Pseudo Diagrams of Knots, Links and Spatial Graphs

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Abstract: Our purpose in this research is to investigate triviality of knots, links and spatial graphs from projections. We introduce a pseudo diagram which is a projection with over/under information at some double points of it and study them. We have DNA knots as a trigger to start this research.

1 Introduction

A knot, a link and a spatial graph are an embedded circle, an embedded disjoint union of some circles, an embedded graph in the 3-sphere $S^3$. A projection $P$ is the image of natural projection of a knot, a link and a spatial graph to the 2-sphere $S^2$ such that its multiple points are only finitely many transversal double points away from vertices. A diagram $D$ is a projection $P$ with over/under information at each double point. A diagram $D$ uniquely represents a knot, a link or a spatial graph up to ambient isotopy. Here a double point with over/under information is called a crossing, in contrast a double point without over/under information is called a pre-crossing.

Question 1 Can we determine from $P$ whether the original knot (link, spatial graph) is trivial or knotted?

We cannot determine it except some special cases. Because we do not know over/under information at each pre-crossing of $P$. For example, let $P$ be a projection of a knot with 3 pre-crossings as illustrated in Fig. 1. Then we have $2^3$ diagrams obtained from $P$. Two diagrams represent nontrivial knots and six diagrams represent the trivial knots.

In this paper, we study the following question. We have DNA knots as a trigger to have this question, namely we cannot determine over/under information at some crossings in some photos of DNA knots.

Question 2 Which pre-crossings of $P$ and which over/under information at them should we know in order to determine that the original knot (link, spatial graph) is trivial or knotted?

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We give new definitions. A pseudo diagram \( Q \) is a projection \( P \) with over/under information at some pre-crossings of \( P \). Here we allow the possibility that a pseudo diagram is a projection or a diagram. We say that a pseudo diagram \( Q' \) is obtained from a pseudo diagram \( Q \) if each crossing of \( Q \) has the same over/under information as \( Q' \). Then \( Q \) is said to be trivial if for any diagram \( D \) obtained from \( Q \), \( D \) represents a trivial knot (link, spatial graph). In contrast, \( Q \) is said to be knotted if for any diagram \( D \) obtained from \( Q \), \( D \) represents a nontrivial knot (link, spatial graph). For example, in Fig. 2, (a) is trivial, (b) is knotted and (c) is neither trivial nor knotted.

We define that the trivializing number of \( P \) is the minimal cardinality of \( C_Q \) where \( Q \) is a trivial pseudo diagram obtained from \( P \) and \( C_Q \) is the set of crossings of \( Q \). Then we denote the trivializing number of \( P \) by \( tr(P) \). In contrast, we define that the knotting number of \( P \) is the minimal cardinality of \( C_Q \) where \( Q \) is a knotted pseudo diagram obtained from \( P \). Then we denote the knotting number of \( P \) by \( kn(P) \). For example, let \( P \) be the projection as illustrated in Fig. 1, then \( tr(P) = 2 \) and \( kn(P) = 3 \).

2 A Theorem and Propositions

Theorem 2.1 Let \( P \) be a projection of a knot. Then \( tr(P) \) is always even.

Proposition 2.2 For any nonnegative even number \( n \), there exists a projection of a knot with \( tr(P) = n \).

Proposition 2.3 There does not exist a projection of a knot whose knotting number is less than 3. For any natural number \( n \geq 3 \) there exists a projection \( P \) of a knot with \( kn(P) = n \).

Proposition 2.4 For any integer \( z \), there exists a projection \( P \) of a knot with \( tr(P) - kn(P) = z \).

In addition, we characterize a projection \( P_1 \) of a knot with \( tr(P_1) = p(P_1) - 1 \), a projection \( P_2 \) of a link with \( tr(P_2) = 2 \), a projection \( P_3 \) of a link with \( tr(P_3) = p(P_3) \) and a projection \( P_4 \) of a link with \( kn(P_4) = p(P_4) \) where \( p(P) \) is the cardinality of the set of pre-crossings of \( P \).

References