On Algebraic Knots II

— An upper bound on their ropelength —

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Abstract: We prove that given a Conway algebraic link diagram D with n crossings then D can be embedded on the cubic lattice with a length bounded above by cn, where c is a positive constant independent of D and n. This implies that the ropelength of alternating Conway algebraic knots growths at most linear with their crossing number.

1 Introduction

The definition of an algebraic knot or link is given in Part I of this mini paper series in this volume. For a knot or link $\mathcal{K}$, let $L(\mathcal{K})$ denote the ropelength of $\mathcal{K}$ and let $Cr(\mathcal{K})$ denote the crossing number of $\mathcal{K}$. An important problem in geometric knot theory concerns the relationship between $L(\mathcal{K})$ and $Cr(\mathcal{K})$ (or intuitively, the relationship between the length of a rope needed to tie a particular knot and the complexity of the knot). We show that there exists a constant $a > 0$ such that for any knot $\mathcal{K}$ that allows a Conway algebraic knot diagram D with n crossings $L(\mathcal{K}) \leq a \cdot n$. In general, it is not known which algebraic knots admit minimal crossing diagrams that have the structure of an algebraic diagram. There exist algebraic knots (for example, the Borromean rings) whose minimal diagrams are not algebraic diagrams. If a given knot or link $\mathcal{K}$ has a minimal alternating diagram D, then it can be determined by an algorithm described in [3] whether $\mathcal{K}$ is algebraic. For the case that $\mathcal{K}$ is indeed algebraic, the algorithm produces an algebraic (possibly non minimal) diagram D' for $\mathcal{K}$ and the number of crossings in D' is less or equal to $(4/3)Cr(\mathcal{K})$. Note, that if D is non-alternating then there is no known practical method to determine whether $\mathcal{K}$ is algebraic and for the case that it is, how many crossings an algebraic diagram of $\mathcal{K}$ would have. The above implies that if $\mathcal{K}$ is an alternating algebraic knot, then $L(\mathcal{K}) \leq a \cdot Cr(\mathcal{K})$. It has been shown in [1] that there exist families of (alternating and non-alternating) algebraic knots $\{\mathcal{K}_n\}$ with the property that $Cr(\mathcal{K}_n) \to \infty$ (as $n \to \infty$) such that $L(\mathcal{K}_n)$ grows as fast as $O(Cr(\mathcal{K}_n))$. Thus the ropelength upper bound given in this paper is sharp up to the power of the crossing number.

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2 Algebraic knot diagrams and their binary trees

Let D be an algebraic knot diagram. with n crossings and let C_i for $i = 1, 2, 3 \ldots n-2$ be the Conway circles decomposing it. The union of the algebraic knot diagram D and $C = \cup C_i$ defines a 4-regular graph on S^2 when the usual over- and under-pass information at each crossing of D is ignored.

In the embedding of $D \cup C$ on S^2 there are regions of three types as shown in Figure 1 on the top right. The type I regions (around a crossing) and type III regions (without crossings and touching three Conway circles) are used to define a tree T. A *shading* of $D \cup C$ is a particular coloring of one of each pair of Conway sub-regions of type III, such that type III regions of adjacent Conway circles share a boundary (segment or point) whenever possible. For an example see Figure 1 on the left. For the details of this construction see [2].

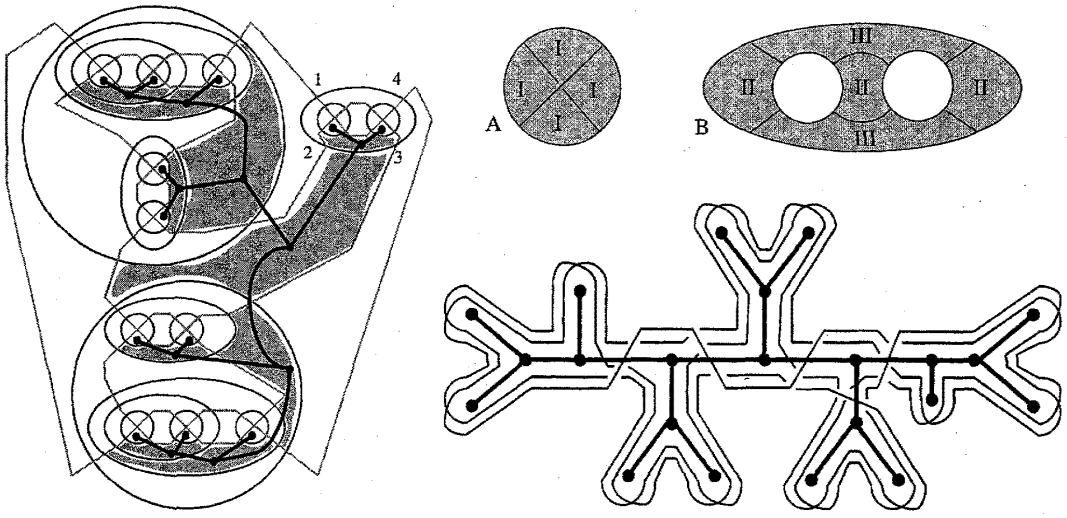

Figure 1: On the left: An algebraic knot diagram D together with a set of Conway circles. Also shown is a shading together with the corresponding tree T. On the right top: Regions bounded by Conway cirles are of these three types. A: type I and B: type II and III. On the right bottom: Deforming this knot so that it is "parallel" to its tree.

A binary tree T is constructed by placing a vertex in each shaded type III Conway sub-region and connecting two such regions if their boundaries intersect. In addition for each crossing a leaf (a vertex and an edge) is attached to the tree by placing the leaf-vertex next to the crossing (within C_i) and by drawing an edge from the leaf to the vertex representing the adjacent shaded region. Notice that if D has n crossings then T has n leafs and $2n-3$ edges. The importance of T lies in the fact that D can be isotoped to a new diagram D' with at most $4n$ crossings that has exactly four strands "parallel" to each edge in the tree as shown in Figure 1 on the right. This is not entirely obvious and we encourage the reader to replicate this isotopy by some drawings. For details see [2]. The above discussion is the basis for the following:

Theorem 2.1 *Let D be an algebraic knot diagram of $\mathcal{K}$ with n crossings, then $\mathcal{K}$ has a diagram D' with at most $4n$ crossings that is "parallel" to a tree T which has less than $2n$ edges and is based on the algebraic knot diagram D.*

The ropelength bound is obtained from embedding D' in the cubic lattice guided by an embedding of T in the plane lattice. For a lattice embedding of a graph G, all vertices must be

at lattice points (points with integer coordinates) and all edges are lattice paths. The embeddings may have intersections of lattice paths, but all such intersections are at lattice points. Once the tree T has a lattice embedding the lattice is expanded by subdividing each lattice square into 12^2 smaller squares. This provides the extent needed to embed the diagram D' onto the now subdivided lattice. Next, the intersections of the plane lattice paths of D' are removed and the over/under information for each crossing is incorporated by creating under- and overpasses in the cubic lattice. This results in the following:

Theorem 2.2 *Let D be an algebraic knot diagram of $\mathcal{K}$ with tree T. There exists a constant $c > 0$ (independent of D and T) such that if $L(T)$ is the length of a lattice embedding of T then the $L(\mathcal{K}) \leq cL(T)$.*

3 Embedding a binary tree in the plane

The arguments given here follow the development given in [4]. Before describing our algorithm embedding a tree into the lattice some preliminary results are provided. Let us denote the number of edges of a tree T by $|T|$.

Lemma 3.1 *[2] Let T be a tree with $n \geq 3$ edges whose maximal degree is less or equal to three, then there exists an edge $e \in T$ such that $T - e$ consists of two trees T_1 and T_2 with $|T_1|/|T| = c$ where $c \in [1/2 - 1/(2n), 2/3] \subseteq [1/3, 2/3]$.*

The above lemma allows for a divide and conquer strategy. The key is how two lattice trees are 'glued back together'. If G is drawn within a rectangle of the form $[0, m-1] \times [0, n-1]$ then we say G has an (m, n)-*embedding*. Consider two binary trees T and T' realized on the lattice as an (m, n)-embedding and an (m', n)-embedding respectively. The two trees can be combined into a new binary tree T'' as an $(m + m' + 1, n + 1)$-embedding as follows: First shift $[0, m'-1] \times [0, n-1]$ (together with T' in it) to the right by m units and then glue it with the right edge to the rectangle $[0, m-1] \times [0, n-1]$. The result is the rectangle $[0, m+m'-1] \times [0, n-1]$ which contains an embedding of T and T'. Next pick any two vertices $v \in T$ and $w \in T'$ with degree(v), degree(w) ≤ 2. Vertically transform v and w by $1/2$ unit to the points v' and w'. If v' is in T, then v' will become a vertex replacing vertex v. If v' is not in T, connect v and v' with a vertical line segment of length $1/2$ and keep the vertex v as a vertex of T. Similarly, connect w and w' with a vertical segment of length $1/2$ if w' is not in T'. Now connect v' to w' with a horizontal straight line segment joining v' and the point v'' on the vertical line $x = m - 1/2$, a horizontal straight line segment joining w' and the point w'' on the vertical line $x = m - 1/2$, and a vertical line segment joining v'' and w'' if needed. Now we expand $[0, m+m'-1] \times [0, n-1]$ to $[0, m+m'] \times [0, n]$ by adding a horizontal row each in $[0, m-1] \times [0, n-1]$ and $[m, m+m'-1] \times [0, n-1]$, and one vertical row such that v' and w' will become lattice vertices and the path connecting v' and w' is on the lattice in the resulting rectangle. This generates a new tree T'' embedded in $[0, m+m'] \times [0, n]$, see Figure 2. This is summarized in the following lemma.

Lemma 3.2 *If two binary trees T and T' have (m, n)- and (m', n)-embeddings respectively (so the total length of the two embedding rectangles in the x-direction is $m + m'$), then a tree T'' obtained by connecting any two vertices $v \in T$ and $v' \in T'$ each with a degree ≤ 2 has an $(m + m' + 1, n + 1)$-embedding.*

The *aspect ratio* of the rectangle $[0, m-1] \times [0, n-1]$ is defined as $r = m/n$. Let $A(k)$ be the minimum over all positive integers p with the property that if the area of a rectangle

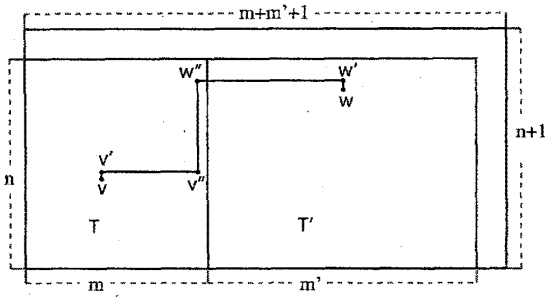

Figure 2: The rectangluar tree embedding for T'' generated by aligning rectangular embeddings of the trees T and T' and connecting one vertex in T to a vertex in T' by a lattice path.

$[0, m-1] \times [0, n-1]$ with aspect ratio between 1 and 3 is equal to or greater than p, then any binary tree T with k edges has an (m, n)-embedding. $A(k)$ can be bounded from above as follows:

Lemma 3.3 *If a function $A_0(k) : \mathbb{Z}^+ \to \mathbb{R}^+$ satisfies the following conditions, then $A(k) \leq A_0(k)$:*
(i) $A_0(3), A_0(4) \geq 6$, $A_0(5) \geq 8$, $A_0(6) \geq 9$, $A_0(7) \geq 10$, $A_0(8) \geq 12$, $A_0(9) \geq 15$;
(ii) $\forall k > 9$ *and* $\forall c$, *where* $1/3 \leq c \leq 2/3$ *and* ck *has an integer value,*
$A_0(ck) \leq c(A_0(k) - 4\sqrt{A_0(k)})$.

In other words, if a rectangle $[0, m-1] \times [0, n-1]$ has aspect ratio between 1 and 3 and $mn \geq A_0(k)$, then any binary tree T with k edges has an (m, n)-embedding.

Theorem 3.1 *The function $F(n)$ defined by $F(n) = 40n - 113\sqrt{n}$ for $n \geq 10$ and $F(n) = A(n)$ for $3 \leq n \leq 9$ satisfies the recursive relationship of Lemma 3.3.*

Theorems 2.2 and 3.1 together now yield the desired linear bound on ropelength. Note that the combined constants obtained are quite large, for example: a Conway algebraic knot $\mathcal{K}$ with a minimal alternating diagram has a ropelength bounded above by $22336Cr(\mathcal{K})$, see [2].

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References

[1] Y. Diao, C. Ernst and M. Thistlethwaite, *The linear growth in the length of a family of thick knots*, J Knot Theory and Ramifications **12**(5) (2003), 709–715.

[2] Y. Diao, C. Ernst and U. Ziegler, *The Linearity of the Ropelengths of Conway Algebraic Knots in Terms of Their Crossing Numbers*, preprint.

[3] M. B. Thistlethwaite, *On the algebraic part of an alternating link*, Pacific J. Math. **151**(2) (1991), 317–333.

[4] L. G. Valiant, *Universality considerations in VSLI circuits*, IEEE Transactions on Computers **30**(2) (1981), 135–140.