Minimal step number of knots with small crossing number
— 交点数の小さい結び目の最小ステップ数について —

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1 Introduction

The steps are unit segments with endpoints in $\mathbb{Z}^3$ in $\mathbb{R}^3$. A simple closed polygonal cycle consisting of steps is called a lattice knot. The minimal step number is the number of steps required to form a lattice knot. The trefoil knot can be constructed by 24 steps. So the minimal step number of the trefoil knot is at most 24. Diao showed the following theorem.

**Theorem 1** ([1]). A lattice knot with a step number less than 24 is an unknot. Therefore, the minimal step number of the trefoil knot is 24.

The figure eight knot can be constructed by 30 steps. So the minimal step number of the figure eight knot is at most 30. Yamaguchi showed the following theorem.

**Theorem 2** ([3]). A lattice knot with a step number less than 30 is an unknot or a trefoil knot. Hence, the minimal step number of the figure eight knot is 30.

By constructing a lattice knot, we can give an upper bound for the minimal step number. For knots with small crossing number, the minimal step numbers are estimated by computer simulation [2].

2 Result

We obtain the following theorem.
Theorem 3. A lattice knot with a step number less than 34 has crossing number at most 4, and so that is an unknot or a trefoil knot or a figure eight knot. Therefore, the minimal step number of the $5_1$ knot is 34.

A lattice $5_1$ knot in the figure in the right gives the minimal step number.

3 Outline of proof

Let $P$ be a lattice knot giving the minimal step number 32 at most. We will show that the crossing number of $P$ is at most 4. Let $G$ be the projection of $P$ to $xy$-plane, that is a graph on $\mathbb{R}^2$. Each edge of $G$ is an unit segment and corresponds to $x$-steps or $y$-steps of $P$. Here a step parallel to $a$-axis ($a \in \{x, y, z\}$) is called an $a$-step. For each edge of $G$, the number of corresponding $x$-steps or $y$-steps is called the multiplicity. The sum of multiplicities for all edges of $G$ is called the total multiplicity, that is the number of all $x$-steps and $y$-steps of $P$. Assuming the number of $z$-steps of $P$ is most, the total multiplicity of $G$ is at most 20. We consider all possibilities of such graphs with multiplicities on $\mathbb{R}^2$. This is a same method as Diao [1]. We add some ideas to this. First, there are some conditions for $G$ where $P$ does not give the minimal step number. For example, if the $3 \times 4$ region does not contain $G$, then we can show $P$ does not give the minimal step number, and so it contradicts the first assumption. Hence, it is enough to consider the graphs in the $3 \times 4$ region. Next, we can give an upper bound for the crossing number of $P$ by calculation for multiplicities of $G$. Using such ideas, we can show that the crossing number of $P$ is at most 4.

Detail proof will soon be appear.

参考文献

