Gaussian, exponential and power-law decays and their time ranges

— An analysis based on the Friedrichs model [1] —

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Abstract

The time ranges of the three typical decay forms in quantum mechanics, i.e., the Gaussian, exponential and power-law decays, are investigated on the basis of the Friedrichs model. The analysis yields a deeper understanding of the decay dynamics of quantum systems.

1 Introduction

It is well known that a quantum system, interacting with another large system with an infinite number of or continuous degrees of freedom and prepared in a state which is not an eigenstate of the total Hamiltonian, undergoes three typical decay forms, that is, the Gaussian decay at the very beginning of the decay process, the exponential decay at long times and finally the power-law decay at very long times [2]. The exponential decay, quite familiar and seen in every decay process of radioactive elements, is not valid both at very short and very long times, because of the unitary evolution of the probability amplitude and of the lower-boundedness of the total Hamiltonian, i.e., the existence of a stable vacuum, in quantum mechanics, respectively. Though the existence of such deviations from the exponential decay is theoretically well known and has been confirmed experimentally at short times [3], it is still not clear when the system stars to show the exponential decay and when it is overridden by the power-law decay. In this short note, we endeavor to clarify the conditions under which such transitions occur [4], on the basis of the analysis of the survival amplitude in the Friedrichs model.

2 Survival amplitude in the Friedrichs model

Let a quantum system be described by the Hamiltonian

$$H = \omega_a|a\rangle\langle a| + \int_0^\infty d\omega\,\omega|\omega\rangle\langle\omega| + \int_0^\infty d\omega g(\omega)(|a\rangle\langle\omega| + |\omega\rangle\langle a|). \tag{1}$$

[1]Dedicated to the late Professor Shuichi Tasaki, who was fond of the Friedrichs model [1].

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The system is composed of a discrete state $|a\rangle$ and the continuous ones $|\omega\rangle$, interacting with each other through a form factor $g(\omega)$ that is assumed to be real for simplicity. These states form a complete orthonormal set

$$\langle a|a\rangle = 1, \quad \langle\omega|\omega'\rangle = \delta(\omega-\omega'), \quad \langle a|\omega\rangle = 0, \quad |a\rangle\langle a| + \int_0^\infty d\omega|\omega\rangle\langle\omega| = 1. \tag{2}$$

If it is prepared in the state $|a\rangle$ at time $t=0$, the system starts to decay into the continuum and the survival amplitude $x(t) = \langle a|e^{-iHt}|a\rangle$ satisfies the equation

$$\dot{x}(t) = -i\omega_a x(t) - \int_0^\infty d\omega g^2(\omega) \int_0^t dt' e^{-i\omega(t-t')} x(t'), \quad x(0) = 1. \tag{3}$$

This is an exact equation and contains all the relevant information about the dynamics of the system.

This equation can be expressed as an algebraic equation in the Laplace space and the analytical property of the Laplace transform, i.e., the existence of a simple pole on the second Riemannian sheet and a cut between the origin and infinity, is known to be responsible for such specific behavior as the exponential and non-exponential decays of the amplitude $x(t)$ [2]. It is, however, quite difficult to clarify the moments at which the transitions from the Gaussian to exponential and from exponential to power-law decays occur in the whole decay process unless one could explicitly carry out the inverse Laplace transform to obtain the analytical expression of the whole amplitude; such an investigation is necessary to compare the different contributions in order to decide the dominant behavior. This is why the following analysis is focused exclusively on the above integro-differential equation itself, without resort to the Laplace transform, to endeavor to clarify and characterize the time domain where a specific behavior dominates over the others.

Let us assume that the form factor $g(\omega)$ characterizing the interaction between the discrete level and the continuum has a semi-finite support $(0,\infty)$, vanishes at $\omega=0$ and is characterized by a high-frequency "cut-off" Λ after which it becomes vanishingly small. The discrete level is assumed to be embedded in the continuum and to lie (far) below the cut-off, $0 < \omega_a \ll \Lambda$. These assumptions are the usual ones.

3 Short-time dynamics

The "short-time" dynamics is easily extracted from the above equation (3). Consider the time region $t \ll 1/\Lambda$ (*short times*). Since the exponential factor in the integrand can be safely replaced with unity in this region, the integro-differential equation (3) can be reduced to a second-order differential equation with constant coefficients and it is easy to show that the amplitude behaves like

$$x(t) \sim e^{-i\omega_a t}\cos(t/\tau_z) \sim e^{-i\omega_a t} e^{-t^2/(2\tau_z^2)}, \quad \tau_z^{-2} \equiv \int_0^\infty d\omega g^2(\omega). \tag{4}$$

This implies that the system starts to decay quadratically, exhibiting non-exponential decay at short times. This quadratic behavior is valid up to $t \sim 1/\Lambda$, which gives a characterization for the word "short."

4 Long-time dynamics

On the other hand, if we consider the behavior of the amplitude at longer times, $t > 1/\Lambda$, we are not allowed to neglect the oscillating behavior of the exponential factor in the integrand in (3). Let us separate the oscillating factor from the amplitude $x(t)$ by setting

$$x(t) = e^{-i\Omega t} y(t), \tag{5}$$

where a real quantity Ω, assumed to be responsible for the oscillating behavior of $x(t)$, has to be determined later. The equation for $y(t)$ now reads as

$$\dot{y}(t) = -i(\omega_a - \Omega)y(t) - \int_0^\infty d\omega g^2(\omega) \int_0^t dt' e^{-i(\omega-\Omega)t'} y(t-t'). \tag{6}$$

This is still an exact equation. The last term contains a memory effect and the derivative of y at time t depends on the previous values of y. In order to evaluate it, we rewrite this term as

$$-\int_0^\infty d\omega g^2(\omega) \int_0^t dt' e^{-i(\omega-\Omega)t'} [y(t) + y(t-t') - y(t)]. \tag{7}$$

to separate the Markov (first term) and non-Markov (the rest) contributions.

4.1 Markov contribution

Since y is evaluated at time t in the first term, its contribution (to the time derivative of y) is independent of the values of y at earlier times. We may call it the Markov contribution. The integration over t' is easily performed for this term to yield the following result

$$\begin{aligned} -\int_0^\infty d\omega g^2(\omega) \int_0^t dt' e^{-i(\omega-\Omega)t'} y(t) &= -i \int_0^\infty d\omega g^2(\omega) \frac{e^{-i(\omega-\Omega)t} - 1}{\omega - \Omega} y(t) \\ &\to \left(-i \int_0^\infty d\omega \mathrm{P} \frac{g^2(\omega)}{\Omega - \omega} - \pi g^2(\Omega) \right) y(t), \end{aligned} \tag{8}$$

for large enough t. Here P stands for Cauchy's principal part. Since the amplitude y is assumed to have no oscillating phase factor, that is, the oscillating behavior of $x(t)$ is solely due to $e^{-i\Omega t}$, the parameter Ω has to satisfy

$$\Omega = \omega_a + \int_0^\infty d\omega \mathrm{P} \frac{g^2(\omega)}{\Omega - \omega}, \tag{9}$$

provided that the remaining terms in (7) representing the non-Markov contributions give rise to non-oscillating behavior. This is exactly the same equation as that determines an eigenvalue of the Hamiltonian H. The second term on the right hand side gives the frequency shift $\Delta\omega$ in perturbation theory where Ω in the denominator is replaced with ω_a. Observe that if the remaining terms in (7) were to be neglected by some reasons, the amplitude y would simply satisfy

$$\dot{y}(t) = -\pi g^2(\Omega) y(t), \tag{10}$$

resulting in the exponential decay

$$x(t) \propto e^{-i\Omega t} e^{-\gamma t/2}, \quad \gamma \equiv 2\pi g^2(\Omega). \tag{11}$$

Needless to say, the quantity γ reproduces the Fermi golden rule in perturbation theory when Ω in g is replaced with ω_a. It is clear that the validity of the exponential behavior of the amplitude is conditioned to the large-t approximation in (8) and to the neglect of the remaining terms in (7).

4.2 Non-Markov contribution

The remaining terms in (7), which are considered to represent a non-Markov effect, are rewritten as

$$\begin{aligned} &-\int_0^\infty d\omega g^2(\omega) \int_0^t dt' e^{-i(\omega-\Omega)t'} [y(t-t') - y(t)] \\ &= -e^{i\Omega t} \int_0^\infty d\mu g^2(\mu/t) \int_0^1 d\xi e^{-i\mu(1-\xi) - i\Omega\xi t} [y(\xi t) - y(t)]. \end{aligned} \tag{12}$$

For large t, dominant contributions are mainly due to those regions where $\xi \sim 0$ and can be estimated as

$$\begin{aligned} &-e^{i\Omega t} \int_0^\infty d\mu g^2(\mu/t) e^{-i\mu} \int_0^\infty d\xi e^{-i\Omega\xi t} [y(0) - y(t)] \\ &= -\frac{e^{i\Omega t}}{i\Omega t} \int_0^\infty d\mu g^2(\mu/t) e^{-i\mu} [1 - y(t)] \\ &\sim \lambda^2 \frac{(-i)^a a!}{\Omega \omega_0^{a-1}} \frac{e^{i\Omega t}}{t^{a+1}} [1 - y(t)], \end{aligned} \tag{13}$$

where the form factor $g(\omega)$ has been assumed to have the asymptotic expansion

$$g^2(\omega) \sim \lambda^2 \omega_0 \left(\frac{\omega}{\omega_0}\right)^a, \quad a > 0 \tag{14}$$

for small $\omega < \omega_0$, with a characteristic frequency ω_0, and t is assumed to be large enough so that $1/t < \omega_0$ holds.

4.3 Dynamics at long and very long times

Collecting all contributions at long times, the amplitude $y(t)$ has thus been shown to follow the differential equation

$$\dot{y}(t) = -\frac{\gamma}{2} y(t) + c \frac{e^{i\Omega t}}{t^{a+1}} [1 - y(t)], \quad c \equiv \lambda^2 \frac{(-i)^a a!}{\Omega \omega_0^{a-1}}. \tag{15}$$

It is apparent that if the first term on the right hand side dominates over the second, the equation approximately reproduces an exponential decay form (see Eq. (11)), while in the opposite case, a power-law decay is realized. Indeed, in the latter case, the equation can be approximated as

$$\frac{d}{dt}[1-y(t)] \sim -c\frac{e^{i\Omega t}}{t^{a+1}}[1-y(t)], \tag{16}$$

which can be solved to yield

$$y(t) \sim -\ln[1-y(t)] \sim c\frac{e^{i\Omega t}}{i\Omega}\frac{1}{t^{a+1}}(1+O(1/t)) \tag{17}$$

or

$$x(t) \sim \frac{c}{i\Omega}\frac{1}{t^{a+1}}(1+O(1/t)). \tag{18}$$

This is the power-law decay with the same exponent as expected [2] and this behavior only shows up when the condition

$$\frac{\gamma}{2}|y(t)| \ll \frac{|c|}{t^{a+1}}|1-y(t)| \sim \frac{|c|}{t^{a+1}} \tag{19}$$

is satisfied. It would define a border between *long* and *very long* times. Unfortunately it depends on the absolute value of the amplitude, which can be obtained only when one completely solves the differential equation. State differently, one can expect the power-law decay to appear if the survival probability has been reduced below a certain value

$$p(t) \equiv |x(t)|^2 \ll \frac{4|c|^2}{\gamma^2 t^{2(a+1)}}. \tag{20}$$

5 Summary

The three time ranges for which the three typical decay forms are valid were clarified on the basis of the Friedrichs model. Even though the results obtained here are based on a particular simple model, the model keeps the general properties of a quantum system in interaction with a large quantum system with continuous degrees of freedom and thus we can think that they reflect the essential properties of the decaying system in quantum mechanics.

We understand that the high-frequency cut-off Λ of the form factor $g(\omega)$, over which the discrete state effectively ceases to interact with the continuous levels, discriminates the short-time range $t \ll 1/\Lambda$ from the long-time range $1/\Lambda \ll t$. Just after the interaction has been turned on, even though the state starts to evolve from the initial state $|a\rangle$ to the continuum, the evolution is rather similar to a unitary oscillation between two discrete levels (see Eq. (4)). We may say that the discrete level starts to interact with the continuum as a whole, without feeling any detailed structure of $g(\omega)$ at $t \ll 1/\Lambda$, resulting in a quadratic behavior of the survival probability. As time elapses $t \gg 1/\Lambda$, the form factor starts to play its role. At this moment, it is interesting to recognize that there are two different contributions, reflecting the Markov and non-Markov effects in the interaction, which give rise to the exponential and power-law decays of the survival probability. Both effects are present; however, the power contribution is naturally anticipated to be of order $O(1/t^{\alpha})$ with some positive exponent $\alpha > 1$ and thus is much smaller than the exponential, which can be of order unity. Therefore, after the initial quadratic (Gaussian) decay, the exponential decay appears first at long times, followed by the power-law decay at very long times. The transition between long and very long times has so far been determined depending on the value of the survival probability. See Eq. (20). It is still not clear whether one can extract a quantity that characterizes the moment of the transition between long and very long times, like the parameter Λ that discriminates the short times from the long times.

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References

[1] Works by the late Professor Shuichi Tasaki on the basis of the Friedrichs model include, M. Fujiyoshi and S. Tasaki, J. Phys. Soc. Jap. Suppl. C **72** (2003) 73; Prog. Theor. Phys. **119** (2008) 883; *ibid* **122** (2009) 1095.

[2] See, for example, H. Nakazato, M. Namiki and S. Pascazio, Int. J. Mod. Phys. B **10** (1996) 247 and references therein.

[3] S. R. Wilkinson, C. F. Bharucha, M. C. Fischer, K. W. Madison, P. R. Morrow, Q. Niu, B. Sundaram and M. Raizen, Nature **387** (1997) 575.

[4] A similar study can be found in, A. Peres, Ann. Phys. **129** (1980) 33. The analysis there is conducted with more general terms, but from a slightly different point of view from that taken in this note.