the present paper, we give a subalgebra \(U_B^{(n)}\) of \(U_h(sl_2)^{\otimes n}\) such that \((\hat{U}_q^{\text{ev}})^{\otimes n} \subset U_B^{(n)} \subset (\hat{U}_q^{\text{ev}})^{\otimes n}\) in which the universal \(sl_2\) invariant of \(n\)-component Brunnian bottom tangles takes values (Theorem 4.6). As an application, we prove a divisibility property of the colored Jones polynomial of Brunnian links (Theorem 5-4). These results are first announced in [16].

The rest of this paper is organized as follows. In Section 2, we recall basic facts of the quantized enveloping algebra \(U_h(sl_2)\). In Section 3, we define the universal \(sl_2\) invariant of bottom tangles. In Section 4, we give the main result for the universal \(sl_2\) invariant of Brunnian bottom tangles. In Section 5, we give an application for the colored Jones polynomial of Brunnian links. Section 6 is devoted to the proofs of the results.

2. Quantized enveloping algebra \(U_h(sl_2)\)

In this section, we recall the definitions of \(U_h(sl_2)\) and its subalgebras. We follow the notations in [3, 15].

2.1. Quantized enveloping algebra \(U_h(sl_2)\)

We recall the definition of the universal enveloping algebra \(U_h(sl_2)\).

We denote by \(U_h = U_h(sl_2)\) the \(h\)-adically complete \(\mathbb{Q}[[h]]\)-algebra, topologically generated by \(H, E, \) and \(F\), defined by the relations

\[
HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},
\]

where we set

\[
q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.
\]

We equip \(U_h\) with the topological \(\mathbb{Z}\)-graded algebra structure such that \(\deg E = 1\), \(\deg F = -1\), and \(\deg H = 0\). For a homogeneous element \(x\) of \(U_h\), the degree of \(x\) is denoted by \(|x|\).

2.2. \(\mathbb{Z}[q, q^{-1}]\)-subalgebras of \(U_h(sl_2)\)

We recall \(\mathbb{Z}[q, q^{-1}]\)-subalgebras of \(U_h\) from [3, 15].

In what follows, we use the following \(q\)-integer notations.

\[
\{i\}_q = q^i - 1, \quad \{i\}_q = \{i\}_q \{i - 1\}_q \cdots \{i - n + 1\}_q, \quad \{n\}_q! = \{n\}_q, \quad \{n\}_q = \{n\}_q = \{n\}_q.
\]

\[
[i]_q = \{i\}_q / \{1\}_q, \quad [n]_q! = [n]_q [n - 1]_q \cdots [1]_q, \quad \{i\}_{n,q} = \{i\}_{n,q} / \{1\}_{n,q}.
\]
for \(i \in \mathbb{Z}, n \geq 0\).

Set

\[
\tilde{E}^{(n)} = (q^{-1/2}E)^n/[n]_q!, \quad \tilde{F}^{(n)} = F^n K^n/[n]_q^1 \in U_h,
\]

\[
e = (q^{1/2} - q^{-1/2})E, \quad f = (q - 1)FK \in U_h,
\]

for \(n \geq 0\).

Let \(U_{\mathbb{Z},q} \subset U_h\) denote the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra generated by \(K, K^{-1}, \tilde{E}^{(n)}, \text{ and } \tilde{F}^{(n)}\) for \(n \geq 1\), which is a \(\mathbb{Z}[q, q^{-1}]\)-version of Lusztig’s integral form (cf. [11, 14]).

Let \(U_q \subset U_{\mathbb{Z},q}\) denote the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra generated by \(K, K^{-1}, e\), and \(\tilde{F}^{(n)}\) for \(n \geq 1\).

Let \(\tilde{U}_q \subset U_q\) denote the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra generated by \(K, K^{-1}, e\) and \(f\), which is a \(\mathbb{Z}[q, q^{-1}]\)-version of the integral form defined by De Concini and Procesi (cf. [1, 14]).

For \(X = U_{\mathbb{Z},q}, U_q, \tilde{U}_q\), let \(X^{ev}\) denote the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra of \(U_h\) defined by the same generators as \(X\) except that \(K^{\pm 2}\) replaces \(K^{\pm 1}\), i.e., \(U_{\mathbb{Z},q}^{ev} \subset U_{\mathbb{Z},q}\) denotes the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra generated by \(K^2, K^{-2}, \tilde{E}^{(n)}, \tilde{F}^{(n)}, n \geq 1; U_q^{ev} \subset U_q\) denotes the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra generated by \(K^2, K^{-2}, e, \tilde{F}^{(n)}, n \geq 1; \text{ and } \tilde{U}_q^{ev} \subset \tilde{U}_q\) denotes the \(\mathbb{Z}[q, q^{-1}]\)-subalgebra generated by \(K^2, K^{-2}, e, f\).

To summarize, we have the following inclusions of the subalgebras of \(U_h\).

\[
\begin{align*}
U_{\mathbb{Z},q}^{ev} & \subset U_q^{ev} \subset U_{\mathbb{Z},q}^{ev} \subset U_h, \\
\tilde{U}_q & \subset U_q \subset U_{\mathbb{Z},q} \subset U_h.
\end{align*}
\]

2.3. Completions

In this section, we recall from [3] the completion \(\tilde{U}_q^{ev}\) of \(U_q^{ev}\) in \(U_h\) and its completed tensor powers \((\tilde{U}_q^{ev})\hat{\otimes}_n\) for \(n \geq 0\).

For \(p \geq 0\), let \(F_p(U_q^{ev})\) be the two-sided ideal in \(U_q^{ev}\) generated by \(e^p\). Let \(\tilde{U}_q^{ev}\) be the completion of \(U_q^{ev}\) in \(U_h\) with respect to the decreasing filtration \(\{F_p(U_q^{ev})\}_{p \geq 0}\), i.e., we define \(\tilde{U}_q^{ev}\) as the image of the homomorphism

\[
\lim_{\substack{\text{Lim} \, \text{p} \geq 0 \, F_p(U_q^{ev}) \rightarrow U_h}}
\]

induced by \(U_q^{ev} \subset U_h\).

For \(n \geq 1\) and \(p \geq 0\), set

\[
F_p((U_q^{ev})\hat{\otimes}_n) = \sum_{i=1}^{n} (U_q^{ev})\otimes(U_q^{ev})\otimes(U_q^{ev})\otimes(n-i).
\]

For \(n \geq 1\), we define \((\tilde{U}_q^{ev})\hat{\otimes}_n\) as the completion of \((U_q^{ev})\hat{\otimes}_n\) in \(U_h\) with respect to the decreasing filtration \(\{F_p((U_q^{ev})\hat{\otimes}_n)\}_{p \geq 0}\), i.e., we define

\[
(\tilde{U}_q^{ev})\hat{\otimes}_n = \text{Im} \left( \lim_{\substack{\text{Lim} \, \text{p} \geq 0 \, F_p((U_q^{ev})\hat{\otimes}_n) \rightarrow U_h}} \right).
\]

For a \(\mathbb{Z}[q, q^{-1}]\)-subalgebra \(A\) of \((U_q^{ev})\hat{\otimes}_n\), we denote by \(\{A\}\) the closure of \(A\) in \((\tilde{U}_q^{ev})\hat{\otimes}_n\), i.e., we set

\[
\{A\} = \text{Im} \left( \lim_{\substack{\text{Lim} \, \text{p} \geq 0 \, F_p((U_q^{ev})\hat{\otimes}_n) \cap A) \rightarrow U_h} \right).
\]

For \(n = 0\), we define \((\tilde{U}_q^{ev})\hat{\otimes}_0 = \mathbb{Z}[q, q^{-1}]\).
3. Universal $sl_2$ invariant of bottom tangles

In this section, we recall the definition of the universal $sl_2$ invariant of bottom tangles.

3.1. Bottom tangles

A bottom tangle (cf. [2, 3]) is an oriented, framed tangle in a cube consisting of arc components such that each boundary point is on a line on the bottom, and the two boundary points of each component are adjacent to each other. We give a preferred orientation of the tangle so that each component runs from its right boundary point to its left boundary point. For example, see Figure 2 (a), where the dotted lines represent the framing. We draw a diagram of a bottom tangle in a rectangle assuming the blackboard framing, see Figure 2 (b).

The closure link $\text{cl}(T)$ of a bottom tangle $T$ is defined as the link in $\mathbb{R}^3$ obtained from $T$ by closing, see Figure 1 again. For each $n$-component link $L$, there is an $n$-component bottom tangle whose closure is $L$. For a bottom tangle, we can define its linking matrix as that of the closure link.

3.2. Universal $R$-matrix of $U_h$

Set

$$D = q^{\frac{1}{4} H \otimes H} = \exp \left( \frac{\hbar}{4} H \otimes H \right) \in U_h^{\otimes 2}.$$  

We use the following universal $R$-matrix of $U_h$,

$$R^\pm = \sum_{n \geq 0} \alpha_n^\pm \otimes \beta_n^\pm \in U_h^{\otimes 2},$$

where we set formally

$$\alpha_n \otimes \beta_n = \alpha_n^+ \otimes \beta_n^+ = D \left( q^{\frac{1}{2} n(n-1) \tilde{F}(n)} K^{-n} \otimes e^n \right),$$

$$\alpha_n^- \otimes \beta_n^- = D^{-1} \left( (-1)^n \tilde{F}(n) \otimes K^{-n} e^n \right).$$

(Note that the right hand sides are sums of infinitely many tensors of the form $x \otimes y$ with $x, y \in U_h$. We denote them by $\alpha_n^\pm \otimes \beta_n^\pm$ for simplicity.)

3.3. Universal $sl_2$ invariant of bottom tangles

For an $n$-component bottom tangle $T = T_1 \cup \cdots \cup T_n$, we define the universal $sl_2$ invariant $J_T \in U_h^{\otimes n}$ in four steps as follows. We follow the notation in [15].

**Step 1. Choose a diagram.** We choose a diagram $\hat{T}$ of $T$ obtained from the copies of the fundamental tangles depicted in Figure 3, by pasting horizontally and vertically. We denote by $C(\hat{T})$ the set of the crossings of $\hat{T}$. For example, for the bottom tangle $B$
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Fig. 3. Fundamental tangles, where the orientations of the strands are arbitrary

Fig. 4. (a) A bottom tangle $B$, (b) A diagram $\tilde{B}$ of $B$, (c) The labels associated to a state $t \in S(B)$

depicted in Figure 4 (a), we can take a diagram $\tilde{B}$ with $C(\tilde{B}) = \{c_1, c_2\}$ as depicted in Figure 4 (b). We call a map

$$s : C(\tilde{T}) \rightarrow \{0, 1, 2, \ldots\}$$

a state. We denote by $S(\tilde{T})$ the set of states of the diagram $\tilde{T}$.

**Step 2. Attach labels.** Given a state $s \in S(\tilde{T})$, we attach labels on the copies of the fundamental tangles in the diagram following the rule described in Figure 5, where “$S^m$” should be replaced with the identity if the string is oriented downward, and with $S$ otherwise. For example, for a state $t \in S(\tilde{B})$, we put labels on $\tilde{B}$ as in Figure 4 (c), where we set $m = t(c_1)$ and $n = t(c_2)$.

**Step 3. Read the labels.** We read the labels we have just put on $\tilde{T}$ and define an element $J_{\tilde{T},s} \in U_b^{\otimes n}$ as follows. Let $\tilde{T} = \tilde{T}_1 \cup \cdots \cup \tilde{T}_n$, where $\tilde{T}_i$ corresponds to $T_i$. We define the ith tensorand of $J_{\tilde{T},s}$ as the product of the labels on $\tilde{T}_i$, where the labels are read off along $T_i$ reversing the orientation, and written from left to right. For example, for the bottom tangle $B$ and the state $t \in S(\tilde{B})$ in Figure 4, we have

$$J_{\tilde{B},t} = S(\alpha_m)S(\beta_n) \otimes \alpha_n\beta_m.$$ 

Here, we identify the labels $S'(\alpha_i^\pm)$ and $S'(\beta_i^\pm)$ with the first and the second tensorands, respectively, of the element $S'(\alpha_i^\pm) \otimes S'(\beta_i^\pm) \in U_b^{\otimes 2}$. Also we identify the label $K^{\pm 1}$ with the element $K^{\pm 1} \in U_h$. Thus $J_{\tilde{T},s}$ is a well-defined element in $U_b^{\otimes n}$. For example, we

Fig. 5. How to place labels on the fundamental tangles
have

\[ J_{\tilde{T},t} = S(\alpha_m)S(\beta_n) \otimes \alpha_n \beta_m \]

\[ = \sum q^{\frac{m(m-1)}{2}} q^{\frac{n(n-1)}{2}} S(D'_1 \tilde{F}(m) K^{-m})S(D'_2 e^n) \otimes D'_3 \tilde{F}(n) K^{-n} D'_4 e^m \]

\[ = (-1)^{m+n} q^{-n+2mn} D^{-2} (\tilde{F}(m) K^{-2n} e^n \otimes \tilde{F}(n) K^{-2m} e^m) \in U_h \hat{\otimes} 2, \]

where \( D = \sum D'_1 \otimes D'_2' = \sum D'_2 \otimes D'_2' \). Note that \( J_{\tilde{T},s} \) depends on the choice of the diagram.

**Step 4. Take the state sum.** Set

\[ J_T = \sum_{s \in \mathcal{S}(-T)} J_{\tilde{T},s}. \]

For example, we have

\[ J_B = \sum_{t \in \mathcal{S}(-B)} J_{\tilde{B},t} = \sum_{m,n \geq 0} (-1)^{m+n} q^{-n+2mn} D^{-2} (\tilde{F}(m) K^{-2n} e^n \otimes \tilde{F}(n) K^{-2m} e^m). \]

As is well known \([12]\), \( J_T \) does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

4. **Results for the universal \( sl_2 \) invariant of bottom tangles**

In this section, we give the main result for the universal \( sl_2 \) invariant of Brunnian bottom tangles. In what follows, we assume that bottom tangles are 0-framed.

4.1. **Universal \( sl_2 \) invariant of algebraically-split bottom tangles, ribbon bottom tangles and boundary bottom tangles**

We recall several results for the value of the universal \( sl_2 \) invariant of algebraically-split bottom tangles. Recall the sequence of the subalgebras \( \hat{U}_q^{ev} \subset \hat{U}_q^{ev} \subset U_{Z,q}^{ev} \subset U_h \).

**Theorem 4.1** ([14, Proposition 4.2, Remark 4.7]). Let \( T \) be an \( n \)-component algebraically-split bottom tangle. For every diagram \( \tilde{T} \) of \( T \) and every state \( s \in \mathcal{S}(\tilde{T}) \), we have

\[ J_{\tilde{T},s} \in (\hat{U}_q^{ev})^{\otimes n}. \]

More precisely, the proof of \([14, \text{Proposition 4.2}]\) implies the following.

**Proposition 4.2.** Let \( T \) be an \( n \)-component algebraically-split bottom tangle. For any diagram \( \tilde{T} \) and any state \( s \in \mathcal{S}(\tilde{T}) \), we have

\[ J_{\tilde{T},s} \in \mathcal{F}| \{(U_q^{ev})^{\otimes n}\}, \]

where we set \( |s| = \max \{s(e) \mid e \in C(\tilde{T})\} \).

Recall from Section 2.3 the completion \( (\hat{U}_q^{ev})^{\otimes n} \) of \( (U_q^{ev})^{\otimes n} \). Theorem 4.1 and Proposition 4.2 imply the following, which was first proved by Habiro \([3]\) in a different way.

**Theorem 4.3** (Habiro \([3]\)). For an \( n \)-component algebraically-split bottom tangle \( T \), we have

\[ J_T \in (\hat{U}_q^{ev})^{\otimes n}. \]

In \([3]\), Habiro denoted by \( (U_q^{ev})^{\otimes n} \) the closure \( \{(U_q^{ev})^{\otimes n}\} \) of \( (U_q^{ev})^{\otimes n} \) in \( (\hat{U}_q^{ev})^{\otimes n} \). In \([14]\) and \([15]\), we defined a refined completion \( (\hat{U}_q^{ev})^{\otimes n} \subset (U_q^{ev})^{\otimes n} \), and proved the
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following theorem, which is an improvement of Theorem 4-3 in the case of ribbon bottom tangles and boundary bottom tangles.

**THEOREM 4.4 ([14, 15]).** Let \( T \) be an \( n \)-component ribbon or boundary bottom tangle. Then we have

\[
J_T \in (U_q^{\text{ev}})^{\otimes n}.
\]

**REMARK 4-5.** Theorem 4.4 with \((U_q^{\text{ev}})^{\otimes n}\) replaced with \((U_q^{\text{ev}})^{\otimes n}\) for boundary bottom tangles had been conjectured by Habiro [3, Conjecture 8.9]. Here, we do not know whether the inclusion \((U_q^{\text{ev}})^{\otimes n}\) \( \subset \) \((U_q^{\text{ev}})^{\otimes n}\) is proper or not, but the definition of \((U_q^{\text{ev}})^{\otimes n}\) is more natural than that of \((U_q^{\text{ev}})^{\otimes n}\) in the settings in [14, 15].

4.2. Results for the universal \( \mathfrak{sl}_2 \) invariant of Brunnian bottom tangles

The following is the main result of the present paper, which is an improvement of Theorems 4-1 and 4-3 in the case of Brunnian bottom tangles.

**THEOREM 4.6.** Let \( T \) be an \( n \)-component algebraically-split Brunnian bottom tangle with \( n \geq 2 \).

(i) For each \( i = 1, \ldots, n \), there is a diagram \( \tilde{T}^{(i)} \) of \( T \) such that

\[
J_{\tilde{T}^{(i)}, s} \in (U_q^{\text{ev}})^{\otimes i-1} \otimes U_{Z_q} \otimes (U_q^{\text{ev}})^{\otimes n-i}
\]

for any state \( s \in S(\tilde{T}^{(i)}) \).

(ii) We have \( J_T \in U_{Br}^{(n)} \), where we set

\[
U_{Br}^{(n)} = \bigcap_{i=1}^{n} \left\{ \left( (U_q^{\text{ev}})^{\otimes i-1} \otimes U_{Z_q} \otimes (U_q^{\text{ev}})^{\otimes n-i} \right) \cap (U_q^{\text{ev}})^{\otimes n} \right\}.
\]

Note that the condition “algebraically-split” in Theorem 4.6 is for 2-component Brunnian bottom tangles, since every \( n \)-component Brunnian bottom tangle with \( n \geq 3 \) is algebraically-split by the definition.

We prove Theorem 4.6 (i) in Section 6. Theorem 4.6 (ii) is derived from Theorem 4.6 (i) and Proposition 4.2 as follows.

**Proof of Theorem 4.6 (ii) by assuming Theorem 4.6 (i)** For each \( i = 1, \ldots, n \), by Theorem 4.6 (i) and Proposition 4.2, there is a diagram \( \tilde{T}^{(i)} \) of \( T \) such that

\[
J_{\tilde{T}^{(i)}, s} \in \left( (U_q^{\text{ev}})^{\otimes i-1} \otimes U_{Z_q} \otimes (U_q^{\text{ev}})^{\otimes n-i} \right) \cap S(\tilde{T}^{(i)})
\]

for any state \( s \in S(\tilde{T}^{(i)}) \). Hence we have

\[
J_T = \sum_{s \in S(\tilde{T}^{(i)})} J_{\tilde{T}^{(i)}, s} \in \left\{ \left( (U_q^{\text{ev}})^{\otimes i-1} \otimes U_{Z_q} \otimes (U_q^{\text{ev}})^{\otimes n-i} \right) \cap (U_q^{\text{ev}})^{\otimes n} \right\}
\]

for all \( i = 1, \ldots, n \).

To compare Theorem 4.6 (ii) with Theorems 4.3 and 4.4 for \( n \geq 2 \), we have the following.
Fig. 6. (a) The Borromean bottom tangle $T_B$, (b) A bottom tangle $T'_B$

\[ \{n\text{-comp. alg. split bottom tangles}\} \xrightarrow{J} (U_q^{ev})^\otimes n \]
\[ \{n\text{-comp. alg. split Brunnian bottom tangles}\} \xrightarrow{J} U_{Br}^{(n)} \]
\[ \{n\text{-comp. ribbon or boundary bottom tangles}\} \xrightarrow{J} (U_q^{ev})^\otimes n \]

**Example 4-7.** For the Borromean bottom tangle $T_B$ depicted in Figure 6 (a), we have

\[
J_T \in \left\{ \left( U_{Z,q}^{ev} \otimes (U_q^{ev})^{\otimes 2} \right) \cap (U_q^{ev})^{\otimes 3} \right\} \\
\cap \left\{ \left( U_q^{ev} \otimes U_{Z,q}^{ev} \otimes U_q^{ev} \right) \cap (U_q^{ev})^{\otimes 3} \right\} \\
\cap \left\{ \left( (U_q^{ev})^{\otimes 2} \otimes U_{Z,q}^{ev} \right) \cap (U_q^{ev})^{\otimes 3} \right\}.
\]

See Example 6-2 for explicit expressions of $J_T$.

**Example 4-8.** Let us add a trivial arc to the Borromean bottom tangle as in Figure 6 (b), and denote it by $T'_B$. Note that the bottom tangle $T'_B$ is not Brunnian but algebraically-split. We have

\[ J_{T'_B} = J_{T_B} \otimes 1 \notin \left\{ \left( U_q^{ev} \otimes U_{Z,q}^{ev} \right) \cap (U_q^{ev})^{\otimes 4} \right\}. \]

5. Application to the colored Jones polynomial

In this section, we give an application of Theorem 4-6 to the colored Jones polynomial of Brunnian links (Theorem 5-4). In what follows, we assume that links are 0-framed.

5.1. Colored Jones polynomials of algebraically-split links, ribbon links and boundary links

We recall results for the colored Jones polynomials of algebraically-split links.

For $m \geq 1$, let $V_m$ denote the $m$-dimensional irreducible representation of $U_h$. Let $\mathcal{R}$ denote the representation ring of $U_h$ over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

\[ \mathcal{R} = \text{Span}_{\mathbb{Q}(q^{\frac{1}{2}})} \{ V_m \mid m \geq 1 \} \]

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

For an $n$-component link $L$, take a bottom tangle $T$ whose closure is $L$. For $X_1, \ldots, X_n \in \mathcal{R}$, the colored Jones polynomial $J_{L;X_1,\ldots,X_n}$ of $L$ with the $i$th component $L_i$ colored by $X_i$ is given by

\[ J_{L;X_1,\ldots,X_n} = (\text{tr}_q^{X_1} \otimes \cdots \otimes \text{tr}_q^{X_n})(J_T) \in \mathbb{Q}(q^{\frac{1}{2}}). \]
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where, for $Y = \sum_j y_j V_j \in \mathcal{R}$ and $u \in U_h$, we set
$$\text{tr}^Y_q(u) = \text{tr}^Y(K^{-1}u) = \sum_j y_j \text{tr}^{V_j}(K^{-1}u).$$

Habiro [3] studied the following elements in $\mathcal{R}$
$$P_l = \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R},$$
$$\tilde{P}_l = \frac{q^{2l}}{\{1\}_q} P_l \in \mathcal{R},$$
for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres.

Recall the notation $\{l\}_q = \{l\}_q \{l-1\}_q \cdots \{l-i+1\}_q$ for $l \in \mathbb{Z}$, $i \geq 0$. Theorem 4.3 implies the following.

**Theorem 5.1 (Habiro [3]).** Let $L$ be an $n$-component algebraically-split link. For $l_1, \ldots, l_n \geq 0$, we have
$$J_{L, \tilde{P}_{l_1}, \ldots, \tilde{P}_{l_n}} \in \mathcal{Z}_{a}^{(l_1, \ldots, l_n)}.$$ Here we set
$$\mathcal{Z}_{a}^{(l_1, \ldots, l_n)} = \frac{\{2l_{\max} + 1\}_q, l_{\max} + 1, \mathbb{Z}[q, q^{-1}]}{\{1\}_q},$$
where $l_{\max} = \max(l_1, \ldots, l_n)$.

For $l \geq 0$, let $I_l$ denote the ideal in $\mathbb{Z}[q, q^{-1}]$ generated by $\{l-k\}_q \{k\}_q!$ for $k = 0, \ldots, l$.

Theorem 4.3 implies the following, which is an improvement of Theorem 5.1 in the cases of ribbon links and boundary links.

**Theorem 5.2 ([14, 15]).** Let $L$ be an $n$-component ribbon link or boundary link. For $l_1, \ldots, l_n \geq 0$, we have
$$J_{L, \tilde{P}_{l_1}, \ldots, \tilde{P}_{l_n}} \in \mathcal{Z}_{a}^{(l_1, \ldots, l_n)}.$$ Here we set
$$\mathcal{Z}_{a}^{(l_1, \ldots, l_n)} = \left( \prod_{1 \leq i < j \neq l \leq m} I_{l_i} \right) \cdot \mathcal{Z}_{a}^{(l_1, \ldots, l_n)}$$
$$= \frac{\{2l_{\max} + 1\}_q, l_{\max} + 1, \mathbb{Z}[q, q^{-1}]}{\{1\}_q} \prod_{1 \leq i < j \neq l \leq m} I_{l_i},$$
where $l_{\max} = \max(l_1, \ldots, l_n)$ and $i_m$ is an integer such that $l_{i_m} = l_{\max}$.

For $m \geq 1$, let $\Phi_m = \prod_{d|m}(q^d - 1)^{\mu(\frac{d}{m})} \in \mathbb{Z}[q]$ denote the $m$th cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all positive divisors $d$ of $m$, and $\mu$ is the M"{o}bius function. For $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to $r$.

In [16], we study the ideal $I_l$ and prove the following result, which we use later.

**Proposition 5.3 ([16]).** For $l \geq 0$, the ideal $I_l$ is the principal ideal generated by
$$g_l = \prod_{m \geq 1} \Phi_{l,m}^{t_m},$$
(5.5)
Thus, comparing Theorem 5

where

\[ t_{l,m} = \begin{cases} \frac{l+1}{m} - 1 & \text{for } 1 \leq m \leq l, \\ 0 & \text{for } l < m. \end{cases} \]

5.2. Result for the colored Jones polynomial of Brunnian links

The following is an application of Theorem 4.6 to the colored Jones polynomial of Brunnian links, which we prove in Section 6.2.

**Theorem 5.4.** Let \( L \) be an \( n \)-component algebraically-split Brunnian link with \( n \geq 2 \). For \( l_1, \ldots, l_n \geq 0 \), we have

\[ J_{L, P_{l_1} \cdots P_{l_n}} \in Z^{(l_1, \ldots, l_n)}_{Br}. \quad (5.6) \]

Here we set

\[ Z^{(l_1, \ldots, l_n)}_{Br} = \frac{\{2l_{\text{max}} + 1\}q^{l_{\text{max}}+1}}{\{1\}q^{\{l_{\text{min}}\}\{1\}}} \prod_{1 \leq i \leq n, i \neq i_M, i_m} I_{l_i}, \]

where \( l_{\text{max}} = \max(l_1, \ldots, l_n), \) \( l_{\text{min}} = \min(l_1, \ldots, l_n) \) and \( i_M, i_m, i_M \neq i_m \), are two integers such that \( l_{\text{max}} = l_{\text{max}}, l_{\text{min}} = l_{\text{min}}, \) respectively.

Note that an algebraically-split Brunnian link satisfies both (5.3) and (5.6). In fact, there is no inclusion which satisfies for all \( l_1, \ldots, l_n \geq 0 \) between \( Z^{(l_1, \ldots, l_n)}_a \) and \( Z^{(l_1, \ldots, l_n)}_{Br} \).

For example, we have \( Z^{(2,2,2,2)}_a \not\subseteq Z^{(2,2,2,2)}_{Br} \) and \( Z^{(2,2,2,2)}_{Br} \not\subseteq Z^{(2,2,2,2)}_a \) since

\[ Z^{(2,2,2,2)}_a = \frac{\{5\}q^3}{\{1\}q} Z[q, q^{-1}] \]

\[ = (q-1)^2(q+1)(q^2 + q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)Z[q, q^{-1}], \]

\[ Z^{(2,2,2,2)}_{Br} = \frac{\{5\}q^3}{\{1\}q^2 q^2} \frac{1}{\{1\}q^2} Z[q, q^{-1}] \]

\[ = (q-1)^4(q^2 + q + 1)(q^2 + 1)(q^4 + q^3 + q^2 + q + 1)Z[q, q^{-1}]. \]

For \( l_1, \ldots, l_n \geq 0 \), set

\[ \tilde{Z}^{(l_1, \ldots, l_n)}_{Br} = Z^{(l_1, \ldots, l_n)}_a \cap Z^{(l_1, \ldots, l_n)}_{Br}. \]

The above argument implies the following refinement of Theorem 5.4.

**Theorem 5.5.** Let \( L \) be an \( n \)-component algebraically-split Brunnian link with \( n \geq 2 \). For \( l_1, \ldots, l_n \geq 0 \), we have

\[ J_{L, P_{l_1} \cdots P_{l_n}} \in \tilde{Z}^{(l_1, \ldots, l_n)}_{Br}. \]

For \( n \geq 2 \), we have

\[ Z^{(l_1, \ldots, l_n)}_{r,b} = \prod_{1 \leq i \leq n, i \neq i_M, i_m} I_{l_i} \cdot Z^{(l_1, \ldots, l_n)}_{Br}, \]

\[ = \left( \{l_{\text{min}}\}q^{l_{\text{min}}} \right) Z^{(l_1, \ldots, l_n)}_{Br}. \]

Thus, comparing Theorem 5.5 with Theorems 5.1 and 5.2 for \( n \geq 2 \), we have the following.
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\[
\{ \text{n-comp. alg. split links} \} \xrightarrow{J_{r_1 \cdots r_n}} Z^{(l_1, \ldots, l_n)}_a \\
\{ \text{n-comp. alg. split Brunian links} \} \xrightarrow{J_{r_1 \cdots r_n}} \tilde{Z}^{(l_1, \ldots, l_n)}_{Br} \\
\{ \text{n-comp. ribbon or boundary links} \} \xrightarrow{J_{r_1 \cdots r_n}} Z^{(l_1, \ldots, l_n)}_{r,b}
\]

Remark 5-6. By Proposition 5-3, the ideals \( Z^{(l_1, \ldots, l_n)}_a, \tilde{Z}^{(l_1, \ldots, l_n)}_{r,b}, Z^{(l_1, \ldots, l_n)}_{Br} \) are principal, each generated by a product of cyclotomic polynomials. See [16] for details and examples.

6. Proofs

In this section, we prove Theorem 4.6 (i) and Theorem 5.4.

6.1. Proof of Theorem 4.6 (i)

We use the following lemma.

Lemma 6.1. For \( m \geq 0 \) and \( k, l \in \mathbb{Z} \), we have

\[
S^k(\alpha_m^\pm) \otimes S^l(\beta_m^\pm) \in D^{-2k}((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q})).
\]

Proof. For \( m \geq 0 \), we have

\[
\begin{align*}
\alpha_m \otimes \beta_m &= D(q^{\frac{1}{2}m(m-1)} \hat{E}(m) K^{-m} \otimes e^m) \\
&= D(q^{m(m-1)} f^m K^{-m} \otimes \hat{E}(m)) \\
&\in D((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q})), \\
\alpha_m^{-} \otimes \beta_m^{-} &= D^{-1}((-1)^m \hat{E}(m) K^{-m} e^m) \\
&= D^{-1}((-1)^m q^{\frac{1}{2}m(m-1)} f^m \otimes K^{-m} \hat{E}(m)) \\
&\in D^{-1}((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q})).
\end{align*}
\]

For \( k, l \in \mathbb{Z} \), we have

\[
\begin{align*}
(S^k \otimes S^l)(D^{\pm}) &= D^{\pm(-1)^{k+l}}, \\
(S^k \otimes S^l)((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q})) &= (U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q}).
\end{align*}
\]

For \( x \in U_h \) homogeneous, we have

\[
(x \otimes 1)D^{\pm} = D^{\pm(x \otimes K^\pm(x))}.
\]

Now, (6.1)–(6.5) imply the assertion. For example, we have

\[
S(\alpha_m) \otimes S(\beta_m) = (S \otimes S)(\alpha_m \otimes \beta_m)
\]

\[
\in (S \otimes S)\left(D((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q}))\right)
\]

\[
\in ((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q}))D
\]

\[
= D((U_{Z,q} \otimes \hat{U}_q) \cap (U_q \otimes U_{Z,q})).
\]

Proof of Theorem 4.6 (i) Let \( T = T_1 \cup \cdots \cup T_n \) be an \( n \)-component algebraically-split Brunian bottom tangle with \( n \geq 2 \). We prove the assertion for \( i = 1 \), i.e., we prove that there is a diagram \( \tilde{T} \) of \( T \) such that

\[
J_{\tilde{T},x} \in U_{Z,q}^{\text{ev}} \otimes (U_q^{\text{ev}})^{\otimes n-1}
\]
Fig. 7. Borromean rings $T_B$ and its diagram $\tilde{T}_B = \tilde{T}_{B,1} \cup \tilde{T}_{B,2} \cup \tilde{T}_{B,3}$ such that $\tilde{T}_{B,2} \cup \tilde{T}_{B,3}$ has no crossing. (Here, in order to show examples of self crossings, we put trivial ones on the leftmost strand.)

For any state $s \in \mathcal{S}(\tilde{T})$. The other cases $2 \leq i \leq n$ are similar.

Since $T$ is Brunnian, the subtangle $T_2 \cup \cdots \cup T_n$ is trivial. Thus $T$ has a diagram $\tilde{T} = \tilde{T}_1 \cup \tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ whose subdiagram $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ has no crossing. See Figure 7 for an example of such a diagram for the Borromean rings $T_B$.

We prove that $\tilde{T}$ satisfies (6-6). Note that $\tilde{T}$ has only two types of crossings as follows.

Type A: Crossings between $\tilde{T}_1$ and $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$

Type B: Self crossings of $\tilde{T}_1$

Recall from the definition of $J_{\tilde{T},s}$ in Section 3-3 the labels which are put on the diagram. For the crossings of type A, by Lemma 6-1, we can assume that the labels on $\tilde{T}_1$ are legs of copies of $D^{\pm 1}$ and elements of $U_{Z,q}$, and the labels on $\tilde{T}_2 \cup \cdots \cup \tilde{T}_n$ are legs of copies of $D^{\pm 1}$ and elements of $\tilde{U}_q$. For the crossings of type B, we assume that the labels on $\tilde{T}_1$ are legs of copies of $D^{\pm 1}$ and elements of $U_{Z,q}$. See Figure 8 for example, where $\circ$ denote elements in $U_{Z,q}$ and $\ast$ denote elements in $U_q$.

Now, except copies of $D^{\pm 1}$, all labels on $\tilde{T}_1$ are elements of $U_{Z,q}$, and all labels on $\tilde{T}_2 \cup \cdots \tilde{T}_n$ are elements of $\tilde{U}_q$, see Figure 9 for example. We gather every copy of $D^{\pm 1}$ to the leftmost of the expression of $J_{\tilde{T},s}$ by using (6-5), and cancel these since the linking
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matrix of $T$ is 0. Then we have

$$J_{T,a} \in U_{\mathbb{Z},q} \otimes \bar{U}_{q}^{\otimes n-1}.$$ 

By Theorem 4.3, we have

$$J_{T,a} \in (U_{\mathbb{Z},q} \otimes \bar{U}_{q}^{\otimes n-1}) \cap (U_{q}^{ev})^{\otimes n}$$

\[ < U_{\mathbb{Z},q} \otimes (U_{q}^{ev})^{\otimes n-1}. \]

This completes the proof.

**Example 6.2.**

The following is the universal $sl_2$ invariant of the Borromean bottom tangle calculated by using the diagram Figure 6 (a) (cf. [3]).

$$J_{T_B} = \sum_{m_1, m_2, n_1, n_2, n_3 \geq 0} q^{m_3+n_3} (-1)^{n_1+n_2+n_3} \sum_{i=1}^{3} \left(-\frac{i}{2}n_1(m_i+1)-n_1+m_i+1-2n_i, 1\right)$$

\[ \mathcal{F}(n_3) e^{m_1} \mathcal{F}(m_3) e^{n_1} K^{-2m_2} \otimes \mathcal{F}(n_1) e^{m_2} \mathcal{F}(m_1) e^{n_2} K^{-2m_3} \otimes \mathcal{F}(n_2) e^{m_3} \mathcal{F}(m_2) e^{n_3} K^{-2m_1} \]

\[ \in \left(\bar{U}_{q}^{ev}\right)^{\otimes 3}, \]

where the index $i$ should be considered modulo 3.

By using the diagram $P$ in Figure 7, after canceling the two self crossings of the leftmost strand, we also have the following expression of $J_{T_B}$.

$$J_{T_B} = \sum_{g,h,k,l,m,n \geq 0} \sum_{i=0}^{m} \sum_{j=0}^{n} t_{g,h,i,j,k,l,m,n}(q)$$

\[ K^{-2(h+k)} \mathcal{F}(g) \mathcal{F}(h) \mathcal{F}(i) \mathcal{F}(l) \mathcal{F}(m-i) \mathcal{F}(n-j) \otimes K^{2(k-l-m)} f^{n} e^{m} \otimes K^{-2(h-i+j+k)} e^{j} f^{h+k} e^{g} \]

\[ \in \left(U_{\mathbb{Z},q} \otimes (U_{q}^{ev})^{\otimes 2}\right) \cap (U_{q}^{ev})^{\otimes 3} \]

where

$$t_{g,h,i,j,k,l,m,n}(q) = \left(-1\right)^{g+h+m+n+i+j} q^{-2g(3h+k)+\frac{1}{2}h(h-1)+h(2l-1)+k(2l+2n-i-j)-\frac{1}{2}l(l-1)}$$

$$\times q^{-l(2n+i-3j)-m(6n-i-j)+\frac{1}{2}n(n-1)-n(j+1)-\frac{1}{2}l(i-1)+\frac{1}{2}j(j-1).}$$

**6.2. Proof of Theorem 5.4.**

In this section, we prove Theorem 5.4.

First of all, we recall generators of $\bar{U}_{q}^{ev}$ and $U_{\mathbb{Z},q}^{ev}$ as $\mathbb{Z}[q,q^{-1}]$-modules. The following Lemma is a variant of a well-known fact about the integral form defined by De Concini and Procesi (cf. [1, 3]).

**Lemma 6.3.** $\bar{U}_{q}^{ev}$ is freely $\mathbb{Z}[q,q^{-1}]$-spanned by the elements $f^{i} K^{2j} e^{k}$ with $i,k \geq 0$ and $j \in \mathbb{Z}$.

For the elements $P_i, \bar{P}_i \in \mathcal{R}$ defined in (5-1), (5-2) in Section 5-1, we have the following results.

**Lemma 6.4** (Habiro [3, Lemma 8.8]).

1. If $i, i', l \geq 0$, $i \neq i'$, and $j \in \mathbb{Z}$, then we have $\text{tr}_{\bar{P}_i}^{P_i}(\bar{F}^{(i)} K^{2j} e^{i'}) = 0$.
2. If $0 \leq l < i$ and $j \in \mathbb{Z}$, then we have $\text{tr}_{\bar{P}_i}^{P_i}(\bar{F}^{(i)} K^{2j} e^{i}) = 0$. 
(3) If $0 \leq i \leq l$ and $j \in \mathbb{Z}$, then we have
\[
\text{tr}_q^l (\tilde{F}(i) K^2 j e_i) = q^{\frac{1}{2} j - j - 2 j i + i^2 - i} \cdot [l]_q! \cdot [l - i]_q! \cdot \left[ j + l - 1 \right]_q \cdot \left[ j - l - i \right]_q .
\]

For $l \geq 0$, recall the ideal $I_l$ in $\mathbb{Z}[q, q^{-1}]$, which is generated by $\{ l - i \} q! \cdot \{ i \} q!$ for $i = 0, \ldots, l$.

**Corollary 6.5** (Habiro [3]). For $l \geq 0$, we have $\text{tr}_q^l (U_{Z, q}) \subset I_l$.

**Proof.** The assertion follows from Lemma 6.3, Lemma 6.4 (1), (2), and
\[
\text{tr}_q^l (f^i K^2 j e_i) = q^{-\frac{1}{2} (i - 1)} \cdot \{ i \} \cdot \text{tr}_q^l (\tilde{F}(i) K^2 e_i)
\]

\[
=q^{-\frac{1}{2} (i - 1)} \cdot \{ i \} \cdot \frac{q^{\frac{1}{2} j} \cdot \text{tr}_q^l (\tilde{F}(i) K^2 e_i)}{\{ l \} q!}
\]

\[
=q^{-\frac{1}{2} (i + i - 2 j + i^2 - i)} \cdot \{ l - i \} q! \cdot \left[ j + l - 1 \right]_q \cdot \left[ j - l - i \right]_q \in I_l
\]

for $0 \leq i \leq l$ and $j \in \mathbb{Z}$.

**Proposition 6.6.** For $l \geq 0$, we have $\{ l \} q! \cdot \text{tr}_q^l (U_{Z, q}) \subset \mathbb{Z}[q, q^{-1}]$.

**Proof.** Since we have $\tilde{F}(i) = q^{\frac{1}{2} (i - 1)} f^i / \{ i \} q!$, $\tilde{E}(i) = e^i / \{ i \} q!$ for $i \geq 0$, we have
\[
U_{Z, q}^t \subset U_{q}^t \otimes \mathbb{Z}[q, q^{-1}] Q(q).
\]

This and Corollary 6.5 imply
\[
\text{tr}_q^l (U_{Z, q}^t) \subset \mathbb{Q}(q).
\]

(6.7)

In what follows, we prove
\[
\{ l \} q! \cdot \text{tr}_q^l (U_{Z, q}) \subset \mathbb{Z}[q^{1/2}, q^{-1/2}],
\]

(6.8)

which and (6.7) imply
\[
\{ l \} q! \cdot \text{tr}_q^l (U_{Z, q}^t) \subset \mathbb{Q}(q) \cap \mathbb{Z}[q^{1/2}, q^{-1/2}] = \mathbb{Z}[q, q^{-1}].
\]

(6.9)

Let $U_z$ be Lusztig’s integral form, which is the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-subalgebra of $U_h$ generated by $K^{\pm 1}$, $F^i / \{ i \} q!$ and $E^i / \{ i \} q!$ for $i > 1$, where $[i]! = [i][i - 1] \cdots [1]$ with $[n] = q^{n/2 - n^{-1/2}}$ for $n \geq 0$. Here, we have
\[
U_Z = U_{Z, q} \otimes \mathbb{Z}[q, q^{-1}] \mathbb{Z}[q^{1/2}, q^{-1/2}].
\]

(6.10)

Recall that $V_m$ denotes the $m$-dimensional irreducible representation of $U_h$. It is well-known that there is a $U_Z$-submodule $V_{m, Z}$ of $V_m$ such that
\[
V_m = V_{m, Z} \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] Q[[h]].
\]

Since
\[
\text{tr}_q^l (V_{Z, m}) \subset \mathbb{Z}[q^{1/2}, q^{-1/2}]
\]

and
\[
\{ l \} q! P_l = q^{1/2} P_l \in \text{Span}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \{ V_m \mid m \geq 0 \},
\]

\[
\text{tr}_q (V_{Z, m}) \subset \mathbb{Z}[q^{1/2}, q^{-1/2}]^{1/2}
\]

and
\[
\text{tr}_q^l (U_Z) \subset \mathbb{Z}[q^{1/2}, q^{-1/2}]^{1/2}
\]

and
\[
\{ l \} q! P_l = q^{1/2} P_l \in \text{Span}_{\mathbb{Z}[q^{1/2}, q^{-1/2}]} \{ V_m \mid m \geq 0 \},
\]
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it follows that

$$\{l\}_q \cdot \text{tr}^{P_1}_{q^i}(U_2) \in \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

This and (6-10) imply (6-8). Hence we have the assertion.

In [3], Habiro proved that if $T$ is an $n$-component algebraically-split bottom tangle, then we have

$$(\text{id}^\otimes n - 1 \otimes \text{tr}^{P_1}_{q^i} \otimes \text{id}^\otimes n - i)(J_T) \in (\tilde{U}_q^{\text{ev}})^{\otimes n - 1}$$

for $1 \leq i \leq n$ and $l_i \geq 0$. We use the following proposition.

**Proposition 6.7.** Let $T$ be an $n$-component algebraically-split Brunnian bottom tangle with $n \geq 2$. For $1 \leq i \leq n$ and $l_i \geq 0$, we have

(i) $$(\text{id}^\otimes n - 1 \otimes \text{tr}^{P_1}_{q^i} \otimes \text{id}^\otimes n - i)(J_T) \in (\tilde{U}_q^{\text{ev}})^{\otimes n - 1},$$

(ii) $$\{l_i\}_q \cdot (\text{id}^\otimes n - 1 \otimes \text{tr}^{P_1}_{q^i} \otimes \text{id}^\otimes n - i)(J_T) \in (\tilde{U}_q^{\text{ev}})^{\otimes n - 1}.$$

**Proof.** We prove the assertion with $i = 1$. The other cases are similar. Let $\hat{T} = \hat{T}^{(1)}$ be a diagram of $T$ as in Theorem 4.6 (i). By the proof of Theorem 4.6 (i), we can assume that $\hat{T} = \hat{T}_1 \cup \cdots \cup \hat{T}_n$ has only two types of crossings as follows.

Type A: Crossings between $\hat{T}_1$ and $\hat{T}_2 \cup \cdots \cup \hat{T}_n$

Type B: Self crossings of $\hat{T}_1$

Let $s \in S(\hat{T})$. Set $|s| = \max\{|s(c)| \mid c \in C(\hat{T})\}$. Note that, for $0 \leq m < p$, the elements $E^p$ and $F^p$ act as 0 on the $m$-dimensional irreducible representation $V_m$ of $U_h$. Since each crossing of either type involves the strand $\hat{T}_1$, there is a crossing $c$ on $\hat{T}_1$ such that $s(c) = |s|$. Since $\hat{P}_1 \in \text{Span}_{\mathbb{Q}[q^{1/2}]}\{V_0, \ldots, V_{l_1}\}$, if $|s| > l_1$, then we have

$$(\text{tr}^{P_1}_{q^1} \otimes \text{id}^\otimes n - 1)(J_{\hat{T},s}) = 0. \quad (6.11)$$

By (6.11), Theorem 4.1 implies (i), and Theorem 4.6 (i), Proposition 6.6 imply (ii).

For a subalgebra $X$ of $U_h$, let $Z(X)$ denote the center of $X$. Habiro [3, Proposition 8.6] proved that for an $n$-component algebraically-split bottom tangle, we have

$$(\text{id} \otimes \text{tr}^{P_1}_{q^{l_1}} \otimes \text{tr}^{P_3}_{q^{l_3}} \otimes \cdots \otimes \text{tr}^{P_n}_{q^{l_n}})(J_T) \in Z(\tilde{U}_q^{\text{ev}}).$$

We can improve this result in the case of Brunnian bottom tangles as follows.

**Proposition 6.8.** Let $T$ be an $n$-component algebraically-split Brunnian bottom tangle with $n \geq 2$. For $l_1, \ldots, l_n \geq 0$, we have

$$(\text{id} \otimes \text{tr}^{P_1}_{q^{l_1}} \otimes \text{tr}^{P_3}_{q^{l_3}} \otimes \cdots \otimes \text{tr}^{P_n}_{q^{l_n}})(J_T) \in Z(\tilde{U}_q^{\text{ev}}).$$

**Proof.** By Proposition 6.7 (i) and $\text{tr}^{P_1}_{q^i}(\tilde{U}_q^{\text{ev}}) \subset \mathbb{Z}[q, q^{-1}]$ for $i \geq 0$, we have

$$(\text{id} \otimes \text{tr}^{P_1}_{q^{l_1}} \otimes \text{tr}^{P_3}_{q^{l_3}} \otimes \cdots \otimes \text{tr}^{P_n}_{q^{l_n}})(J_T) \in (\text{id} \otimes \text{tr}^{P_1}_{q^{l_1}} \otimes \cdots \otimes \text{tr}^{P_n}_{q^{l_n}})(\tilde{U}_q^{\text{ev}})^{\otimes n - 1} \subset \tilde{U}_q^{\text{ev}}.$$
It is well-known that $J_T$ is contained in the invariant part of $U_h^{\otimes n}$ (cf. Kerler [8, Corollary 12]). This fact implies
\[
(id \otimes \text{tr}_{q_2} \otimes \text{tr}_{q_3} \otimes \cdots \otimes \text{tr}_{q_n})(J_T) \in Z(U_h).
\]
Since $U_q^{ev} \cap Z(U_h) \subset Z(U_q^{ev})$, we have the assertion.

Let $C = (q^{1/2} - q^{-1/2})^2FE + q^{1/2}K + q^{-1/2}K^{-1}$ denote the Casimir element. Recall from [3] that $Z(U_q^{ev})$ is freely generated by $C^2$ as a $Z[q,q^{-1}]$-algebra, and thus, freely spanned by the following monic polynomials in $C^2$ as a $Z[q,q^{-1}]$-module.
\[
\sigma_p = \prod_{i=1}^{p} (C^2 - (q^i + 2 + q^{-i})), \quad p \geq 0.
\]
Habiro proved the following.

**Proposition 6.9** (Habiro [3, Proposition 6.3]). For $l, m \geq 0$, we have
\[
\text{tr}_{q_1}^{P_{1,m}}(\sigma_m) = \delta_{l,m},
\]
where $P_{l,m} = q^{l(l+1)} \frac{(1)_l}{(2l+1)_l} \tilde{P}_l'$. Proposition 6.9 implies the following.

**Corollary 6.10** (Habiro [3]). For $l \geq 0$, we have
\[
\text{tr}_{q_1}^{P_l'}(Z(U_q^{ev})) \subset \frac{(2l+1)q^{l+1}Z[1,q,q^{-1}]}{\{1\}_q}.
\]
Now, we prove Theorem 5.4.

**Proof of Theorem 5.4** Let $L$ be an $n$-component algebraically-split Brunnian link with $n \geq 2$ and $T$ a Brunnian bottom tangle whose closure is $L$. Let $l_1, \ldots, l_n \geq 0$. Without loss of generality, we assume $l_1 = \max(l_1, \ldots, l_n)$ and $l_2 = \min\{l_i \mid 1 \leq i \leq n\}$. By Proposition 6.7 (ii) and Corollary 6.5, we have
\[
\{l_2\}_q!(\text{id} \otimes \text{tr}_{q_1} \otimes \text{tr}_{q_3} \otimes \cdots \otimes \text{tr}_{q_n})(J_T)
\]
\[
\in (\text{id} \otimes \text{tr}_{q_1} \otimes \text{tr}_{q_3} \otimes \cdots \otimes \text{tr}_{q_n})(U_q^{ev})^{\otimes n-1}
\]
\[
\subset \left( \prod_{3 \leq i \leq n} l_i \right) \cdot U_q^{ev}
\]
\[
\subset \left( \prod_{3 \leq i \leq n} l_i \right) \cdot g_1 \cdots g_n U_q^{ev},
\]
where the last equation is follows from Proposition 5.3.

Since $U_q^{ev}$ has no non-trivial zero divisor, we have
\[
(g_3 \cdots g_n U_q^{ev}) \cap Z(U_q^{ev}) \subset g_3 \cdots g_n Z(U_q^{ev}).
\]
By (6.12), (6.13) and Proposition 6.8, we have
\[
\{l_2\}_q!(\text{id} \otimes \text{tr}_{q_1} \otimes \text{tr}_{q_3} \otimes \cdots \otimes \text{tr}_{q_n})(J_T) \subset g_3 \cdots g_n Z(U_q^{ev}).
\]
By (6.14) and Corollary 6.10, we have
\[
\{l_2\}_{q}J_{L, \tilde{r}_1, \ldots, \tilde{r}_n} = \left\{ l_2 \right\}_{q}^{1} \left( \text{tr}_{q}^{1} \otimes \text{tr}_{q}^{2} \otimes \cdots \otimes \text{tr}_{q}^{n} \right) (J_{T})
\]
\[
eq \text{tr}_{q}^{1} \left( g_{3} \cdots g_{n} Z(l_{q}^{n}) \right)
\]
\[
\subseteq \left\{ 2l_{1} + 1 \right\}_{q}^{l_{1}+1} g_{3} \cdots g_{n} Z[q, q^{-1}]
\]
\[
= \left\{ 2l_{1} + 1 \right\}_{q}^{l_{1}+1} \prod_{3 \leq i \leq n} I_{i}.
\]
Hence we have
\[
J_{L, \tilde{r}_1, \ldots, \tilde{r}_n} = \frac{\left\{ 2l_{1} + 1 \right\}_{q}^{l_{1}+1}}{\left\{ 1 \right\}_{q}^{l_{2}} \prod_{3 \leq i \leq n} I_{i}}.
\]
This completes the proof.

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