Anomalous behavior of singularities of solution on stable and/or unstable manifolds

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Abstract

Anomalous behavior of singularities of solution on stable and/or unstable manifolds is discussed and an aspect understanding complexity of chaotic trajectories from the view point of complex dynamical systems is given.

1 Introduction

Shuichi Tasaki was one of my classmates during student days in Waseda University. Before he was affected by a disease, I met him a few times every year mainly in conferences. His student, Shigeru Ajisaka, and he wrote a paper titled 'Reconnection of Unstable/Stable Manifolds of the Harper Map' five years ago[1]. Concerning with this problem, they seemed to be interested in the nature of singularities of solution on stable and unstable manifolds in continuous time systems[2].

During last fifteen years, I have worked in collaboration with Kensuke S. Ikeda on the problem of multidimensional tunneling by using the complex semiclassical method[3]. Complex stable and unstable manifolds play an important role in understanding multidimensional tunneling; namely they guide tunneling trajectories, which are governed by anomalous behavior of singularities of solution on them[3, 4]. In this paper, I address the issue of singularities of solution on stable and/or unstable manifolds and give an aspect understanding complexity of chaotic trajectories from the view point of complex dynamical systems.

2 Divergence behavior of singularities of trajectories on stable or unstable manifold

The minimum model system to study anomalous properties of stable and unstable manifolds is a 1D barrier potential. Here we choose the Eckart barrier potential and the Hamiltonian is written by

$$H(Q,P) = \frac{1}{2}P^2 + \mathrm{sech}^2(Q). \tag{1}$$

The solution in the range $(0 < E_1 < 1)$ is given by[5]

$$Q(t) = \sinh^{-1}\Big(\lambda \cosh\Big(\sqrt{2E_1}(t-t_0)\Big)\Big), \tag{2}$$

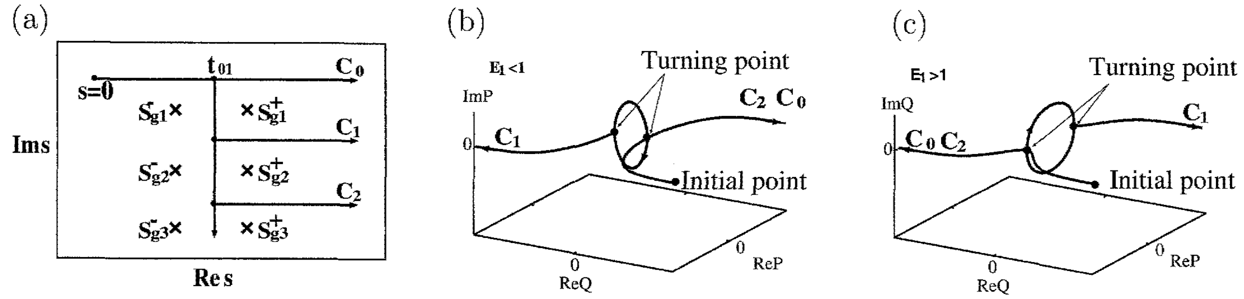

Figure 1: Singularities, integration paths and complex trajectories for the Eckart potential. (a) Singularities and representative integration paths on the lapse time plane. (b) Complex trajectories for $E < 1$. (c) Complex trajectories for $E > 1$.

where $\lambda \equiv \sqrt{1/E_1 - 1}$ and t_0 is the time at which the trajectory hits the turning point at $Q_{turn} = \pm\log\left[\lambda + \sqrt{\lambda^2+1}\right]$. For a given initial condition $(Q = Q_1(\gg 1), P = P_1(= -\sqrt{2E_1} < 0))$ at $t = t_1$, the interval between t_0 and t_1 is determined by $t_{01} \equiv t_0 - t_1 = (Q_1 - \log\lambda)/\sqrt{2E_1}$.

The solution has singularities in the complex lapse time plane s, i.e., $s = t - t_1$, at

$$Sg_n^{\pm} = \frac{1}{\sqrt{2E_1}}\left(Q_1 - \log\lambda \pm \log\left(1/\lambda + \sqrt{1/\lambda^2+1}\right)\right) + i(-n+1/2)\Delta t_I/2, \tag{3}$$

where $\Delta t_I \equiv 2\pi\big/\sqrt{2E_1}$[3, 4, 5]. There are two types of singularities, namely entrance singularities Sg_n^- and exit singularities Sg_n^+. The singularities correspond, but not one-to-one, to the singularities of the potential $V = \mathrm{sech}^2(Q)$ at $Q_m = i(2m+1)\pi/2$, and around a singularity the solution has a form, $Q - Q_m \propto \sqrt{s - Sg_n^{\pm}}$, which is a branch point of solution.

Fig.1(a) shows the singularities in the complex lapse time plane s. Representative integration paths C_n with different topologies with respect to the singularities are also shown in this picture. Fig.1(b) shows trajectories corresponding to the integration paths for the case $E < 1$ in the complex phase space projected on the space $(\mathrm{Re}Q, \mathrm{Re}P, \mathrm{Im}Q)$.

As shown in Fig.1(b), the trajectory starting at an initial point in an asymptotic side hits the turning point at $s = t_{01}$ and goes around a cycle in the classically forbidden region with imaginary time evolution along the vertical line in Fig.1(a). This cycle is nothing more than an instanton path used for the semiclassical analysis of quantum tunneling in classically integrable systems[6]. If it takes an odd integration path C_{2n+1}, it reaches the opposite turning point after a half-integer times rotations and goes toward the transmitted side with real time evolution. But it, for an even path C_{2n}, goes back to the same turning point after n-th rounds and is scattered to the reflective side.

From eq.(3), the singularities Sg_n^+ diverge logarithmically at $E_I = 1$, though the singularities Sg_n^- remain in finite ranges due to the cancellation of the logarithmic terms[3, 4] . The solution Q_s on the stable manifold at $E_I = 1$ is given by[3, 4],

$$Q_s(t,\mu) = \sinh^{-1}\left(\mathrm{e}^{-\sqrt{2}(t-\mu)}\right), \tag{4}$$

where the parameter μ denotes the initial phase or initial time of the solution. Note that for the solution on the unstable manifold, Sg_n^-'s diverge but Sg_n^+'s remain in finite ranges.

The solution for the case $E_1 > 1$ is obtained by the analytical continuation of the solution eq.(2) along one of the contours on the complex energy plane avoiding the critical energy

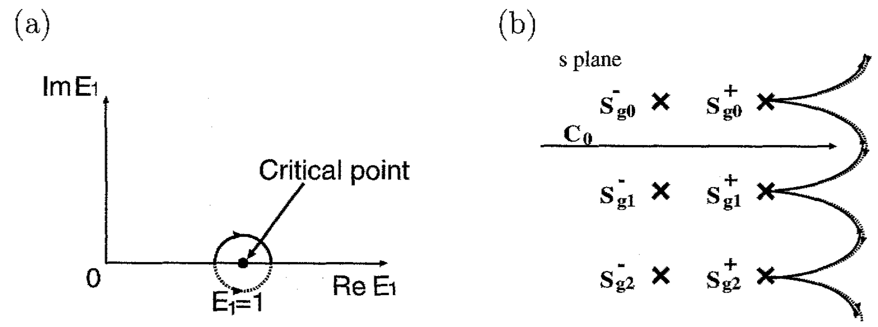

Figure 2: Critical point and movement of the singularities. (a) Critical point(critical energy) at $E_1 = 1$ on the complex energy plane and two topologically different contours. (b) Movement of the singularities Sg_n^+ along the contours in (a) and integration path C_0 on the lapse time plane.

$E_I = 1$ as shown in Fig.2(a). During this process the singularities Sg_n^+ go up or down depending on the choice of contour in the complex E_1 plane(see Fig.2(b)). As a result, the location of singularities in the lapse time plane is similar to that for $E_1 < 1$, but the topology of integration paths with respect to the singularities changes from transmissive one to reflective one, and vice versa, as shown in Fig.1(c): trajectories along odd integration paths go back to the reflective side, though those for even ones reach the transmissive side.

The logarithmic divergence of some group of singularities giving rise to the topological change of the Riemann surface is an important nature of stable and unstable manifolds extended to the complex domain and it also occurs for a periodically perturbed system[3, 4],

$$H_\epsilon(Q, P, \omega t) = \frac{1}{2}P^2 + (1 + \epsilon \sin \omega t)\mathrm{sech}^2(Q). \tag{5}$$

Let us consider a set of initial points with $E_I = 1$ in a asymptotic region in Fig.3. By the effect of the perturbation, trajectories starting from the part indicated by A are bounced off the potential wall, though trajectories of the other part B go over it, except for the intersections with the stable manifold W_S^+. Since the intersections are on the stable manifold, they asymptotically approach the unstable periodic orbit U_P on the top of the potential. The solution in A, which goes along W_S^+ and is scattered along W_U^+, has the Riemann surface with the essentially same topology as the solution for $E_I < 1$ of the unperturbed system. On the other hand, the solution in B, which is going along W_S^+ and W_U^-, has the opposite Riemann surface topologically same as that for $E_I > 1$ of the unperturbed system. The singularities Sg_n^+ disappear for the solution starting at the intersection, namely they diverse for the solution on the stable manifold.

3 Anomalous behavior of singularities of solution on separatricies

Let us consider behavior of singularities of solution on separatricies. An invariant manifold that separates the phase space into two distinct areas is called separatrix. In a 1D system, the separatrix exists at a critical energy, below and above which the geometrical change occurs to the Riemann surface of the solution. Examples of separatricies for several potentials are shown in Fig.4. As shown in the upper left picture, the stable and unstable manifolds of

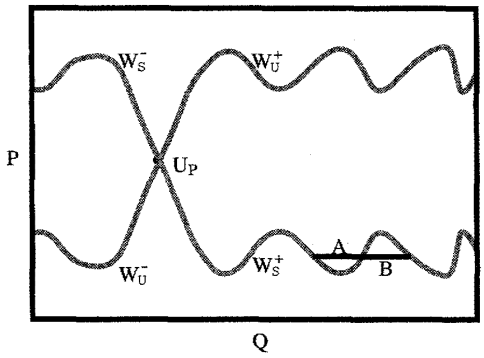

Figure 3: Schematic picture of the Poincaré map of the periodically perturbed system(eq.(5)). U_P denotes the unstable saddle. $W_S^{\pm}$ and $W_U^{\pm}$ indicate the stable and unstable manifolds of U_P, respectively.

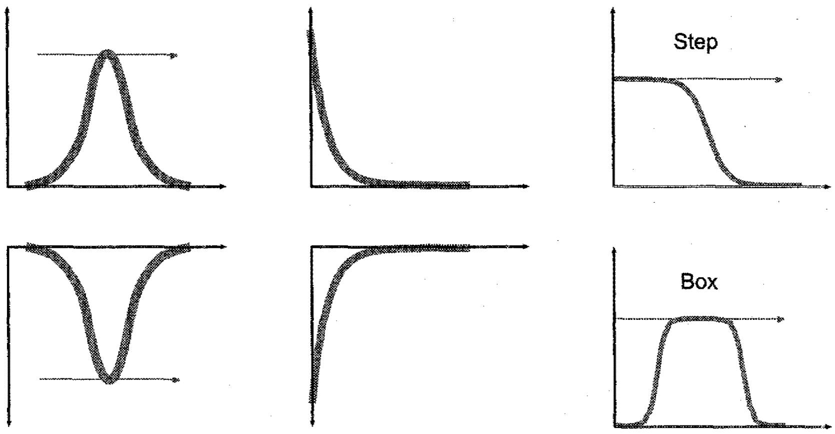

Figure 4: Separatrices of various potentials.

the potential saddle are separatricies. At the potential minimum in the lower left picture, the elliptic fixed point exists, but the trajectory goes in imaginary time evolution along a complex stable or unstable manifold, namely complex separatrix.

There exists other type of separatrix, on which the solution asymptotically approaches the saddle that moves to infinity. Several examples are shown in Fig.4. In the case of the rounded off step potential at the upper right picture, the separatrix exists at the energy of the potential top, i.e., saddle, which moves to negative infinity[7]. The trajectory on the separatrix logarithmically approaches the saddle at negative infinity. Like stable and unstable manifolds, anomalous behavior of singularities of the solution on this type of separatrix is observed. Fig.5 shows the location of singularities for the rounded off step for the three cases, $E_I < 1$, $E_I = 1$ and $E_I > 1$. Below and above the critical energy $E_I = 1$, the location of singularities completely changes and some of singularities diverge to infinity at $E_I = 1$, i.e., on the separatrix. Anomalous behavior of singularities of solution always occur on separatrices of every potential and it should be generic.

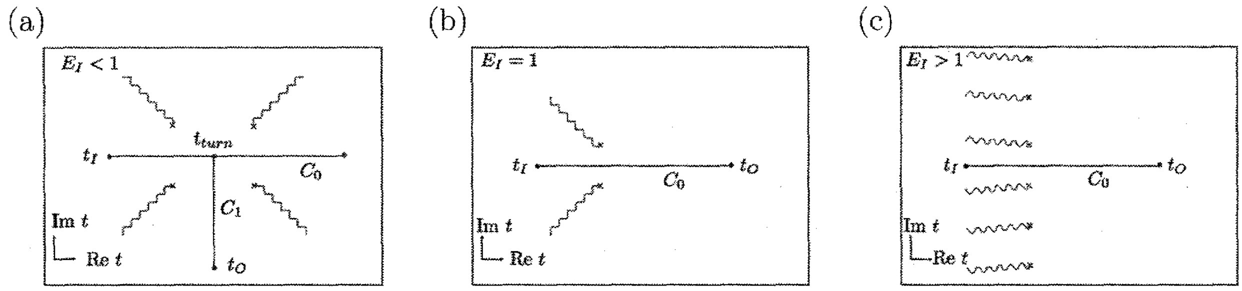

Figure 5: Location of singularities of the solution for the rounded off step potential with the unit height. (a) For $E_1 < 1$, the trajectory with the path C_0 is bounced by the potential and the path C_1 of imaginary time evolution induces instanton. (b) At $E_I = 1$, the solution is on the separatrix and some singularities disappear. (c) For $E_I > 1$, the trajectory with C_0 goes over the potential.

4 Riemann surface of chaotic solution

Let us see how the Riemann surface of chaotic solution is constructed in non-integrable systems. There are infinitely many unstable periodic orbits in a chaotic sea: A few of them are dominant periodic orbits which survive in the unperturbed limit, i.e., integrable system, and infinitely many others are created by resonance bifurcations, e.g, unstable periodic orbits forming a resonance chain in cooperation with elliptic orbits. Since a trajectory wanders in the chaotic sea passing close to one unstable periodic orbit after another, then its history is designated by

$$... \rightarrow \{U_P(n), W_S^{\{+,-\}}, W_U^{\{+,-\}}\} \rightarrow \{U_P(n+1), W_S^{\{+,-\}}, W_U^{\{+,-\}}\} \rightarrow ..., \tag{6}$$

where $U_P(n)$ denotes n-th unstable periodic orbit which it visits and , $W_S^{\{+,-\}}$ and $W_U^{\{+,-\}}$ are the stable and unstable manifolds of $U_P(n)$, respectively. One of + and − in superscript of $W_S^{\{+,-\}}$ ($W_U^{\{+,-\}}$) is chosen at each step and the sequence in eq.(6) defines the history of an individual trajectory. Roughly speaking every trajectory has an individually different history designated by eq.(6) due to instability of chaos. According to the discussion in section 2, at least every when passing close to a dominant unstable periodic orbit, it makes a choice of Riemann sheet, e.g., $W_S^+ \rightarrow W_U^+$ or $W_S^+ \rightarrow W_U^-$. As a result, every solution starting at a different initial point has topologically different structure of the Riemann surface from each other. In other words, every trajectory has a topologically different integration path with respect to singularities.

I believe that this is an explanation of instability and complexity of chaotic solution from the view point of complex dynamical systems of continuous time. For more complete discussion, we need to the knowledge of behavior of singularities of stable and unstable manifolds of sub-dominant periodic orbits caused by resonance bifurcation and have to take into account every entanglement among stable and unstable manifolds of dominant and sub-dominant periodic orbits together with generalized separatorices introduced in section 3. This is not easy task, but it seems to me that it gives fruitful results in understanding chaos and quantum chaos from the view points of complex dynamical systems and complex semiclassical method, if it succeeds.

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