Universal Characteristics of Time-irreversibility in Quantum Normal Diffusion [1]

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abstract: Time-irreversibility of apparently irreversible phenomena observed in quantum systems with a very small number of degrees of freedom is studied quantitatively by a time-reversal test: the system evolved in the forward-time direction is time-reversed after applying a small perturbation at the reversal time and the separation between the time-reversed perturbed and the time-reversed unperturbed states is measured as a function of perturbation strength. If the system exhibits a normal diffusion, the time-reversal characteristics has a universal threshold of perturbation strength above which a complete loss of memory is realized. A remarkable fact is that, for almost all the model systems we examined, including discrete-time/continuous-time systems and deterministic/stochastic systems, the time reversal characteristics as well as the time-reversal dynamics itself converge asymptotically to universal behaviors independent of the details of the systems, which implies the prototype of quantum irreversibility. Only the diffusive behavior of the critical states in Harper equation deviates significantly from the universal behavior.

1 Introduction

In quantum systems which is chaotic in the classical limit, apparently irreversible phenomena such as normal diffusion [1, 2], stationary energy absorption [3] and so on do occur

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even though the number of degrees of freedom is small. Even in quantum systems with no classical limit, normal diffusion can be realized in two- or three-dimensional disordered systems [4, 5, 6].

Irreversiblity was considered as a peculiar behavior of macroscopic systems composed of infinitely many degrees of freedom [7]. On the other hand, at least within the framework of classical dynamics, chaos which exists in non-integrable dynamical systems even with a small number of degrees of freedom, has been considered to be a more generic origin of time irreversibility and dissipation, as was emphasized by Prigogine [8], and the relationship between irreversibility and chaos was the very lifework of Prof.Tasaki [9, 10]. He was thinking that the links among chaos, fractal nature of distribution function, and time irreversibility are essential in classical transport phenomena [10, 11].

Compared with classical dynamics, the origin of time irreversibility in quantum dynamics is not very well understood. There is no trajectory concepts in quantum dynamics, and it is impossible to introduce the concept of chaos on the basis of the instability of trajectories. On the other hand, as mentioned above, there exist apparently irreversible phenomena even in quantum systems with small number of degrees of freedom. Sometimes they have even no classical counterparts. However, very few works have been done on the direct characterization of time irreversibility in apparently irreversible quantum systems [12].

The purpose of this article is to present some attempts to quantitatively characterize the time ireversibility by a simple examination we call the time-reversal test. As will be discussed again in the final section in connection with Tasaki's work, our basic motivation was to explore the relationship between quantum irreversibility and analyticity of quantum wavefuncton. For this purpose, we have to know what is the nature of time irreversibility in apparently irreversible quantum phenomena realized in simple quantum systems. This is the motivation of quantifying the time irreversibility (or time-reversality) of the irreversible quantum dynamics. As an apparently irreversible phenomenon we take the quantum normal diffusion, and we examine the time-reversal test in several quantum systems which have contrastively different characters although the systems commonly exhibit a normal diffusion.

In section 2, we describe the method of time reversal test and introduce what we call time-reversal characteristics. The $8 = 2^3$ simple models are selected based upon 3 criteria, and they are examined by the time-reversal test. In section 3 we discuss the classical counterpart of time-reversal characteristics, and a quantum parameter called the least quantum perturbation unit (LQPU) is introduced. We examine time-reversal test for two typical samples, and show that the quantum time-reversal characteristics deviate drastically from the classical counterpart. It is suggested that, if the characteristics is scaled by LQPU, the time reversal-characteristics of the two models converge to a common limit. In section 4, we examine the time-reversal test for all the 8 models, and the scaled time-reversal characteristics all converges to a universal limit independent of the details of the system except for only one counterexample. Introducing an additional plausible assumption, the universality of time-reversal characteristics can be interpreted as the universality of time-reversed dynamics itself, which is confirmed numerically for all the samples except for the counterexample. The last section is devoted to the summary and perspectives in connection with Tasaki's work.

2 Method and models

2.1 Time-reversal test

The systems we examine here exhibits a normal diffusion either in momentum (p) or in position space (q) with the mean square displacement (MSD),

$$M(t) = \sum_{x} |\psi(x,t))|^2 x^2 - (\sum_{x} |\psi(x,t))|^2 x)^2, \tag{1}$$

increasing linearly in time as $M(t) = Dt$ at the diffusion constant D, where $\psi(x,t) =< x|\psi(t) >$ is the time-dependent wavefunction of the quantum system and x is either q or p. The purpose of this manuscript is to report the result of an extensive numerical examination exploring the time-irreversibility of quantum normal diffusion. The reason why we are interested in normal diffusion is that it seems to be an irreversible quantum phenomenon.

We would like to characterize the instability underlying quantum dynamics in terms of the sensitivity of time-reversed dynamics to the perturbation. Unlike the classical dynamics, there is no trajectory concepts which allows us to define the sensitivity of the motion to the external perturbation. However, if we use the time reversed dynamics which enables the system to return exactly to the initial state, we can measure the sensitivity of the motion as the time reversibility of the system. This kind of fictious time-reversal examination is called *time-reversal test*, which was first proposed in Ref.[13], and used in Ref.[14, 15, 16].

The time-reversal test consists of the following three processes. First the initial point-wise localized state $|x_0 >$, i.e.., $\psi(x,0) =< x|x_0 >= \delta(x - x_0)$ is evolved forward in time by operating the time-evolution operator U^t until the reversal time $t = T$. At $t = T$, a perturbation $\hat{P}(\eta)$, where η denotes the strength of perturbation and $\hat{P}(0) = 1$, is applied, and finally the perturbed state is evolved backward in time by operating the time-reversed evolution operator $U^{-(t-T)}$. Using MSD of the time-reversed state $|\psi(2T) >= U^{-T}\hat{P(\eta)}U^T|x_0 >$, the relative irreversibility

$$\mathcal{R}(\eta) = \frac{|M_\eta(2T) - M_0(2T)|}{M_0(T)}, \tag{2}$$

is defined as a function of the perturbation strength η. Here $M_\eta(t)$ means MSD when the perturbation with the strength η is applied at the reversal time T, and so $M_0(t)$ denotes MSD in the unperturbed case, $\eta = 0$. It is used as a measure of the sensitivity of the quantum dynamics to the external perturbation, and the η-dependence of $\mathcal{R}(\eta)$ is referred as the *time-reversal characteristics* [13, 15].

The fidelity has been frequently used by many works to characterize the sensitivity of quantum systems [17, 18, 19] since the proposal by Peres[20]. However, we use $\mathcal{R}$ as the measure of quantum irreversibility. The reason is that the fidelity measures the memory associated with the phase of wavefunction against the applied perturbation and thus it does not have classical counterpart and does not allow a direct quantum-classical comparison. On the other hand, $\mathcal{R}$ allows an immediate comparison between quantum irreversibility and its classical counterpart, but its disadvantage is that it does not contain direct information about quantum phase.

We here discuss a measurement-theoretic meaning of $\mathcal{R}$. The system starts at any eigenstate $|x_0>$ of the observable $\hat{x}$, and so if there is no perturbation, i.e., $\eta = 0$, the most definite measurement process for the time-reversed wavepacket $|\psi(2T)>= U^{-T}U^{T}|x_0>= |x_0>$ which can measure the associated observable without any fluctuation is the measurement of x itself. So the observation of the probability of the time reversed wavepacket at $t = 2T$ by the coordinate x, namely $|<\psi(2T)|x>|^2$ is the probability of the "best measurement" in the above sense. It may be more legitimate to use the entropy of measurement

$$S(\eta) = \sum_x |<\psi(2T)|x>|^2 \log |<\psi(2T)|x>|^2 \tag{3}$$

where $|\psi(2T)>= U^{-T}\hat{P}_\eta U^{T}|x_0>$, instead of MSD as the parameter characterizing the uncertainty of the "best measurement". It is of course possible to do so, and we can define $\mathcal{R}$ by S_η [16], but we would like to use MSD, which is the most convenient index characterizing the diffusion process, rather than the entropy as the measure of the uncertainty in measurement which is the best without the perturbation.

As the perturbation $\hat{P}$ applied at $t = T$, we can propose typically two sorts of perturbations. One is the "perpendicular-shift", and the other is the "parallel-shift". The former shifts the wavepacket by η in the $y-$space canonically conjugate to the diffusion space x, and the latter shifts the wavepacket by η in the $x-$space:

$$\hat{P}_y(\eta) = \exp\{\eta\partial/\partial y\} = \exp\{\pm i\eta\hat{x}/\hbar\}, \tag{4}$$

$$\hat{P}_x(\eta) = \exp\{\eta\partial/\partial x\} = \exp\{\mp i\eta\hat{y}/\hbar\}, \tag{5}$$

where $+$ and $-$ corresponds to $x = p$ or $x = q$, respectively. This method provides a simple and powerful tool when we measure an instability of quantum dynamics which has no counterpart of classical orbital instability [13].

2.2 Models: a zoo of normally diffusive quantum systems

We examine various kinds of quantum map and continuous-time quantum systems with quite different nature. Only the common feature that all the models have is that they exhibit a normal diffusion under appropriate conditions. As typical examples, we take discrete-time systems (quantum maps) and continuous-time systems with one active degrees of freedom in which a normal diffusion occurs. The quantum maps are described by the following common form of the discrete-time(t)-dependent unitary operator,

$$\hat{U} = \mathrm{e}^{-i\frac{H_0(\hat{p})}{2\hbar}} \mathrm{e}^{-i\frac{V(\hat{q},t)}{\hbar}} \mathrm{e}^{-i\frac{H_0(p)}{2\hbar}}, \quad (6)$$

and the time-continuous model is the continuous-time-dependent unitary operator,

$$\hat{U} = \mathcal{T}\exp\{\frac{-i\int_0^t dt(H_0(\hat{p}) + V(\hat{q},t))}{\hbar}\}, \quad (7)$$

where $\mathcal{T}$ is time-ordering operator. $H_0(\hat{p})$ and $V(\hat{q})$ represent kinetic energy and potential energy, respectively. Here $\hat{p}$ and $\hat{q}$ are momentum and position operators, respectively.

In the classical terms, the system is defined on a cylindrical (x,y) phase space, where x and y form a canonical pair, and so either of them corresponds to q(position) or p(momentum). The diffusion occurs along the direction of the infinitely extended cylinder axis which we define as the x-coordinates, while the periodic boundary condition is imposed for the perpendicular y-coordinate.

We are most interested in diffusion phenomena exhibited by deterministic quantum dynamical systems which are not perturbed by external noise. However, the number of known quantum examples showing deterministic normal diffusion is not so many. Their number increases if we include the stochastic quantum systems, namely, quantum systems driven by the classical noise, into the object of our investigation. In the following we summarize the dynamical properties of all the models we are to examine. They are classified upon the three basic features: being defined on discrete time or on continuous time, being deterministic or stochastic, and having classical counterpart or not.

Discrete time quantum systems (quantum map)

<u>Deterministic systems</u>

- Standard map (SM)

$$H_0 = \frac{p^2}{2}, \quad V(q,t) = K\cos q. \quad (8)$$

Normal diffusion occurs in $p-$space (p is quantized as p =integer$\times\hbar$, and the $q-$space is periodic with the period 2π) in the limit $K \gg 1$ or $\hbar \to 0$, and it has a classical counterpart. The diffusion of SM saturates due to the localization effect, and so we examine the time-reversal test before the saturation sets in.

• Perturbed Anderson map (PAM) [21]

$$H_0(p) = \cos(\frac{p}{\hbar}), \quad V(q,t) = v_q\{1 + \epsilon \sum_{i=1}^{M} \cos\omega_i t\}. \tag{9}$$

Here v_q is random values uniformly distributed over an adequate range. Normal diffusion occurs in $q-$space if the number of incommensurate frequencies M is more than one and ϵ is large enough, where q takes integer values and $p-$space is periodic with the period $2\pi\hbar$. It has no classical counterpart.

Stochastic systems

•Stochastic standard map(SSM) [15]

$$H_0(p) = p^2/2, \quad V(q,t) = \epsilon_n n_t \cos q, \tag{10}$$

where the potential part $\cos q$ is perturbed by uncorrelated noise $< n_t n_{t'} >= \delta_{tt'}$. Normal diffusion occurs in $p-$space irrespective of the magnitude of ϵ_n and it has a classical counterpart. The conditions for q and p are the same as SM.

•Haken-Strobl map(HSM) [15]

$$H_0(p) = \cos(\hat{p}/\hbar), \; V(q,t) = \epsilon_n n_{qt}, \tag{11}$$

where the potential part is generated by spatio-temporal uncorrelated noise $< n_{qt} n_{q't'} >= \delta_{qq'}\delta_{tt'}$. Normal diffusion always occurs in $q-$space and it has no classical counterparts. The Conditions for q and p are the same as PAM.

Continuous time quantum systems

Deterministic systems

• Perturbed continuous Harper equation(PHE) [22]

$$H_0(p) = \cos(\hat{p}), \quad V(q,t) = \epsilon \sum_{i=1}^{M} \cos(\omega_i t)\cos(q). \tag{12}$$

Normal diffusion occurs in $q-$space for sufficiently large $|\epsilon|$. So, the $p-$space is periodic with the period 2π. It has a classical counterpart, which shows a classical chaotic diffusion if $M \geq 2$ and $|\epsilon|$ large enough, but in quantum systems with $M \geq 3$ is necessary for an unlimited diffusion to occur.

•Critical Harper equation(CHE) [23, 24, 25]

$$H_0(p) = 2\cos(\hat{p}/\hbar), \quad V(q,t) = 2\cos(q). \tag{13}$$

The dynamical behavior is "very close to" a normal diffusion in $q-$space [26, 27, 28]. The conditions for q and p are the same as PAM. It has no classical counterparts.

Stochastic systems

• Random pendulum equation(RPE) [22]

$$H_0(p) = p^2/2, \quad V(q,t) = \epsilon_n n_t \cos q. \tag{14}$$

The same as in SSM. The noise is temporally delta correlated: $< n_t n_t' >= \delta(t-t')$

• Haken-Strobl equation(HSE) [29, 30]

$$H_0(p) = cos(\hat{p}/\hbar), \quad V(q) = \epsilon_n n_{qt}. \tag{15}$$

The same as in HSM. The noise is spatio-temporally delta-correlated $< n_{qt} n_{q't'} >= \delta(t-t')\delta_{qq'}$.

SM, SSM, PHE, and RPE have classical counterparts and so it is expected that they mimic their classical behavior in the limit of $\hbar \to 0$. An important fact is that all the classical counterparts of the models are characterized by orbits which are exponentially unstable, and the origin of diffusion in classical counterparts are the exponentially unstable motion. However, PAM, HSM, CHE and HSE do not have classical counterparts, since the transfer operator $i\cos(\hat{p}/\hbar)/2$ has no classical limit.

3 Typical examples: deterministic quantum maps

We first investigate time-reversal characteristics of diffusing quantum state by using deterministic quantum maps SM and PAM, which shows typical well-behaved normal diffusions without stochastic perturbation. In the case of PAM appropriate sample average is taken for different random configurations of v_q. A main result obtained in this section will be examined in all other systems introduced above in the next section.

SM has a classical limit showing a typical exponentially unstable chaotic behavior, while PAM has not. Before going on to quantum time-reversal characteristics, we first consider the classical time-reversal characteristics $\mathcal{R}_{cl}$ by using SM, which is easier to understand than the quantum counterpart. Classical dynamics of quantum map is given by the mapping rule $(q_k, p_k) \to (q_{k+1}, p_{k+1})$, namely,

$$p_{k+1} = p_k - V'(q_k + H_0'(p_k)/2), \quad q_{k+1} = q_k + (H_0'(p_k) + H_0'(p_{k+1}))/2. \tag{16}$$

If the dynamics is integrable, the canonical pair is represented by action-angle variables (θ, I) via a canonical transformation $(q,\ p) = (Q(\theta, I),\ P(\theta, I))$, where Q and P are 2π periodic functions of θ. (Precisely SM is integrable only when $K = 0$, but it is quasi-integrable if $K \ll 1$ and its phase space is almost covered by the KAM tori, which supports the angle representation.) I is conserved and θ changes linearly in time at the constant frequency $\omega(I)$ decided by I, as $\theta_t = \theta_0 + \omega(I)t$. In this case, the only meaningful change due to the perturbation η applied at the reversal time of time-reversal test is a change in action. It changes the frequency $\omega(I)$ in proportion to η and the perturbed and unperturbed orbits makes a linear difference at $t = 2T$. Thus the time-reversal characteristics are

$$\mathcal{R}_{cl} = \frac{|M_\eta(2T) - M_0(2T)|}{M_0(T)} \sim \eta T. \tag{17}$$

Therefore, we can control the accuracy of the system's return to the initial state by controlling the magnitude of the perturbation strength as

$$\eta \sim 1/T. \tag{18}$$

On the other hand, in the case of chaotic motions, the deviation of the perturbed orbit following the unperturbed one grows exponentially as $d(\tau) \sim \eta e^{\lambda\tau}$, where $\tau \equiv t - T$ for $t > T$ and λ is the Lyapunov exponent, up to the time τ_d. τ_d is defined as time when $d(\tau)$ grows up to $O(1)$, namely $d(\tau_d) \sim O(1)(= C)$, which is referred to as the *delay-time* hereafter. It means the time required for the loss of memory in the dynamics, beyond which diffusion motion is recovered in backward time-evolution at the same diffusion constant as the forward time-evolution. After the reversal-time the difference increases like $M_\eta(t) - M_0(T) = D(\tau - \tau_d)$ for $\tau > \tau_d$. Consequently,

$$\mathcal{R}_{cl} \sim 2 - \frac{\tau_d(\eta)}{T} = 2 - \frac{\log C}{\lambda T} \tag{19}$$

where $\tau_d(\eta) = \frac{\log(C/\eta)}{\lambda}$. Thus we have to keep η exponentially as small as

$$\eta \sim C e^{-\lambda T}, \tag{20}$$

if we would like to control the system to recover the time-reversibility.

Figs.1(a) and (b) compare the time-reversed dynamics of classical and quantum SM in the chaotic regime. One can easily recognize that, after following the complete time-reversed dynamics for the the period of the delay time, a switch to the forward diffusion occur. It is found that the delay time of the time-reversed dynamics increases as $\tau_d(\eta) \propto \log \eta$ when the perturbation strength η decreases geometrically. In contrast to the classical dynamics, as is seen in Fig.1(b), the restoration of diffusion suddenly terminates as the perturbation strength become too small. It seems that a certain threshold η_{th} below

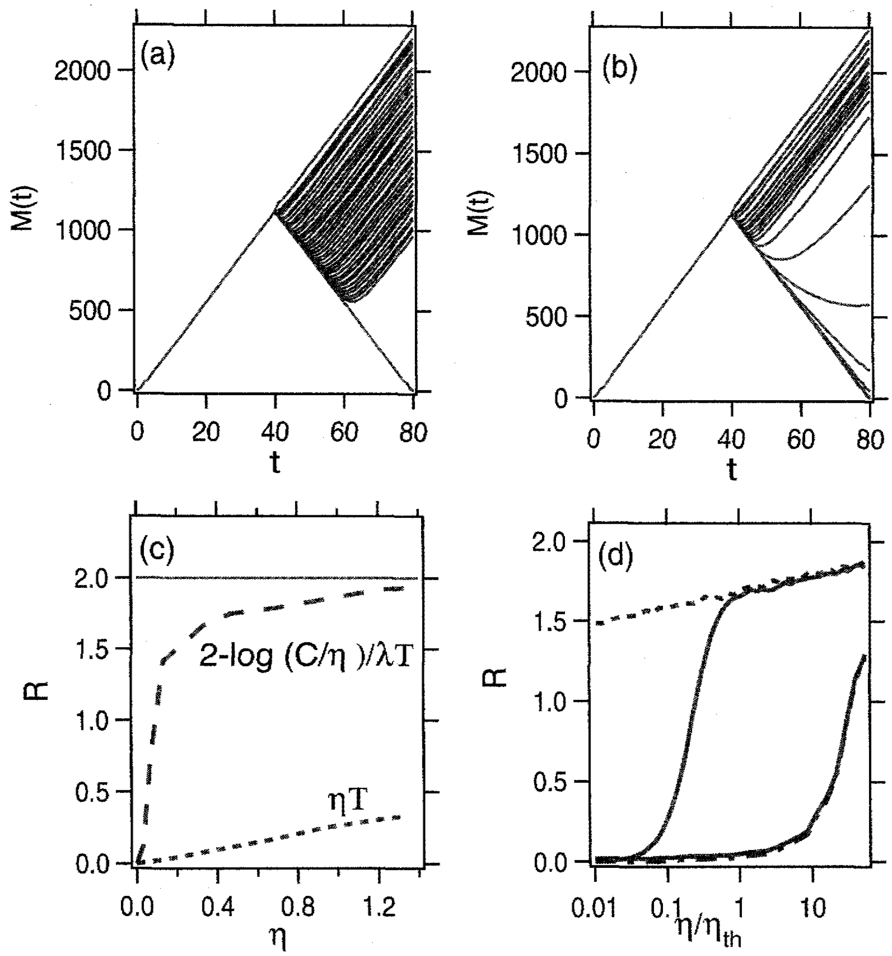

図 1: Typical time-reversal tests for the normal diffusion in (a)classical and (b)quantum SM with $K = 6$, $\hbar = \frac{2\pi 121}{2^{17}}$, where η is decreasing geometrically, and the delay time increases linearly in (a). (c) Schematic illustration of time-reversal characteristics $\mathcal{R}_{cl}$ as a function of the perturbation strength for some typical cases in classical dynamics. See the main text for the explanation of the function form. (d) Typical time-reversal characteristics for quantum (red full) and classical(blue broken) SM with normal diffusion obtained by time-reversal tests. The bottom curve denotes the typical result for the quasi-integrable quantum and classical SM. Note that the horizontal axis is in logarithmic scale in arbitrary unit.

which the quantum reversed motion loses the irreversible feature and restores the time-reversibility.

To demonstrate the quantum feature of time-reversal characteristics we compare in Fig.2 the classical and quantum time-reversal characteristics of SM measured at increasing T. The perturbation strength η is scaled by a fundamental unit η_{th}, which will be discussed later. The classical characteristics clearly follows the classical $\log(C/\eta)/(\lambda T)$ dependence of Eq.(19) in a relatively large regime of η, but a striking difference between quantum and classical time-reversibility is seen in the low η-regime. Below η_{th} the relative irreversibility approaches zero very rapidly. The presence of the threshold is a direct manifestation of quantum uncertainly in the quantum time-irreversibility.

We suppose that the wavepacket diffuses to cover the range of x with width $x(T) = M(T)$ at the reversal time. Then the perpendicular perturbation shifting the quantum state in the y space by η sweeps the phase space area $A = \eta\sqrt{M(T)}$. The shifted quantum state is classically distinguishable from the original state, if η is large enough such that the sweeped area contains more than one quantum state, namely $A/h > 1$, which defines the least quantum perturbation unit (LQPU),

$$\eta_{th} = \frac{2\pi\hbar}{\Delta X} = \frac{2\pi\hbar}{\sqrt{M(T)}} \tag{21}$$

as the threshold perturbation strength. If $\eta > \eta_{th}$, the orbit from the shifted state separates from the orbit from the original state in the classical mechanical way. the curves $\mathcal{R}$ versus the scaled perturbation η/η_{th} measured at different T shows a strong tendency of convergence

In Fig.2, $\mathcal{R}$ is displayed as functions of η scaled by the LQPU, and different curves measured at different Ts, which should be characterized by different $\eta_{th}(T)$ given by Eq.(21), are well ordered and indicate a strong tendency to converge to a limit. We refer to the regions $\eta/\eta_{th} < 1$ and $\eta/\eta_{th} > 1$ as *quantum region* and *post quantum region*, respectively. The convergence occurs in different manner in these two regions, and in Fig.2 one can see that the classical convergence according to Eq.(19) certainly occurs in the post quantum region.

In the case of the parallel perturbation shifting the quantum state in the $x(= p)$ space by η, the wavepacket fills the full domain of definition of $y(= q)$. Recall that it is defined by $0 \leq q \leq 2\pi$. Then the sweep area $A = 2\pi\eta$, and the threshold perturbation strength is given by

$$\eta_{th} = \hbar. \tag{22}$$

An another explanation for the presence of LQPU is based upon a semiclassical theory. Semiclassical wavepacket is constructed by considering all the trajectories which starts from the initial x_0 and return to the final x_0. Every trajectory contribute with the phase decided by the action S along the trajectory over the Planck constant $\hbar$, namely, $S/\hbar$. The condition that the time-reversed wavepacket behaves as if classical ensemble of trajectories is that the phase difference between the perturbed and unperturbed trajectories exceeds 2π and no quantum interference takes place. There is no space to give the detailed theory here, but the condition results in the same threshold given above.

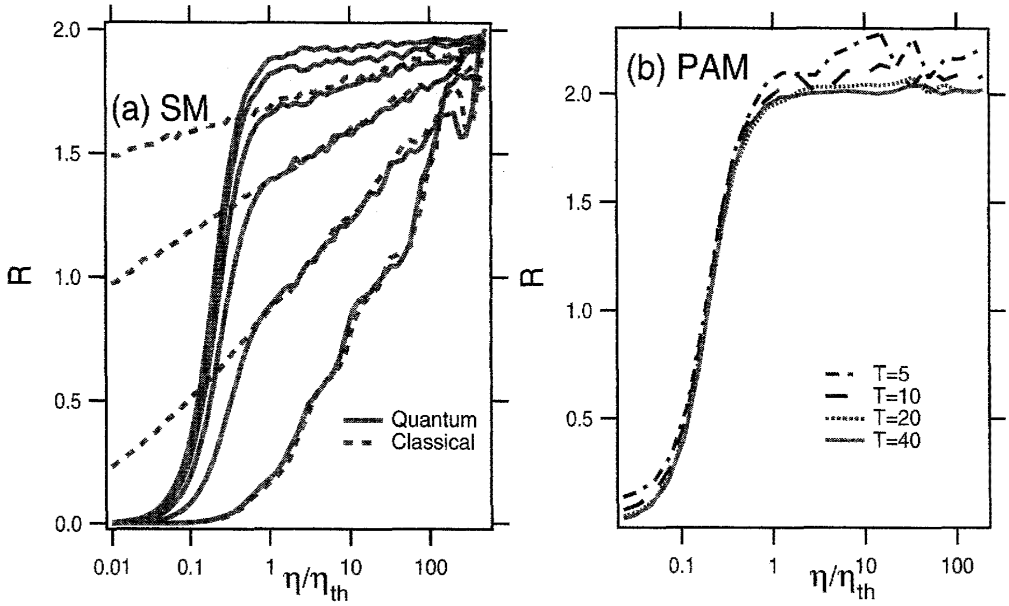

図 2: Time-reversal characteristics of (a) quantum (solid) and classical(broken) SM and (b) PAM, as functions of the scaled perturbation strength η/η_{th} at $T = 5, 10, 20, 40$ (in order from below in (a) and as marked in (b)). $K = 6$, $\hbar = \frac{2\pi 121}{2^{17}}$ for SM and $W = 1.0, M = 3, \epsilon = 0.5$ for PAM. In (a) the classical time-reversal characteristics agrees with Eq.(19).

Figure 3(a) shows $\mathcal{R}$ measured at various T of SM. In the post quantum region, $\eta > \eta_{th}$, $\mathcal{R}$ approaches the common line $\mathcal{R} = 2$ in proportion to $1/T$ slowly. Also in the quantum region $\eta < \eta_{th}$, the results suggest that $\mathcal{R}$ as a function of η/η_{th} converges to a limit following the convergence in the post quantum region. Figure 3(a) shows the asymptotic limit of $\mathcal{R}$ measured for various value of the nonlinear parameter K. All the curves seems to ride on a common curve insensitive to of K. These facts strongly suggest that in SM the time-reversal characteristics represented by the scaled η becomes the same common curve in the limit of $T \to \infty$. We also numerically confirmed the curves of $\mathcal{R}$ vs η/η_{th} with various different periodic potential $V(q)$ of SM and confirmed that all curves coincide in the large limit of T if the normal diffusion is maintained in the range $0 < t < 2T$. We hereafter use a well-converged curve of $\mathcal{R}$ vs η/η_{th} of SM as the *reference curve* to compare with $\mathcal{R}$ of other systems.

Next, we investigate the time-reversal characteristics of PAM. It has no classical limit

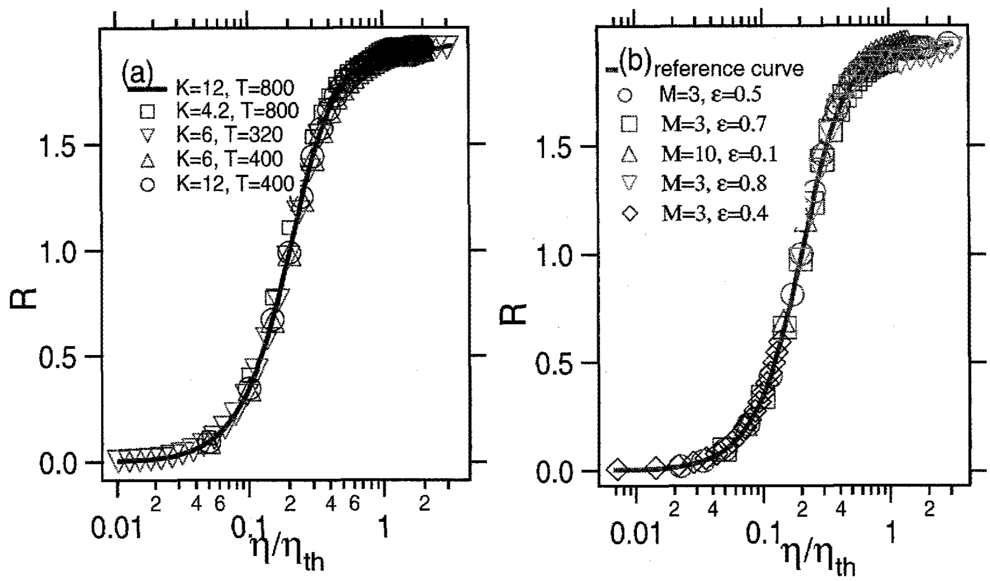

図 3: Time-reversal characteristics of (a)SM and (b) PAM as a function of the scaled perturbation strength with some different parameter sets. Hereafter we quote the SM data of $K = 12, T = 800$ as a reference curve. For PAM, T is fixed at $T = 200$.

and is thus free from chaotic instability. The physical origin of the normal diffusion in PAM is not chaotic instability like SM but destruction of Anderson localization by harmonic perturbations. The diffusion occurs in the position space (namely $x = q$) quite differently from SM.

As seen in Fig.3(b) the time-reversal characteristics of PAM also have the asymptotic limit for $T \to \infty$. The asymptotically converged $\mathcal{R}$ for some typical parameter sets in PAM that generate normal diffusion is given in Fig.3(b). The plots of time-reversal characteristics coincides very well with the reference curve. Recalling the basic difference of physical mechanisms causing the normal diffusion in the two systems, such a coincidence is very surprising. Moreover it follows that in the post quantum region the time-reversal characteristics is accompanied by erratic fluctuation as is the case in the post quantum region of SM.

However, the way of convergence with increase in T depends on the details of the system. This is related to the short time scale dynamics discarded in the above universality argument.

As seen in Fig.3(a) the time-reversal characteristics of SM slowly approaches the universal curve from below according to the classical $1/T$ rule in the post quantum region, reflecting the exponential chaotic instability. On the other hand, the convergence to the asymptotic curve is very rapid in the case of PAM, but one can recognize that the characteristics curve converges to the common limit from the above. In the very short time scale such that $\tau = t - T \ll T$, the deviation,

$$\Delta M_\eta(T,\tau) = M_\eta(T+\tau) - M_0(T+\tau), \tag{23}$$

due to the perturbation, grows much more rapidly than that of SM, although the growth seems to be algebraic rather than exponential. The rapid growth of $\Delta M_\eta(\tau)$ indicates that PAM is excessively sensitive than SM to the perturbation, and thus the convergence to the common curve occurs from the above i.e., the excessive side, in contrast to SM [22]. The way of convergence to the common curve thus reflects the details of the sensitivity to the external perturbation.

In the two examples discussed above, the value $\mathcal{R} = 2$ means that the time-reversed dynamics follows the normal diffusion at $t = 2T$, which can not be distinguished from the forward diffusion without the time-reversal operation, and therefore it means a complete loss of memory. The time-reversal characteristics of the above discussed two typical examples means that without the control at the level of LQPU the system completely lose their past memory. This quantum level of control for recovering the initial state seems to be the most severe request. The fact that the characteristics of the two examples completely coincide seems to imply that the common characteristics is the ultimate universal one irrespective of the detail of systems. In the next section we examine our hypothesis by other normally diffusive quantum systems.

4 The universality

In this section we extend our investigation of time-reversal test to more wide class of systems. First, we examine examples of time-discrete stochastic systems having classical counterpart (SSM) and no classical counterpart(HSM), and next we study systems examplifying deterministic continuous-time systems with classical counterpart(PHE) and no classical counterpart(CHE) and stochastic continuous-time systems with classical counterpart(RPE) and no classical counterpart (HSE). In case of stochastic systems, a sample average over noise process is necessary to reduce fluctuation around mean diffusive motion and to obtain a definite result at finite Ts.

4.1 Convergence to a common characteristics

Here we have included stochastic models as the objects of our time-irreversibility study. Then one can easily construct exactly solvable stochastic quantum systems which shows a rigorous normal diffusion. A simple example is obtained by replacing the quadratic kinetic energy $T(p) = p^2/2$ of SSM by a linear photonic energy ωp. Then in the mapping rule (19) the classical variables (q,p) can be looked upon as operators $(\hat{q},\hat{p})$, and in this particular model the solution can easily be obtained for the forward evolution:

$$\hat{p}_t = \hat{p}_0 + \sum_i^t \epsilon_n n_i \sin\{(i-1/2)\omega + \theta_0\}. \tag{24}$$

Taking both quantum mechanical average and the ensemble average, the mean MSD exactly obeys the diffusion law

$$M(t) =< (\hat{p}_t - < \hat{p}_t >)^2 >=< \epsilon_n^2 > t/2. \tag{25}$$

After applying the perpendicular perturbation, the dynamics is time-reversed, then the $\mathcal{R}$ can be obtained analytically as

$$\mathcal{R} = 4\sin^2\frac{\eta}{2}. \tag{26}$$

If we compare the above integrable result with the $\mathcal{R}$ characteristics of SM and PAM discussed in the previous section, we can recognize again how the latter is anomalous compared with the former integrable case. The former contain no microscopic parameter $\hbar$, and the irreversibility changes slowly from the maximum 4 to minimum 0 and has no particular scale. This shows the most typical feature of the time-reversal characteristics exhibited by integrable normal diffusion. In this case, the value 2, which indicate the complete loss of memory, has no particular significance, while in the latter case $\mathcal{R}$ does never drops to 0 from the value 2 unless the perturbation strength is reduced less than the small microscopic quantum parameter $\eta_{th} \propto \hbar$. It, of course, manifests the difficulty of recovering the initial memory in intrinsically unstable quantum normal diffusion. However, as will be shown below, except for such an exactly solvable case, even the stochastic model which at first sight seems not to have a mechanism of instability shows the same time-reversible characteristics as the deterministic systems demonstrated in Sect.2.

We return to the SSM by restoring the quadratic kinetic energy $T(p) = p^2/2$, and consider its classical behavior. The trajectory of nonlinear classical system whose parameters are modulated by stochastic variables in general shows unstable behavior, which can be understood easily in our example. (See, for example, Ref.[31].) It is more convenient to take the unitary evolution operator $\hat{U}_t' = \mathrm{e}^{-iV(q,t)/\hbar}\mathrm{e}^{-iH_0(p)/\hbar}$, which is a simple unitary transformation of the original evolution operator as $\hat{U}_t' = \mathrm{e}^{iH_0(p)/2\hbar}\hat{U}_t\mathrm{e}^{-iH_0(p)/2\hbar}$, instead of $\hat{U}_t$. The corresponding classical map connecting the steps t and $t+1$ is then given by

$$q_{t+1} - q_t = p_t, \qquad p_{t+1} - p_t = \epsilon_n n_{t+1} \sin q_{t+1}. \tag{27}$$

Let the small deviation from the classical trajectory be δq_t and $\delta p_t = \delta q_{t+1} - \delta q_t$, then it satisfies the linearlized equation of motion

$$\delta q_{t+1} + \delta q_{t-1} - 2(1 + \epsilon_n n_t \cos q_t)\delta q_t = 0. \tag{28}$$

Since n_t is random variable, $2n_t\cos(q_t)$ is also a random variable. Eq.(28) can thus be regarded as a multiplicative random process and $\delta q(t)$ should follow an exponential instability, namely

$$|\delta q_t| \propto \mathrm{e}^{\lambda t}, \tag{29}$$

where λ is a certain positive exponent, due to Furstenberg's theorem for nonlinear stochastic systems[31, 32].

An advantage of this model is that Lyapunov exponent λ and the diffusion constant can be made arbitrarily small by reducing ϵ_n because $\lambda \propto \log \epsilon_n^2$. In quantum model, we have numerically confirmed that normal diffusion is realized for arbitrarily small ϵ_n and there is no saturation of the normal diffusion seen in non-stochastic standard map.

We executed the time reversal test for the quantum SSM system, and confirmed that the time-reversal characteristics converges to to the reference curve of Fig.2 following the classical convergent rule (19) in the post quantum region. We have further confirmed that the classical convergence toward the reference curve is observed up to the smallest ϵ_n for which the time-reversal test is executable. We could not practically find any threshold of ϵ_n below which the normal diffusion terminates, and a nice convergence to the reference characteristics is observed as long as the normal diffusion is maintained. This fact implies that the the classical convergence toward the common time reversal characteristics also occurs for the continuous-time random pendulum equation (RPE). Indeed, by setting

$$\hbar/\sqrt{\epsilon} \to \hbar_{eff}, \qquad \delta t \to \sqrt{\epsilon_n}, \tag{30}$$

the discrete-time operator can be well approximated by the continuous time operator for $\delta t-$step as

$$\hat{U}(\delta t) = \mathrm{e}^{-i\delta t\hat{p}^2/4\hbar_{eff}}\mathrm{e}^{-i\delta t V(q,t)/\hbar_{eff}}\mathrm{e}^{-i\delta t\hat{p}^2/4\hbar_{eff}} = \exp\{-i\delta t\frac{\hat{p}^2/2+V(q,t)}{\hbar_{eff}}\} + O(\delta t^3), \tag{31}$$

and the effective Hamiltonian

$$H_{eff} = \hat{p}^2/2 + V(q,t) = \hat{p}^2/2 + n_t\cos q, \tag{32}$$

is nothing more than the random pendulum Hamiltonian.

The similar relationship also holds between Haken-Strobl map (HSM) and Haken-Strobl equation (HSE). They both are fully quantum systems and do not have the classical counterpart, and we have no classical basis for predicting the time-reversal characteristics. We examined the HSM reducing the parameter ϵ_n as small as the time-reversal test executable. We observed that the time reversal characteristics converges to the common curve from below as T increases. Similarly to the case of SSM, we could not find any indication that decrease in the potential strength suppresses the normal diffusion and/or the convergence to the reference characteristics. Thus we conclude that the convergence to the common curve occurs on the continuous-time version of HSM as well in HSE. But we have to stress that the smaller the potential strength ϵ_n reduces the longer the time scale of the convergence to the universal characteristics becomes.

Finally we go to the continuous-time deterministic system which we do not still discuss. Two examples, the perturbed continuous Harper equation (PHE) having a classical counterpart and the critical Harper equation (CHE) having no classical counterpart are given. Since the CHE gives a definite counterexample to the universal characteristics, it is discussed later, and we consider first the PHE.

Origin of normal diffusion of PHE can be explained in the classical limit. It is due to the formation of separatrix chaos at the nonlinear resonances at p satisfying $dH_0(p)/dp = \sin p = \omega_i$ due to the interaction between them. In particular, for sufficiently large ϵ, overlap of resonances removes dynamical inhomogeneity of the mixed phase space composed of, roughly speaking, ballistic component and chaotic diffusive component, and enables a uniform diffusion in $q - space$. We examined its quantum version, and confirmed the occurrence of normal diffusion. To diminish fluctuation, we take an average over the phase of driving force, which corresponds to supposing that the driving source is prepared in a eigenstate of the action variable conjugate to the phase (or angle) variable. The time-reversal characteristics converges following nicely the classical $1/T$ rule (19) in the post quantum regime, and finally converges to the reference curve. Thus we conclude that, besides SM and PAM discussed in the previous section, in SSM, HSM, RPE, HSE, and PHE, time-reversal characteristics converges to a common curve, when they exhibit a well-behaved normal diffusion. We show in Fig.4 the summarized result. We here call the common limit as the *universal curve*. All the above results are for the perturbation of the perpendicular shift, where the LQPU depends on the reversal time T. We examined the perturbation of parallel shift for all models. In this case the LQPU is essentially the Planck constant itself, and the functional form of the asymptotic time-reversal characteristics is slightly different from the perpendicular shift. It is interesting to note that the universal curve has a small oscillation at the border of quantum and post quantum regions. We confirmed for all the above 7 models that the time-reversal characteristics converges to a universal curve, as is shown in Fig.5.

4.2 A counterexample: the critical Harper diffusion

However, a very interesting fact is that the time-reversal characteristics of critical Harper equation (CHE) deviates significantly from the universal behavior. This is the only counter example we encountered in our examination. There have been some controversies on the normality of diffusion of CHE, and some authors claim that the MSD deviates from the normal diffusion as $M(t) \propto t^{0.97}$ [27, 28] ; on the other hand, other authors claims it exactly follow the $M(t) \propto t$ law [26]. According to our limited sim-

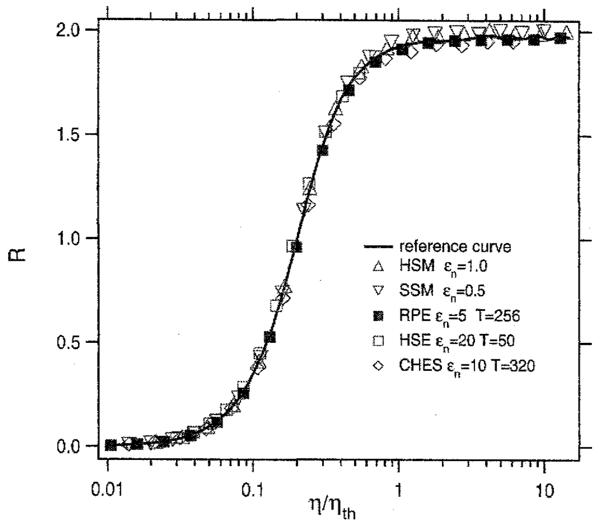

図 4: Time-reversal characteristics of all the models examined. Case of the perpendicular shift perturbation. All curves are those obtained as the converged curves in a large limit of T. CHES denotes a result for CHE with stochastic perturbation. The bold line is the reference curve.

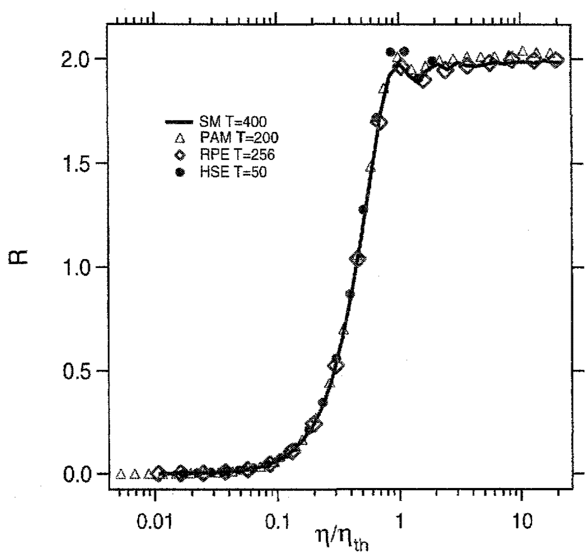

図 5: Time-reversal characteristics of several models examined. Case of parallel shift perturbation. All curves are those obtained as the converged curves in a large limit of T. The bold line is the reference curve.

ulation it seems that the clear deviation from the $M(t) \propto t$ law is undetectable. The difference from the normal diffusion seems to be, if any, very delicate. However, seen from the time-reversal test, the characteristics of CHE converges to a different characteristics deviating significantly from the universal characteristics. Fig.6 shows the convergence of time-reversal characteristics of CHE at increasing T. It seems that the time-reversal characteristics converges to a limit, but it shifts significantly upward from the universal curve. In particular $\mathcal{R}$ exceeds from the value 2, which mean complete loss of initial memory, even in the post quantum region. This is manifestation that the critical diffusion of CHE does not achieve a complete loss of memory.

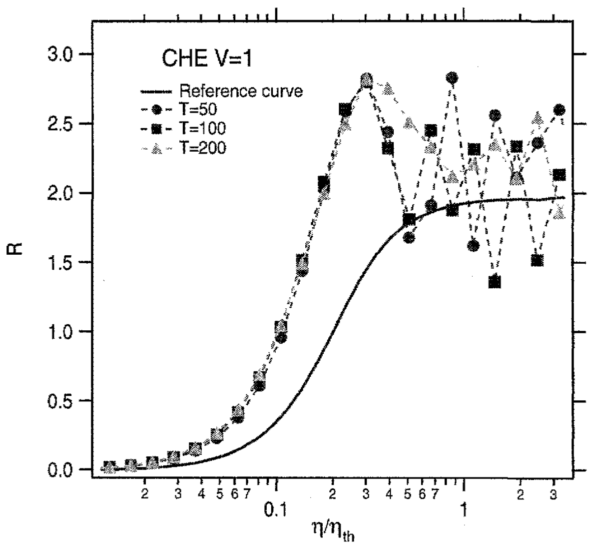

図 6: Time-reversal characteristics of critical Harper equation. It converges to a curve in the quantum region, while it is accompanied by a wild fluctuation in the post quantum region. In both regions, however, the characteristics is shifted significantly above the universal curve. $T = 50, 100, 200$ are displayed.

Another interesting fact is that the time-reversal characteristics of CHE recovers the universal characteristics by applying such a weak noise that does not significantly change the diffusion rate [22]. A result of the CHE with stochastic perturbation is plotted in Fig.4. This fact implies an application of weak noise destroys the long time scale memory without changing the short time scale memory which dominates the diffusion rate. We guess that the noise with arbitrary weak strength works as the source which remove the long term memory and the universal characteristic is recovered on a long time scale depending upon the noise strength.

4.3 Universality in time-reversed dynamics

The universality of quantum normal diffusion discussed above is not only a feature of the irreversibility defined at the returning time $t = 2T$. It is a general property of the time-reversed dynamics itself. To demonstrate this, we consider the separation $\Delta M_\eta(T,\tau) = M_\eta(\tau+T) - M_0(\tau+T)$ between the perturbed time-reversed process from the unperturbed one defined in Eq.(23). In Fig.7, the scaled separation $\Delta M_\eta(T,\tau)/M_0(T)$ is shown as a function of the scaled time τ/T for SM and PAM. Evidently, they are on a common curve independent of the system in the asymptotic limit $T \to \infty$ if the scaled perturbation strength η/η_{th} is the same.

Such a universal behavior can be explained, if we admit a further assumption in addition to the universality of the time-reversal characteristics $\mathcal{R}$ discussed above, which can be written as

$$\mathcal{R} = F(\frac{\eta}{\eta_{th}(T)}), \tag{33}$$

by introducing the universal function F. The further assumption we do is the stationarity of the time-reversed dynamics, which means that for the same η the difference ΔM_η does not depend on the reversal time T, i.e.,

$$\Delta M_\eta(T,\tau) = G(\eta,\tau), \tag{34}$$

where G is a function depending only on η and τ. Indeed, extended numerical examination supports the validity this hypothesis. Equations (33) and (34) claim that $G(\eta,\tau) = D\tau F(\frac{\eta}{\eta_{th}(\tau)})$, which is immediately followed by the relation

$$\mathcal{R} \;=\; \frac{\tau}{T}F(\frac{\eta}{\eta_{th}(T)}\{\frac{\tau}{T}\}^{\chi}), \tag{35}$$

where $\eta_{th}(T) \propto T^{-\chi}$. The index χ is determined by the type of perturbation as, $\chi = 1/2$ for perpendicular η–shift and $\chi = 0$ for parallel η–shift. Thus, $\Delta M_\eta(T,\tau)/M_0(T)$ is determined only by the scaled perturbation strength η/η_{th} and the scaled time τ/T for the reversal time.

In particular, in the case of the parallel shift the time-dependent term in Eq.(35) comes only from the term τ/T and the time-reversed MSD should all form straight lines as a function of $\tau = t - T$. This is indeed confirmed by choosing arbitrary systems from 7 models discussed above, as is shown in Fig.8. The time-reversed curves spreads forming straight lines at various angles like ribs of Japanese fan.

The above result is a natural consequence of the stationarity and the universal scaling of the time-reversed dynamics. The stationarity means that ΔM does depends only on the passed time $\tau = t - T$ from the time T at which time-reversal operation is done and does not depends on T. This means that it does not depends on the past history

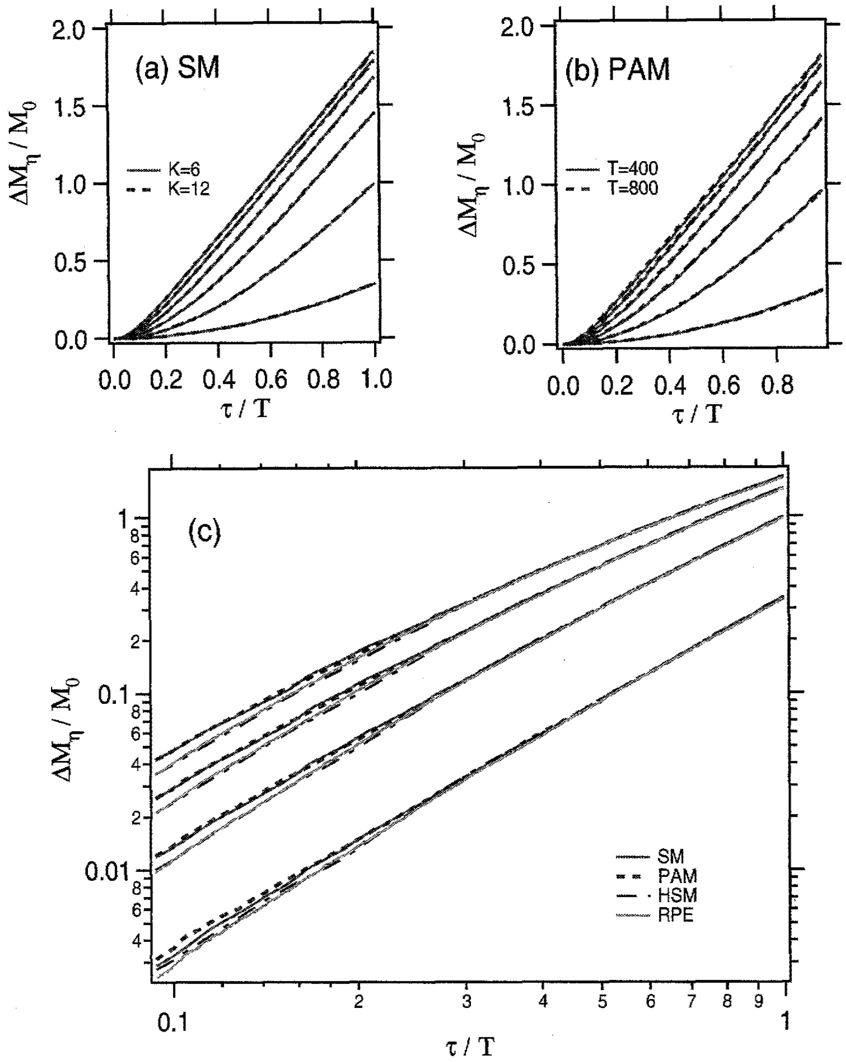

図 7: Scaled separation $(M_\eta(T+\tau)-M_0(T+\tau))/M_0(T)$ as a function of scaled time τ/T for several scaled perturbation strength $\eta/\eta_{th}=0.1, 0.2, 0.3, 0.4, 0.5, 0.6$, respectively, from below. (a)SM with $K=6, 12$ and $\hbar=\frac{2\pi 487}{2^{19}}, \frac{2\pi 975}{2^{20}}$, respectively, at $T=400$. (b)PAM with $M=3, \epsilon=0.5$ at $T=400, 800$. (c)Log-log plots of the data for SM and PAM at $T=400$, HSM($\epsilon_n=20$) at $T=50$ and RPE($\epsilon_n=5$) at $T=256$. The cases $\eta/\eta_{th}=0.1, 0.2, 0.3, 0.4$ from below, respectively, are plotted for each model.

up to T. On the other hand, the scaling property of time-reversality, suggests that the memory from $t=0$ to T is maintained during the time evolved process even for large T is taken. The two apparently contradictory features are unified to yield the universality of the time-reversed dynamics.

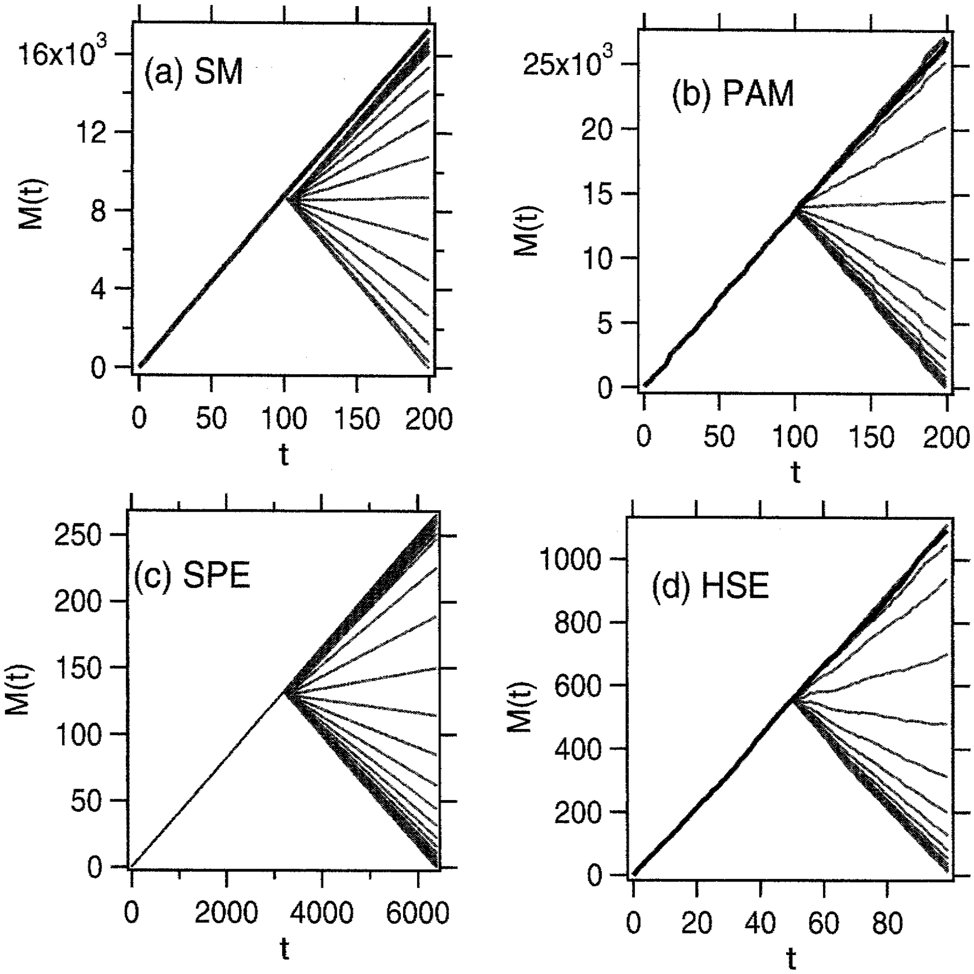

図 8: Time-reversed dynamics of several systems when parallel shift is applied as the perturbation. All the time-reversed curve follows straight lines with different angles. (a) SM, (b)PAM, and (c)RPE, and (d)HSE.

5 Discussions and perspectives

In the present paper, we demonstrated our attempt to measure quantitatively the irreversibility of apparently irreversible quantum phenomena by using a simple examination we called time-reversal test. As a typical example of irreversible phenomena, the normal quantum diffusion is taken, and various kinds of quantum systems exhibiting normal diffusion which are either deterministic or stochastic, either a discrete-time system or a continuous-time systems, and either the system having classical counterpart or no classical counterpart, are examined. As a result, almost all the quantum systems, except for trivially solvable models, shows universal time-reversal characteristics.

We are thinking that the universal time-reversal characteristics demonstrated here represents a prototype of quantum irreversibility: in order that the initial quantum state is recovered by the time-reversed dynamics, the perturbation (or disturbance) applied at the reversal time is required to be less than a microscopic quantum parameter called LQPU, and the strength of perturbation η exceeds the LQPU leads to a complete loss of memory. The characteristic curve which measures the deviation from the exactly time-reversed state as the function of the perturbation strength converges to a universal curve in the long limit of the reversal time. The only nontrivial counterexample that do not follow the universal curve is the critical Harper equation (CHE).

In addition to the universality, if we suppose a stationarity of time reversed dynamics, we can conclude that the time-reversed dynamics itself follows a universal dynamics, which was confirmed numerically for all the systems except for the CHE

As was mentioned in introduction, our motivation of examining the time-reversal test is to extract "the most typical" quantum irreversible systems out of a number of quantum systems with small number of degrees of freedom. Except for trivial integrable stochastic model, almost all the systems exhibiting the normal diffusion follow universally the most irreversible prototype. Thus the underlying quantum dynamics of almost all models following the universal time-reversal characteristics has the same degree of instability, or the same degree of sensitivity to external disturbance. Therefore, we focus our attention to deterministic quantum systems in which irreversibility is self-organized without the assist of external stochasticity.

We are expecting that the occurrence of irreversibility has a radical connection with the analyticity (or singular properties) of wavefunction. In order for the irreversibility to comes out, the eigenstate represented by the diffusing coordinate should be extended. (The dimension of the Hilbert space is of course infinite.) However, we have to stress that the extended states do not necessarily mean the occurrence of irreversibility, which is typically exemplified by the ballistic motion of Bloch-like state. (See Ref.[16] for the reversible feature of Bloch state measured by the time-reversal characteristics.) The eigenstate which can generate the universal irreversibility revealed in the present paper will not have any well-defined limit in the infinite limit of the dimension of Hilbert space. It is no doubt that such an eigenfunction is very singular functions which is far from analytical.

How can we capture the singular properties of eigenfunctions of irreversible quantum systems? It seems to be too difficult to treat the extended state which is supported by an infinite range. On the other hand, in some systems, the existence of phase transition from the localized state (or non-dissipative state) to the extended state (or dissipative-state) is known. If the continuity from the localized state to the extended one exists, it will be more fruitful to start our study from the analyticity of localized eigenfunction. Indeed, in case of Harper equation, we can show that the localized eigenfunction have a remarkably singular structure when its argument is continued analytically to the complex plane. In this particular example, the singular structure existing in the complex domain falls down to the real axis, which result in the occurrence of critical Harper diffusion [33]. At this instant the eigenfunctions have multi-fractal structure [34]. We do not, however, know whether or not the above scenario toward the onset of irreversibility is general or not, because, as is demonstrated in the previous section, the critical Harper diffusion do not follow the universal time-reversal characteristics. Tasaki and Gaspard first presented an example manifesting that the stationary distribution function of chaotic diffusion system is a fractal function (the Takagi function) [9]. They expect a close link among chaos, fractal nature and irreversibility [10]. The example they gave is a non-equilibrium state in a classical chaotic system, whereas what the present authors would like to study is a relaxation process toward an equilibrium state in quantum systems. The situations under consideration may be slightly different, but there are common basis for the understanding of time-irreversibility.

The present authors had several chances of exchanging our opinions with Prof.Tasaki, but it is our regret that we could not deepen our idea on the analyticity and irreversibility in quantum systems, because, at that time, we did not have a very clear picture on the characteristics of quantum irreversibility. With the present article, we believe that we have had some basis to understand quantum irreversibility. We now return to our original purpose and would like to pursue relationship between the irreversibility and analyticity of quantum states. We believe it would be the best memorial to Prof.Tasaki and his soul, passed away too early.

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