Magnetic Writhe and Self-Organized Braiding

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Abstract : Knot theory and the geometry of curves have important applications in astrophysics and fluid mechanics. This paper presents two. First, the *writhe* number, which measures the buckling and coiling of a closed curve, arises in the study of magnetic structures in the atmosphere of the sun. As these structures have endpoints at the solar surface, the definition of writhe must be modified. We present definitions for open writhe appropriate for both unconstrained open curves, and for curves with endpoints on a physical boundary. Secondly, *braids* occur naturally in the solar atmosphere: magnetic field lines in x-ray loops can become braided owing to motions of the endpoints at the surface. Reconnection in the atmosphere reduces the topological complexity of the magnetic field, and releases magnetic energy in the form of flares. We conjecture that the braid pattern evolves to a self-organized state with power law statistical properties.

1 Open Writhe

Many objects in nature, from DNA molecules to interplanetary magnetic clouds, exhibit kinked or coiled geometries. For example, figure 1 shows two examples of kinked structures formed by magnetic fields in the atmosphere (corona) of the sun. Mathematicians, Physicists, and Biologists seeking to measure and analyze this geometric structure often use a quantity called the *writhe.*

Formally, for a three-dimensional closed curve $\mathbf{x}(s)$ (where s is arc length from some point along the curve) the writhe is [6, 1]

$$\mathcal{W} = \frac{1}{4\pi} \oint \oint \frac{\mathbf{x}(s) - \mathbf{x}(s')}{|\mathbf{x}(s) - \mathbf{x}(s')|^3} \cdot \frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} \times \frac{\mathrm{d}\mathbf{x}(s')}{\mathrm{d}s'}\, \mathrm{d}s\, \mathrm{d}s'. \tag{1}$$

If we project the curve onto a plane, we will see a certain number of (signed) crossings. If we average this number over all projections, then we obtain the writhe [1, 13, 7, 9].

Consider a ribbon or double-stranded DNA molecule with one edge or strand following the curve $\mathbf{x}(s)$. Suppose the second edge or strand mostly follows $\mathbf{x}(s)$, but with a small perpendicular offset $\hat{V}(s)$. Then there will be additional structure not captured by the writhe: the two strands can twist about each other by some amount $\mathcal{T}$; also we can measure the net linking $\mathcal{L}$ of the two strands. We then have Călugăreanu 's famous formula: $\mathcal{L} = \mathcal{T} + \mathcal{W}$ [6].

How do we define the writhe for open curves? One way is simply to use the double integral of Eqn. 1, except that the line integrals simply extend from endpoint to endpoint [11]. This choice preserves the relation between writhe and average crossing number. However, the relation to linking number is more problematic: we would have to accept a definition of linking (i.e. the Gauss linking integral applied to open curves) which is not topologically invariant. Other proposed definitions involve comparing the open curve to some reference curve, e.g. a straight line or part of a circle [10, 15]. However, this comparison only gives in general a number defined mod 2, and requires that the reference curve can be deformed into the open curve with some awkward constraints on possible deformation paths.

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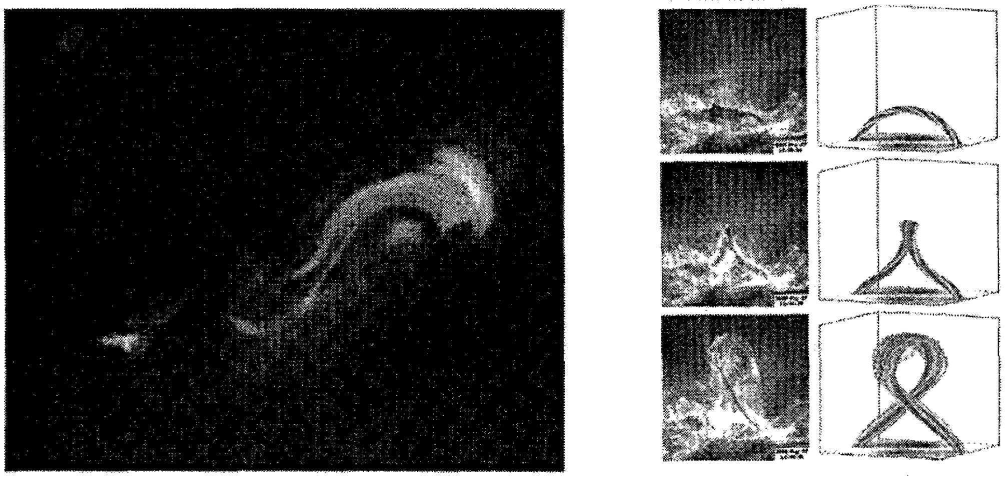

Figure 1: Left: Kinked loops observed by the Hinode X-Ray Telescope. Hot plasma in the solar atmosphere emits x-rays detected by the telescope, which is looking down toward the solar surface. The plasma traces the shape of the magnetic field lines in the atmosphere. Right: a kinked loop seen rising through the corona by the SOHO satellite. The diagrams show a numerical simulation of this event [17]. (Image Credits: JAXA/NASA/SAO)

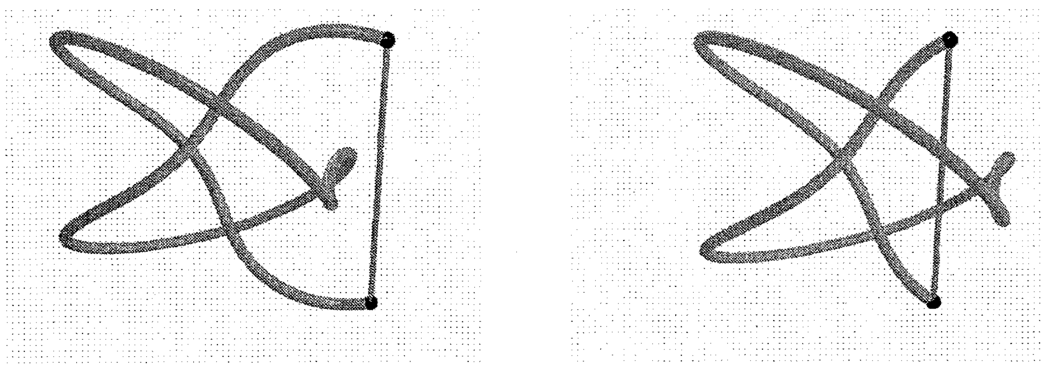

Figure 2: An open curve which has been closed by a straight line. If the endpoints of the curve are moved slightly, then the linkage of the closure with parts of the open curve may change.

Alternatively we can close the open curve by adding a new curve from one end to the other (or by closing the tangent indicatrix curve) [16]. Consider the simplest way to do this: employ a straight line. Figure 2 shows an attempt to do this. Unfortunately, the resulting number is strongly dependent on the positions of the endpoints.

There is, however, another way to proceed. Suppose there is a probability distribution of closures. We can then average the writhe over all closure curves in the distribution (see Fig. 3). This average writhe is remarkably easy to calculate when the distribution follows a vacuum magnetic field — equivalently, a field $\mathbf{B} = \nabla\psi$ where ψ is a harmonic function (satisfying the Laplace equation $\nabla^2\psi = 0$).

We imagine that the open curve is embedded in a thin tube of magnetic flux (i.e. a solenoid). Note that the average linking number between pairs of field lines in a magnetic field is called the magnetic helicity [12, 3, 2]. For a unit flux magnetic tube the helicity equals the twist plus writhe [3, 13], just like Călugăreanu 's theorem for ribbons. If there is no electric current inside the tube, then the average twist of magnetic lines of force about the axis curve of the tube is zero [5]. Similarly pairs of field lines in the vacuum magnetic closure do not twist about each

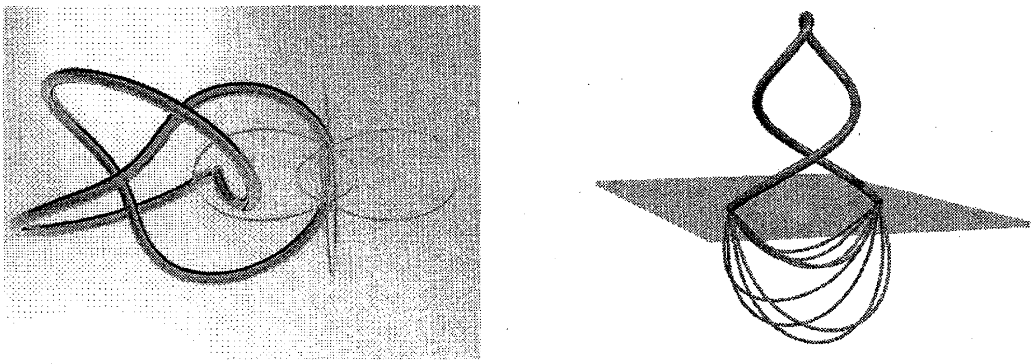

Figure 3: We can calculate the average writhe of a probability distribution of closures. In both figures, a vacuum magnetic field completes a tube of magnetic flux following an open curve. On the right, the magnetic field is modified by the constraint that all closures must remain below the boundary plane.

other. As a consequence, the average writhe of the system equals its magnetic helicity.

We can also apply this technique to curves confined to some region of space, with endpoints on the boundary of the region. For example, some DNA molecules have endpoints fixed to the cell walls. Magnetic flux elements in the solar atmosphere have endpoints fixed at the solar surface, as in Fig. 1. In this case we can calculate the magnetic helicity assuming that the vacuum field does not cross the boundary, as in the right side of Fig. 3.

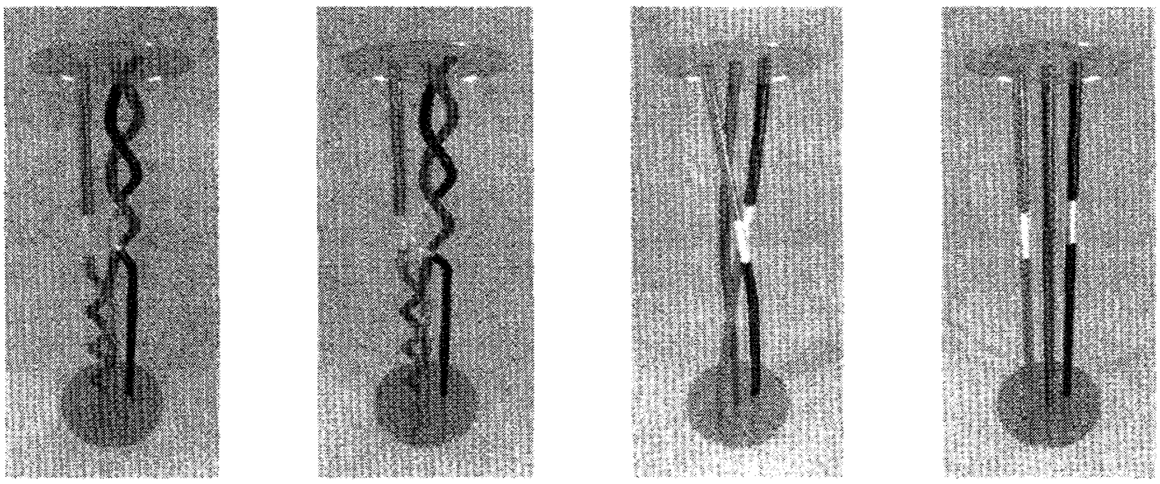

Figure 4: Reconnection can reduce topological complexity and liberate energy. The braid on the left has both positive and negative twists, but not stored on the same pair of curves. A reconnection near where the curves are shown in white puts the positive and negative twists on the same pair (second braid). This allows the braid to relax to a lower energy state (third braid). A second reconnection restores the initial connectivity, resulting in a trivial braid.

2 Self-organized braiding

Magnetic field lines in the solar atmosphere have the shape of loops, with two endpoints rooted at the surface. Fluid motions at the surface can braid the field lines about each other. When a certain amount of topological complexity is reached, the lines can reconnect, thereby releasing magnetic energy as a solar flare [14, 4]. Figure 4 illustrates this process. The distribution of energies of flares obeys a power law [8]. Ongoing work shows how the combination of random endpoint motions and selective reconnection can lead to a self-organized braid structure consistent with power law flares.

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