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On a complexity of a spatial graph

Osaka City University Akio Kawauchi 1

Abstract: In a research of proteins, molecules, or polymers, it is important to understand geometrically and topologically spatial graphs possibly with degree one vertices including knotted arcs. In this article, we introduce a concept of a complexity and related topological invariants for a spatial graph without degree one vertices, called the \( \gamma \)-warping and warping degrees as well as the \( \gamma \)-unknotting and unknotting numbers generalizing the usual unknotting number of a knot. These invariants define geometric invariants for a spatial graph with degree one vertices, meaningful even for a knotted arc.

1 A spatial graph without degree one vertices and its diagram

For general references of knots, links and spatial graphs, we refer to [3]. First, we consider a compact polygonal graph \( \Gamma \) which does not have any vertices of degrees 0 and 1 and, for simplicity, has at most one component with vertices of degrees \( \geq 3 \). A spatial graph of \( \Gamma \) is a topological embedding image \( G \) of \( \Gamma \) into \( \mathbb{R}^3 \) such that there is an orientation-preserving homeomorphism \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) sending \( G \) to a polygonal graph in \( \mathbb{R}^3 \). We consider a spatial graph \( G \) by ignoring the degree two vertices which are useless in our argument. When \( \Gamma \) is a loop, \( G \) is called a knot, and it is trivial if it is the boundary of a disk in \( \mathbb{R}^3 \). When \( \Gamma \) is the disjoint union of finitely many loops, \( G \) is called a link, and it is trivial if it is the boundary of mutually disjoint disks. A spatial graph \( G \) is equivalent to a spatial graph \( G' \) if there is an orientation-preserving homeomorphism \( h : \mathbb{R}^3 \to \mathbb{R}^3 \) such that \( h(G) = G' \). Let \([G]\) be the class of spatial graphs \( G' \) which are equivalent to \( G \). It is well-known that two spatial graphs \( G \) and \( G' \) are equivalent if and only if any diagram \( D_G \) of \( G \) is deformed into any diagram \( D_{G'} \) of \( G' \) by a finite sequence of the generalized Reidemeister moves, where we call the moves necessary for links the Reidemeister moves (cf. [3]). Let \([D_G]\) be the class of diagrams obtained from a diagram \( D_G \) of \( G \) by the generalized Reidemeister moves, which is identified with the class \([G]\).

2 A monotone diagram and complexity

Our spatial graph \( G \) is obtained from a maximal tree \( T \) (containing all the vertices of degrees \( \geq 3 \) of \( G \)) by adding edges or loops \( \alpha_i \) \((i = 1, 2, \cdots, m)\). Clearly, \( T = \emptyset \) if \( G \) is a link, and \( T \) is meaningful even for a knotted arc.

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1E-mail: kawauchi@sci.osaka-cu.ac.jp
one vertex if \( G \) has just one vertex of degree \( \geq 3 \). Let \( D \) be a diagram of \( G \). The subdiagrams of \( D \) corresponding to \( T \) and \( \alpha_i \) are called the maximal tree diagram \( DT \) and the edge or loop diagram \( D\alpha_i \), respectively. Let \( c_D(DT) \) be the number of crossing points of \( D \) belonging to \( DT \). The diagram \( D \) is a based diagram (on \( T \)) and denoted by \( (D; T) \) if \( c_D(DT) = 0 \). We can deform every diagram into a based diagram by a finite sequence of the generalized Reidemeister moves. Let \( (D; T) \) be a based diagram of \( G \), obtained from \( T \) by adding the edges or loops \( \alpha_i \) \((i = 1, 2, \ldots, m)\). The edge diagram \( D\alpha_i \) is monotone if there is an orientation on \( \alpha_i \) such that a point going along the oriented diagram \( D\alpha_i \) from the origin vertex meets first the upper crossing point at every crossing point (see Figure 1). The loop diagram \( D\alpha_i \) is monotone if there is an orientation on \( \alpha_i \) such that a point going along the oriented diagram \( D\alpha_i \) from a non-crossing point always meets every upper crossing point first. The based diagram \( (D; T) \) on \( T \) is monotone if \( D\alpha_i \) is monotone for every \( i \) and contains the upper crossing point on every crossing point between \( D\alpha_i \) and \( D\alpha_j \) for any \( j > i \) with respect to an oriented ordered sequence of \( D\alpha_i \) \((i = 1, 2, \ldots, m)\). A similar notion of a monotone diagram was used by W. B. R. Lickorish and K. C. Millett in [5] for an oriented ordered link diagram. The warping degree \( d(D; T) \) of a based diagram \( (D; T) \) is the least number of crossing changes on the edge or loop diagrams \( D\alpha_i \) \((i = 1, 2, \ldots, m)\) needed to obtain a monotone diagram from \( D \). For \( T = \emptyset \), we denote \( d(D; T) \) by \( d(D) \). When the edges or loops \( \alpha_i \) \((i = 1, 2, \ldots, m)\) are previously oriented, we can also define the oriented warping degree \( d^+(D; T) \) (or \( d^+(D) \) for \( T = \emptyset \)) of \( D \) by considering only the crossing changes on the oriented edge or loop diagrams \( D\alpha_i \) \((i = 1, 2, \ldots, m)\). For an oriented knot diagram \( D \), A. Shimizu in [7] established the inequality \( d^+(D) + d^+(-D) \leq c(D) - 1 \) with \( c(D) \) the crossing number of \( D \), where the equality holds if and only if \( D \) is an alternating diagram. The complexity of a based diagram \( (D; T) \) is the pair \( cd(D; T) = (c(D; T), d(D; T)) \) together with the dictionary order. This notion was introduced in [4] for an oriented ordered link diagram. A. Shimizu also observed that the dictionary order on \( cd(D; T) \) is equivalent to the numerical order on \( c(D; T)^2 + d(D; T) \) by using the inequality \( d(D; T) \leq c(D; T) \). The complexity \( \gamma(G) \) of \( G \) is the minimum (in the dictionary order) of the complexities \( cd(D; T) \) for all based diagrams \( (D; T) \in [D_G] \). This topological invariant \( \gamma(G) \) is also denoted by \((c_\gamma(G), \delta_\gamma(G))\) where \( c_\gamma(G) \) and \( \delta_\gamma(G) \) are called the \( \gamma \)-crossing number and the \( \gamma \)-warping degree of \( G \), respectively. The minimal crossing number \( c(G) = \min_{D \in [D_G]} c(D) \) of \( G \) has the inequality \( c(G) \leq c_\gamma(G) \). The following properties (1) and (2) on \( G \) gives a reason why we call \( \gamma(G) \) the complexity of \( G \):

(1) \( c_\gamma(G) = 0 \) if and only if \( c(G) = 0 \), i.e., \( G \) is equivalent to a graph in a plane. If \( c_\gamma(G) > 0 \), then there is a spatial graph \( G' \) with \( c_\gamma(G') < c_\gamma(G) \) by a splice on \( G \), so that \( \gamma(G') < \gamma(G) \).

(2) \( \delta_\gamma(G) = 0 \) if and only if \( G \) is equivalent to \( G' \) with a monotone diagram \( (D'; T') \) with \( c(D'; T') = c_\gamma(G) \). If \( \delta_\gamma(G) > 0 \), then by a crossing change on \( G \) there is a spatial graph \( G' \) with \( \gamma(G') < \gamma(G) \).

Figure 2: An unknotted plane graph with a Hopf constituent link.
3 Warping degree and unknotting number

The warping degree \( \delta(G) \) of \( G \) is the minimum of the warping degrees \( \delta(D; T) \) for all based diagrams \( (D; T) \in \mathcal{D}_G \). Then \( \delta(G) \) is a topological invariant and we have \( \delta(G) \leq \delta_r(G) \). A spatial graph \( G \) is unknotted if \( \delta(G) = 0 \), and \( \gamma \)-unknotted if \( \delta_\gamma(G) = 0 \). A link \( L \) is unknotted in this sense if and only if \( L \) is a trivial link, and a spatial plane graph \( G \) is \( \gamma \)-unknotted if and only if \( G \) is equivalent to a graph in a plane. A constituent link of \( G \) is a link contained in \( G \). We note that there is an unknotted plane graph with a non-trivial constituent link. For example, the spatial plane graph \( G \) in Figure 2 has \( \delta(G) = 0 \), but has a Hopf constituent link and \( \delta_\gamma(G) = 1 \).

We also note the Conway-Gordon Theorem in [1]: Every spatial 6-complete graph \( K_6 \) contains a non-trivial constituent link, and every spatial 7-complete graph \( K_7 \) contains a non-trivial constituent knot. Nevertheless, we have the following properties on an unknotted graph: For every graph \( \Gamma \), there are only finitely many unknotted graphs \( G \) of \( \Gamma \) up to equivalences. Further, we have the following properties (1) and (2) on an unknotted graph \( G \): (1) By a sequence of edge reductions illustrated in Figure 3, \( G \) is deformed into a maximal tree. In particular, every edge of \( G \) is contained in a trivial constituent knot. (2) \( G \) is equivalent to a trivial bouquet of circles after some edge contractions. Let \( u(D) \) be the minimal number of crossing changes of a diagram \( D \) needed to obtain a diagram of an unknotted graph. The unknotting number \( \mu(G) \) of \( G \) is the minimum of the numbers \( u(D) \) for all diagrams \( D \in \mathcal{D}_G \). Let \( u_\gamma(D) \) be the minimal number of crossing changes of a diagram \( D \) needed to obtain a diagram of a \( \gamma \)-unknotted graph. The \( \gamma \)-unknotting number \( \mu_\gamma(G) \) of \( G \) is the minimum of the numbers \( u_\gamma(D) \) for all diagrams \( D \in \mathcal{D}_G \). The topological invariants \( \mu(G) \), \( \mu_\gamma(G) \), \( \delta(G) \) and \( \delta_\gamma(G) \) are mutually distinct topological invariants satisfying the following square:

\[
\begin{align*}
\mu_\gamma(G) & \leq \delta_\gamma(G) \\
\mu(G) & \leq \delta(G)
\end{align*}
\]

For example, the spatial graph \( G \) in Figure 2 has \( \mu(G) = \delta(G) = 0 \) and \( \mu_\gamma(G) = \delta_\gamma(G) = 1 \). On the other hand, we see that Kinoshita's \( \theta \)-curve in Figure 4 has \( \mu(G) = \mu_\gamma(G) = 1 < \delta(G) = \delta_\gamma(G) = 2 \) and \( c(G) = 4 < c_\gamma(G) = 7 \). The proof of this assertion is omitted here, but our proof uses H. Moriuchi's classification of algebraic tangles in [6]. Also, we can show the following result by using a technique in [2]: For every graph \( \Gamma \) and any integer \( n \geq 0 \), there are infinitely many spatial graphs \( G \) of \( \Gamma \) such that \( \mu(G) = \mu_\gamma(G) = \delta(G) = \delta_\gamma(G) = n \).

4 A spatial graph with degree one vertices

Let \( \Gamma \) be a finite polygonal graph with degree 1 vertices, for simplicity, which has just one connected component with vertices of degrees \( \geq 3 \). A spatial graph of \( \Gamma \) is a topological embedding image \( G \) of \( \Gamma \) into \( \mathbb{R}^3 \) such that \( h(G) \) is a polygonal graph in \( \mathbb{R}^3 \) for an orientation-preserving

Figure 3: An edge reduction
homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Let $V$ be the set of degree one vertices of $G$. For the line segment $[a, b]$ between $a, b \in \mathbb{R}^3$ and $x \in G$, let $S_v(x) = [v,x] \cup (\bigcup_{v' \in V} [v,v'])$ be a star with origin $v$. Assume that $G_v(x) = G \cup S_v(x)$ is a spatial graph without degree one vertices for every $v \in V$ and $x \in G$. Then the warping degree $\delta(G,x)$ and the unknotting number $\mu(G,x)$ of $(G,x)$ are defined by $\delta(G,x) = \max_{v \in V} \delta(G_v(x))$ and $\mu(G,x) = \max_{v \in V} \mu(G_v(x))$, which are called the warping degree and the unknotting number of $G$ and denoted by $\delta(G)$ and $\mu(G)$, respectively, when $x \in V$. An example is illustrated in Figure 5. In a similar way, the $\gamma$-warping degrees $\delta_{\gamma}(G,x)$, $\delta_{\gamma}(G)$ and the $\gamma$-unknotting numbers $\mu_{\gamma}(G,x)$, $\mu_{\gamma}(G)$ are defined. Different invariants taking the minimum or the average in place of the maximum are also defined.

\begin{align*}
\delta(G) = \mu(G) &= 0 \\
\delta(G) = \mu(G) &= 1
\end{align*}

Figure 5: Knotted arcs

References


