# UNKNOTTING NUMBERS OF DIAGRAMS OF A GIVEN NONTRIVIAL KNOT ARE UNBOUNDED

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## 概要

任意の非自明結び目 $K$ と任意の自然数 $n$ に対して、$K$ のあるダイアグラム $D$ が存在して $D$ の結び目解消数は $n$ 以上となる。$K$ の結び目解消数の 2 倍が $K$ の最小交点数から 1 引いたもの以下であることはよく知られている。ここで等式が成り立つのは $K$ が $(2,p)$-トーラス結び目であるときに限る。

Let $L$ be a link in the 3-sphere $\mathbb{S}^3$ and $D$ a diagram of $L$ on the 2-sphere $\mathbb{S}^2$. It is well known that by changing over/under information at some crossings of $D$ we have a diagram of a trivial link. See for example [3]. Let $u(D)$ be the minimal number of such crossing changes. Namely, there are some $u(D)$ crossings of $D$ such that changing them yields a trivial link diagram, and changing less than $u(D)$ crossings never yields a trivial link diagram. We call $u(D)$ the *unlinking number of* $D$. In the case that $D$ is a diagram of a knot $u(D)$ is called the *unknotting number of* $D$. The *unlinking number* $u(L)$ *of* $L$ is defined by the minimum of $u(D)$ where $D$ varies over all diagrams of $L$. Namely we have the following equality.

$$u(L) = \min\{u(D) \mid D \text{ is a diagram of } L\}.$$

For a knot $K$ $u(K)$ is called the *unknotting number of* $K$. Then it is natural to ask whether or not the set $\{u(D) \mid D \text{ is a diagram of } L\}$ is bounded above. In [1] Nakanishi showed that an unknotting number one knot $6_2$ has an unknotting number two diagram. Then he showed the following theorem in [2].

**Theorem 1** [2]. *Let $K$ be a nontrivial knot. Then $K$ has a diagram $D$ with $u(D) \geq 2$.*

As an extension of Theorem 1, we have the following theorem.

**Theorem 2.** *Let $L$ be a nontrivial link. Then for any natural number $n$ there exists a diagram $D$ of $L$ with $u(D) \geq n$.*
*That is, the set $\{u(D) \mid D$ is a diagram of $L\}$ is unbounded above.*

We note that Theorem 2 is an immediate consequence of the following proposition.

**Proposition 3.** *Let $L$ be a nontrivial link and $D$ a diagram of $L$. Then there exists a diagram $D'$ of $L$ with $u(D') = u(D) + 2$.*

The proof of Proposition 3 is done by using a modification of diagram illustrated in Figure 1 that is essentially the same as that used in [2]. See [4] for the detail.

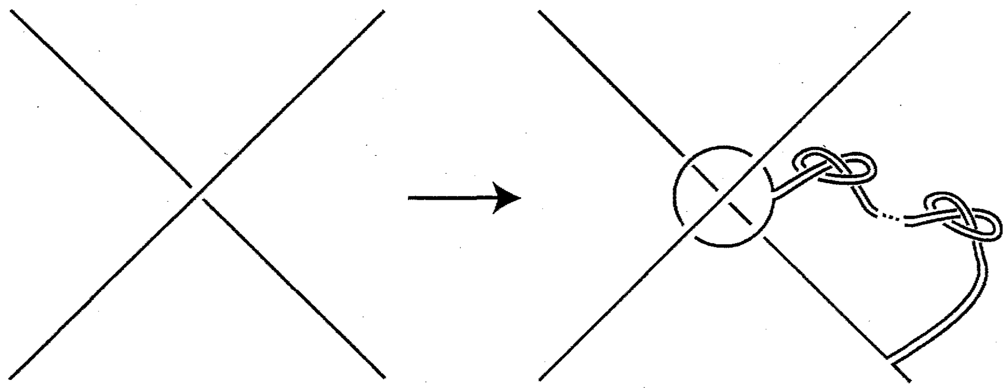

Figure 1

As an immediate consequence of Proposition 3 we have the following corollary.

**Corollary 4.** *Let $L$ be a nontrivial link.*
*Then the set $\{u(D) \mid D$ is a diagram of $L\}$ contains a set*
*$\{u(L)+2m \mid m$ is a non-negative integer$\}$.*

**Question 5.** *Let $L$ be a nontrivial link. Is the set $\{u(D) \mid D$ is a diagram of $L\}$ equals the set $\{u(L)+m \mid m$ is a non-negative integer$\}$?*

The following proposition is a partial answer to Question 5.

**Proposition 6.** *Let $L$ be an alternating link with $u(L)=1$. Suppose that $L$ has an alternating diagram $D_0$ with $u(D_0)=1$.*
*Then the set $\{u(D) \mid D$ is a diagram of $L\}$ equals the set of natural numbers $\{u(L)+m \mid m$ is a non-negative integer$\}$.*

Let $c(D)$ be the number of crossings in $D$. We call $c(D)$ the *crossing number of* $D$. Then the *crossing number* $c(L)$ *of* $L$ is defined by the minimum of $c(D)$ where $D$ varies over all diagrams of $L$. It is natural to ask the relation between $u(D)$ and $c(D)$, or $u(L)$ and $c(L)$. For a diagram $D$ of a knot $K$ other than a trivial diagram the following inequality is well-known. See for example [3].

$$u(K) \leq u(D) \leq \frac{c(D)-1}{2}.$$

In particular this inequality holds for a minimal crossing diagram $D$ of $K$ where $c(D)=c(K)$. Thus for any nontrivial knot $K$ we have the following inequality.

$$u(K) \leq \frac{c(K)-1}{2}.$$

It is also well known that the equality holds for $(2,p)$-torus knots. Conversely we have the following theorem.

**Theorem 7.** *(1) Let $D$ be a diagram of a knot that satisfies the equality*

$$u(D) = \frac{c(D)-1}{2}.$$

*Then $D$ is one of the diagrams illustrated in Figure 2. Namely $D$ is a reduced alternating diagram of some $(2,p)$-torus knot, or $D$ is a diagram with just one crossing.*

*(2) Let $K$ be a nontrivial knot that satisfies the equality*

$$u(K) = \frac{c(K)-1}{2}.$$

*Then $K$ is a $(2,p)$-torus knot for some odd number $p \neq \pm 1$. Namely only 2-braid knots satisfy the equality.*

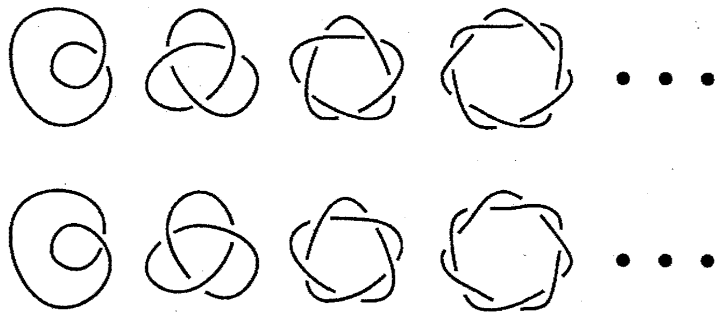

Figure 2

For links the situation is somewhat different. Let $D$ be a diagram of a link. Then the following inequality is well-known.

$$u(L) \leq u(D) \leq \frac{c(D)}{2}.$$

Thus for any link $L$ we have the following inequality.

$$u(L) \leq \frac{c(L)}{2}.$$

The following theorem shows that not only $(2,p)$-torus links but some other links satisfy the equality.

**Theorem 8.** *(1) Let $D = \gamma_1 \cup \cdots \cup \gamma_\mu$ be a diagram of a $\mu$-component link that satisfies the equality*

$$u(D) = \frac{c(D)}{2}.$$

*Then each $\gamma_i$ is a simple closed curve on $\mathbb{S}^2$ and for each pair $i, j$, the subdiagram $\gamma_i \cup \gamma_j$ is an alternating diagram or a diagram without crossings.*

*(2) Let $L$ be a $\mu$-component link that satisfies the equality*

$$u(L) = \frac{c(L)}{2}.$$

*Then $L$ has a diagram $D = \gamma_1 \cup \cdots \cup \gamma_\mu$ such that each $\gamma_i$ is a simple closed curve on $\mathbb{S}^2$ and for each pair $i, j$, the subdiagram $\gamma_i \cup \gamma_j$ is an alternating diagram or a diagram without crossings.*

Two examples of such links are illustrated in Figure 3. We note that for a link described in Theorem 8 the unlinking number equals the sum of the absolute values of all pairwise linking numbers.

The detail will appear in [4].

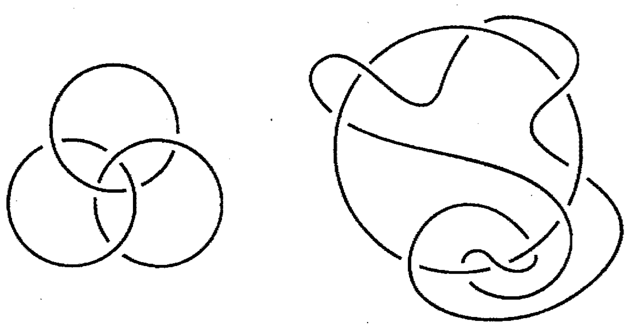

Figure 3

## References


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