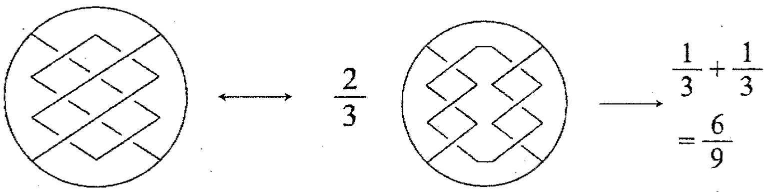
Rational structure on algebraic tangles and closed incompressible surfaces in the complements of algebraically alternating knots and links

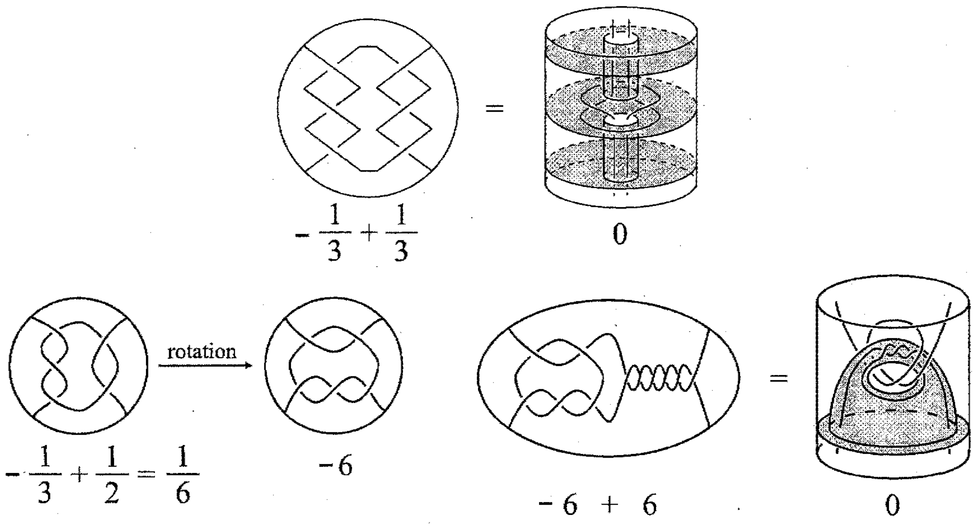
Makoto Ozawa
Komazawa University

In 1970, Conway ([1]) showed that the rational tangles correspond to the rational numbers in one-to-one. In 1999, Krebes ([2]) constructed a map from tangles to formal fractions (not necessarily reduced), and the map on the algebraic tangles is surjective.



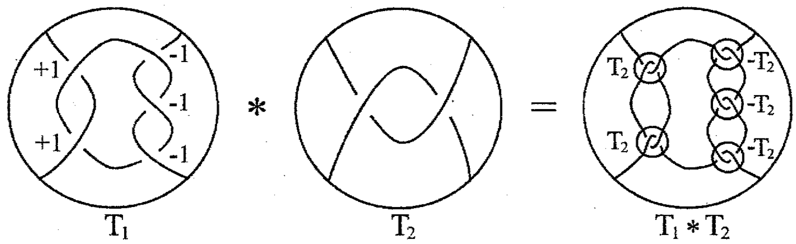
Conway's correspondence   Krebes's correspondence

Theorem 1. *Let (B,T) be an algebraic tangle, $F \subset B - T$ an essential surface. Then, F separates the components of T in B. Moreover, $B-T$ contains at least one essential surface, and the boundary slope of essential surfaces are unique.*



By Theorem 1, a map ϕ from algebraic tangles to the boundary slopes of essential surfaces is defined.

We define the *multiplication* $T_1 * T_2$ of two tangles T_1 and T_2 like a figure.

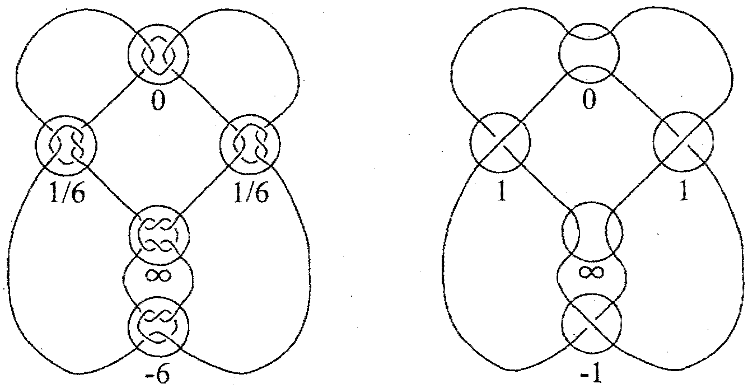


Theorem 2. *The map ϕ is a homomorphism from algebraic tangles to rational numbers. Namely, the following hold.*

- $\phi(T_1 + T_2) = \phi(T_1) + \phi(T_2)$
- $\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$
- $\phi(-T) = -\phi(T)$
- $\phi(T^*) = -\frac{1}{\phi(T)}$

Here, + denotes the tangle sum, $*$ the tangle multiplication, $-$ the reflection and * the rotation.

In Conway notation $\tilde{K}$, we replace each algebraic tangle T with a rational tangle of slope $+1$ (resp. -1, 0, ∞) if $\phi(T) > 0$ (resp. < 0, $= 0$, $= \infty$), and the resultant diagram $\tilde{K}_0$ is called the *basic diagram* of $\tilde{K}$. We say that $\tilde{K}$ is *algebraically alternating* if $\tilde{K}_0$ is alternating, and that K is *algebraically alternating* if K has an algebraically alternating diagram. The class of algebraically alternating links contains both of algebraic links and alternating links.



$\tilde{K}$: algebraically alternating diagram $\quad \tilde{K}_0$: basic diagram of $\tilde{K}$

Theorem 3. *Let (S^3, K) be an algebraically alternating links, $F \subset S^3 - K$ an essential closed surface. Then, F separates the components of K in S^3. Moreover, the basic diagram $\tilde{K}_0$ is split, or F is contained in an algebraic tangle of (S^3, K). In particular, if F is a 2-sphere, then there exists a cut tangle. If F is a torus and there exists no cut tangle, then (S^3, K) contains Q_2.*

Corollary. *Any essential closed surface in the complement of an algebraically alternating knot is meridionally compressible.*

References

[1] J. H. Conway, *An enumeration of knots and links and some of their algebraic properties*, in "Computational Problems in Abstract Algebra", (D. Welsh, Ed.), pp. 329-358, Pergamon Press, New York, 1970.

[2] David A. Krebes, *An obstruction to embedding 4-tangles in links*, Journal of Knot Theory and Its Ramifications **8** (1999) 321–352.

[3] M. Ozawa, *Rational structure on algebraic tangles and closed incompressible surfaces in the complements of algebraically alternating knots and links*, preprint (2008) http://arxiv.org/abs/0803.1302