On Lambda transformation approach to the theory of irreversibility

— With a particular contribution of S. Tasaki —

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1 Introduction

In memory of our friend and colleague Shuichi Tasaki

Problems of irreversibility in classical and quantum mechanics were among the most important objects of research interests of professor Ilya Prigogine, the director of the International Solvay Institutes in Brussels, and the group of researchers gathered around him. That was also one of the main interests of Shuichi Tasaki, especially in the period of his stay in the Solvay Institutes in the beginning of 1990s. The problem of irreversibility in statistical mechanics can be described briefly as the problem of consistent description of observed macroscopic behaviour in terms of irreversible microscopic dynamics. Two different ways to resolve this problem were proposed by I. Prigogine and his collaborators. The first will be called here the Λ transformation theory and the second the complex spectral theory. S. Tasaki contributed to both approaches by resolving some important problems associated with the Λ transformation theory and by a thorough spectral analysis of some class of dynamical systems associated with the so called chaotic maps. He has done, in particular, a comparative case study of these two nonequivalent theories of irreversibility.

Taking the opportunity to review some of Tasaki's contributions to the problems of irreversibility we would like to give an overview of the Λ transformation theory. We will present the main idea of the theory and the most relevant results, pointing out Tasaki's contribution, as well as a further progress in this field. It should be stressed, however, that the presented here theory was only one of the fields of research where S. Tasaki contributed. During his relatively short stay in Solvay Institutes he and one of the authors of this article (Z.S.) collaborated in four

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different fields which resulted in seven common publications. A brief overview of some results of that collaboration will be given in the last section.

2 Λ transformation theory

The problem of irreversibility in statistical physics lies in the understanding the dynamical origin of the observed broken time symmetry which is manifested by the entropy increasing evolution. In the case of dynamical systems exhibiting a high degree of instability the deterministic description of time evolution in terms of phase space trajectories is not sufficient to describe the observed macroscopic behaviour. More adequate, and in fact commonly used, approach is to replace deterministic evolution of phase space points by the Liouville evolution of probability density functions. The Liouville evolution is still time reversible but allows to introduce the broken time symmetry at a fundamental level. The idea behind the Λ transformation theory is to relate the Liouville evolution with an entropy increasing evolution through a similarity transformation Λ (to be defined precisely below).

In an abstract setting, let us consider an abstract dynamical flow given by the quadruple $(\Omega, \Sigma, \mu, \{S_t\})$, where $\{S_t\}$ is a group of one-to-one μ invariant transformations of Ω and either $t \in \mathbf{Z}$ or $t \in \mathbf{R}$. The invariance of the measure μ implies that the transformations U_t

$$U_t\, \rho(\omega) = \rho(S_{-t}\omega), \quad \rho \in L^2 \tag{1}$$

are unitary operators on $L^2 = L^2(\Omega, \Sigma, \mu)$. Let us point out the following, very important properties of U_t as operators on L^2:

(a) $U_t\rho \geq 0$ if $\rho \geq 0$,

(b) $\int_\Omega U_t\rho\, d\mu = \int_\Omega \rho\, d\mu$, for $\rho \geq 0$,

(c) $U_t 1 = 1$.

An abstract operator W on L^2 which satisfies conditions (a)–(c) is called *doubly stochastic operator.*

The Λ transformation theory known also as the Misra-Prigogine-Courbage theory of irreversibility [16] (see [18] for its generalized version) proposes to relate the group $\{U_t\}$, considered on the space L^2, to the irreversible Markov semigroup W_t, $t \geq 0$, through a nonunitary but invertible operator Λ:

$$W_t\Lambda = \Lambda U_t\,, \quad (t \in \mathbb{R}^+)\,, \tag{2}$$

The operators W_t should tend strongly to the equilibrium state, as $t \to \infty$, on some subset of admissible densities. A dynamical system for which such a construction is possible is called *intrinsically random* and the conversion of the reversible group $\{U_t\}$ into the irreversible Markov

semigroup $\{W_t\}$ through a nonunitary transformation Λ is understood as a *change of representation.*

The irreversible Markov semigroup W_t, $t \geq 0$, on the space L^2 has the following properties:

(1) W_t are contractions

$$\|W_t\rho\|^2 \leq \|\rho\|^2 . \qquad (i)$$

(2) W_t preserve probabilities, i.e.,

$$W_t\rho \geq 0 \qquad \text{if} \quad \rho \geq 0 \qquad (ii)$$

$$\int_Y d\mu W_t\rho = \int_Y d\mu\rho . \qquad (iii)$$

The last condition means that the operators W_t^* adjoint to W_t satisfy

$$W_t^* 1 = 1 . \qquad (iv)$$

Properties (iv) and (i) imply on L^2 [13] that

$$W_t 1 = 1 . \qquad (v)$$

(3) Irreversible approach to equilibrium is described by the condition

$$\|W_t\rho - 1\|^2 \to 0 \quad \text{as} \quad t \to \infty , \qquad (vi)$$

for every square integrable density ρ.

As it was pointed out by Misra [19] the conventional approach to the problem of irreversibility introduces extraneously some form of coarse-graining or contraction of description and certain approximation schemes in order to arrive at an evolution equation with broken time symmetry.

Misra, Prigogine and Courbage showed in fact that the unitary evolution U_t can be intertwined by a similarity for dynamical systems which are Kolmogorov systems or K-flows. Let us recall that a dynamical system is a K-flow if there exists a sub-σ-algebra Σ_0 of Σ such that for $\Sigma_t = S_t(\Sigma_0)$ we have

(i) $\Sigma_s \subset \Sigma_t$, for $s < t$

(ii) $\sigma(\cup_{t\in\mathbf{R}}\Sigma_t) = \Sigma$

(iii) $\cap_{t\in\mathbf{R}}\Sigma_t = \Sigma_{-\infty}$ – the trivial σ-algebra, i.e. the algebra of sets of measure 0 or 1

where $\sigma(\cup_{t\in\mathbf{R}}\Sigma_t)$ stands for σ-algebra generated by Σ_t, $t \in \mathbf{R}$.

The main idea of the construction of Λ is the following. With any K-flow we can associate a family of conditional expectations $\{E_t\}$ with respect to the σ-algebras $\{\Sigma_t\}$ (projectors if we confine ourselves to the Hilbert space L^2). These projectors determine the time operator T:

$$T = \int_{-\infty}^{+\infty} t dE_t \ . \tag{3}$$

Then Λ is defined, up to constants, as a function of the operator T:

$$\Lambda = f(T) + E_{-\infty} \, , \tag{4}$$

where $E_{-\infty}$ is the expectation (projection on constants). The function f is assumed to be positive, non increasing, $f(-\infty) = 1$, $f(+\infty) = 0$ and such that lnf is concave on $\mathbf{R}$.

The Markov operators W_t are of the form

$$W_t = \left(\int_{-\infty}^{\infty} \frac{f(s)}{f(s-t)} dE_s + E_{-\infty} \right) U_t \, . \tag{5}$$

2.1 Implementability of the semigroup $\{W_t\}$

The Misra-Prigogine-Courbage approach to the problem of irreversibility has led to a number of other problems related with the characterization of the class dynamical systems for which the construction of Λ transformation is possible and with the studies of properties of both Λ and W_t. One of such problems in which S. Tasaki was involved is the problem of implementability of the semigroup $\{W_t\}$. In order to formulate this problem let us note that not only invertible dynamical system can be associated with evolution operators. In general, we can consider a dynamical system defined as a semigroup S_t, $t \geq 0$, of measurable and measure preserving transformation of a measure space (Ω, Σ, μ). The idea of using operator theory, which is due to Koopmaan [15], is to replace time evolution S_t of single points of Ω by the time evolution of the corresponding Koopman operators V_t defined as

$$V_t f(\omega) = f(S_t \omega) \, , \quad f \in L^2(\Omega, \Sigma, \mu) \, , \quad \omega \in \Omega \, .$$

The object under consideration is the semigroup U_t, $t \geq 0$, of the operators adjoint to W_t, $U_t = W_t^*$, which governs the evolution of probability densities.

It should be clear that in the considered above case of invertible transformations S_t each operator U_t, as defined by (1), is the Frobenius-Perron operator associated with S_t thus the adjoint of the corresponding Koopman operator. The operators W_t preserve the property of double stochasticity characteristic to Frobenius-Perron operators. The question of implementability can be formulated as follows:

Are W_t the Frobenius-Perron operators associated with some measure preserving transformations $\tilde{S}_t$ or, equivalently, is the adjoint W_t^* the Koopman operator

$$W_t^* f(\omega) = f(\tilde{S}_t \omega)$$

Misra and Prigogine [22] have shown that for a specific model, the baker transformation (to be defined below), with a specific choice of Λ transformation the resulting semigroup is nonlocal, i.e. non implementable. In the article Ref. [21] written together with S. Tasaki we have shown that the semigroup

$$W_t = \Lambda U_t \Lambda^{-1}\,, \quad t \geq 0\,,$$

is non-local for all Kolmogorov systems and for all choices of Λ-transformations. More precisely, we have shown the following:

Theorem 1 *The semigroup $W_t = \Lambda U_t \Lambda^{-1}$, $t \geq 0$ is not implementable, i.e., there does not exist a measurable point transformation $\tilde{S}_t$ of Y such that $\tilde{S}_t$ preserves μ and for which*

$$W_t^* \rho(y) = \rho(\tilde{S}_t y) \quad \text{for all } t \geq 0\ .$$

The prooof of Theorem 1 is based on the time operator scaling, i.e on the listed above properties of the function f defining the Λ transformation and on important feature of implementable linear transformations expressed by the followng lemma [21]

Lemma 1 *Let M be a linear operator on L^2 which is implementable by a measure preserving point transformation $\tilde{S}$. Then for each measurable set Δ such that its image $\tilde{S}(\Delta)$ under $\tilde{S}$ is also measurable, the following holds:*

$$\int_{\Omega - \tilde{S}(\Delta)} d\mu M \mathbf{1}_\Delta = 0\ .$$

We have shown that, contrary to this lemma, for an arbitrary measurable subset Δ of Ω the supports of the functions $W_t \mathbf{1}_\Delta$ cover , as t tends to infinity, the whole space Ω.

2.2 Λ transformation theory, time operators and Sz.-Nagy-Foiaş dilations

In an abstract setting, reversible evolutions are expressed in terms of unitary groups $\{U_t\}_{t\in\mathbb{R}}$, whereas *irreversible dynamics* are described by contraction semigroups $\{W_t\}_{t\in\mathbb{R}^+}$, both of them defined on a separable Hilbert space $\mathcal{H}$. In particular, *dissipative systems*, which approach a unique equilibrium for long times, are described by contraction semigroups $\{W_t\}_{t\in\mathbb{R}^+}$ satisfying

$$||W_t h||^2 \text{ tends to zero as } t \to \infty\,. \tag{6}$$

In the exact theory of irreversibility proposed by Misra, Prigogine and Courbage, a unitary group $\{U_t\}_{t\in\mathbb{R}}$ is related by (2) to a contraction semigroup $\{W_t\}_{t\geq 0}$ through an intertwining

transformation Λ, where Λ is a *non-unitary similarity* (or *quasi-affinity*) on $\mathcal{H}$, i.e. a linear, one-to-one and continuous transformation onto a dense subspace of $\mathcal{H}$, so that Λ^{-1} exists on this dense domain, but is not necessarily continuous. In such a case we will say that the unitary group $\{U_t\}_{t\in\mathbb{R}}$ has the *intertwining property.* In the above presented MPC construction Λ is defined as function of the internal time operator (3) associated with the underlying dynamical system. It therefore natural to ask about a characterization of the dynamical systems which admit time operators associated with their unitary evolutions. Such a characterization in the form of a set of equivalent conditions can be found in Ref. [25]. In this section we will show obtained recently analogous results for general unitary evolutions on Hilbert spaces.

In general, *(internal) time operator* for a unitary evolution $\{U_t\}_{t\in\mathbb{R}}$ in a (separable) Hilbert space $\mathcal{H}$ is defined as a self-adjoint operator T on $\mathcal{H}$ satisfying the following relation: for every $\rho \in \mathrm{Dom}(T)$ and $t \in \mathbb{R}$, one has $U_t\rho \in \mathrm{Dom}(T)$ and

$$U_{-t}TU_t\,\rho = (T + tI)\,\rho \tag{7}$$

($\mathrm{Dom}(T)$ denotes the domain of T). The time operator T allows the attribution of an *average age* $(\rho, T\rho)$ to the states $\rho \in \mathcal{H}$ which keeps step with the external clock time t for the evolved state $U_t\rho$: $(U_t\rho, TU_t\rho) = (\rho, T\rho) + t$, where $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$.

The question of the existence of selfadjoint time operators for unitary evolutions in classical and quantum mechanics has been tackled in [6, 10] on the basis of Halmos-Helson theory of invariant subspaces, Sz.-Nagy-Foiaş dilation theory and MPC theory of irreversibility. An extensive set of equivalent conditions to the existence of selfadjoint time operators are obtained in these works. Among them we would like to distinguish the following:

Theorem 2 *Let $\{U_t\}_{t\in\mathbb{R}}$ be a unitary dynamics defined on $\mathcal{H}$. The following assertions are equivalent:*

(a) There exists an internal time operator for $\{U_t\}_{t\in\mathbb{R}}$, i.e. a selfadjoint operator T on $\mathcal{H}$ satisfying (7).

(b) There exists a closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that

$$U_{t_2}\,\mathcal{M} \subseteq U_{t_1}\,\mathcal{M}\,, \quad \textit{if } t_2 \geq t_1\,, \tag{8}$$

$$\bigcap_{t\in\mathbb{R}} U_t\,\mathcal{M} = \{0\}\,, \quad \overline{\bigcup_{t\in\mathbb{R}} U_t\,\mathcal{M}} = \mathcal{H}\,. \tag{9}$$

An interesting recent result is that the intertwining property (2) can be obtain without the use of time operator. The structure of the reversible and irreversible evolutions admitting such a type of change of representation as well as a prototype for the transformations Λ have been

given in [8] on the basis of the Sz.-Nagy-Foiaş dilation theory and the derived functional calculus for contractions on Hilbert spaces.

Let us recall that for every contraction W defined on a Hilbert space $\mathcal{H}'$ there exists an isometric dilation U_+ on some Hilbert space $\mathcal{K}_+ \supset \mathcal{H}'$, which is moreover minimal in the sense that

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} U_+^n \mathcal{H}' .$$

Such a minimal isometric dilation is determined up to an isomorphism. Sz.-Nagy and Foiaş call *residual part* of the minimal isometric dilation $U_+, \mathcal{K}_+$ to the unitary part $R, \mathcal{R}$ of the Wold decomposition of $U_+, \mathcal{K}_+$, i.e.

$$R := U_{+|\mathcal{R}}, \quad \mathcal{R} := \bigcap_{n=0}^{\infty} U_+^n \mathcal{H}' .$$

The following result was obtained in [8]. Its most interesting part says that an intrinsically random group $\{U_t\}$ defined on a Hilbert space $\mathcal{H}$ is unitarily equivalent to an orthogonal summand of the residual part corresponding to the irreversible Markov semigroup $\{W_t\}$.

Theorem 3 *Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. The following assertions are equivalent:*

(a) The unitary group $\{U_t\}$ defined on $\mathcal{H}$ and the contraction semigroup $\{W_t\}$ defined on $\mathcal{H}'$ are related through the intertwining relation (2), with $\Lambda : \mathcal{H} \to \mathcal{H}'$ a quasi-affinity.

(b) The contraction semigroup $\{W_t\}$ is of class $C_{\cdot 1}$, i.e., it verifies

$$\operatorname*{s-lim}_{t\to\infty} W_t^* h \neq 0 \text{ for each non-zero } h \in \mathcal{H}' ,$$

and the unitary group $\{U_t\}$ is unitarily equivalent to an orthogonal summand of the group of residual parts $\{R_t\}$ for $\{W_t\}$.

The analysis of the unitary group $\{U_t\}$ and the contraction semigroup $\{W_t\}$ can be done through their respective cogenerators U and W. Given a semigroup of contractions $\{W_t\}$ on a Hilbert space with infinitesimal generator A, the *cogenerator* W of $\{W_t\}$ is defined by

$$W = (A+I)(A-I)^{-1}, \qquad A = (W+I)(W-I)^{-1}.$$

A semigroup of contractions $\{W_t\}$ consists of normal, selfadjoint, isometric, unitary or completely nonunitary operators iff its cogenerator W is normal, selfadjoint, isometric, unitary or completely nonunitary, respectively (see [17, Sect.III.8-9]).

In the situation described by Theorem 3, the orthogonal summand of $\{R_t\}$ unitarily equivalent to the intrinsically random group $\{U_t\}$ is described by means of a lifting Λ_+ of the operator Λ given in terms of the cogenerator U of $\{U_t\}$ and the minimal isometric dilation U_+ of the cogenerator W of $\{W_t\}$ (see [9] for details):

Proposition 1 *Under the conditions of Theorem 3, if $\Lambda_+ : \mathcal{H} \to \mathcal{K}_+$ is the lifting operator defined by*

$$\Lambda_+ = \text{s-}\lim_{n\to\infty} U_+^n \Lambda U^{-n} ,$$

then $\overline{\Lambda_+\mathcal{H}} \subseteq \mathcal{R}$ is a reducing subspace for the residual group $\{R_t\}$ and the restriction group $\{R_t|\overline{\Lambda_+\mathcal{H}}\}$ is unitarily equivalent to $\{U_t\}$.

Thus, an unitary group $\{U_t\}$ with the intertwining property has the same structure as an orthogonal summand of the residual part $\{R_t\}$ corresponding to the intertwined contraction semigroup $\{W_t\}$ satisfying (2). The corresponding Sz.-Nagy-Foiaş functional model contains additional information about the "intrinsic disorder" of the system and solves the *inverse intertwining problem* in the negative [7, 11]:

Theorem 4 *There exist unitary groups $\{U_t\}$ with the intertwining property which do not admit internal time operators.*

Indeed, for unitary groups with the intertwining property there exists a time operator if and only if the evolution of a simply invariant subspace fills the space where the functional model is defined. This is equivalent to the existence of a rigid operator-valued function with appropriate properties in the functional model. In some sense, the existence of such rigid function is related with the lack of "gaps" in the significant subspaces along the components of the model.

2.3 Λ transformation theory and nested Hilbert spaces

There are close connections between the Λ transformation theory with that of *nested Hilbert spaces* and *generalized eigenvalues* introduced by Grossmann [12]. A nested Hilbert space is a pair of Hilbert spaces each of which is in a certain sense identified with a dense subset of the other. These structures have been used to study analytic continuation into "unphysical sheets" and to discuss non-normalizable states of quantum-mechanical systems.

To be precise, a *nested Hilbert space* $(\mathcal{H}_0, \mathcal{H}_1, E_{01})$ is a structure that consists of two infinite dimensional separable Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, a quasi-affinity E_{01} of $\mathcal{H}_1$ into $\mathcal{H}_0$, and the adjoint quasi-affinity $E_{10} = E_{01}^*$ of $\mathcal{H}_0$ into $\mathcal{H}_1$.

Let $A : \mathcal{H}_0 \to \mathcal{H}_0$ be a linear operator and consider in $\mathcal{H}_1$ the operator

$$j_{10}(A) \equiv E_{01}^{-1} A E_{01}.$$

The domain of $j_{10}(A)$ is the subset of $E_{01}^{-1}\mathcal{D}(A)$ consisting of vectors f which are also such that $AE_{01}f \in E_{01}\mathcal{H}_1$. The matrix elements of $j_{10}(A)$ are different from the matrix elements of A, because of the different definition of the scalar product.

The eigenvalues of $j_{10}^*(A^*)$ include the "improper" eigenvalues of A, long familiar in quantum mechanics:

Definition 1 *Let A be an operator in $\mathcal{H}_0$ such that A is densely defined, so that the adjoint A^* exists, and $j_{10}(A^*)$ is also densely defined, so that its adjoint $j_{10}^*(A^*)$ also exists. The complex number z is said to be a "generalized eigenvalue" of A if there exists in $\mathcal{H}_1$ a non-zero vector f such that*

$$j_{10}^*(A^*)f = zf.$$

This definition includes the usual one: Let $Ah = zh$, $(h \in \mathcal{H}_0,\ h \neq 0)$. Notice that $j_{10}^*(A^*) \supseteq E_{10}A^{**}E_{10}^{-1} \supseteq E_{10}AE_{10}^{-1}$. So $j_{10}^*(A^*)f = zf$, with $f = E_{10}h$.

On the other hand, in Λ transformation theory, the relations of interest are of the form

$$W\Lambda = \Lambda U, \tag{10}$$

where W is a contraction defined on a Hilbert space $\mathcal{H}'$, U is a unitary operator on a Hilbert space $\mathcal{H}$ and Λ is a quasi-affinity from $\mathcal{H}$ into $\mathcal{H}'$. The intertwining relation (10) induces a nested Hilbert space $(\mathcal{H}, \mathcal{H}', \Lambda^*)$ where the generalized eigenvalues of U are just the eigenvalues of W. Indeed, taking adjoints in (10) we have

$$W^* = (\Lambda^*)^{-1}U'^*\Lambda^*.$$

Thus, identifying in the previous framework

$$\begin{array}{lll} E_{01} : \mathcal{H}_1 \to \mathcal{H}_0 & \text{with} & \Lambda^* : \mathcal{H}' \to \mathcal{H}, \\ A : \mathcal{H}_0 \to \mathcal{H}_0 & \text{with} & U^* : \mathcal{H} \to \mathcal{H}, \\ j_{10}(A) \equiv E_{01}^{-1}AE_{01} & \text{with} & W^* = (\Lambda^*)^{-1}U^*\Lambda^*, \end{array}$$

we get the result [9]:

Proposition 2 *In the nested Hilbert space $(\mathcal{H}, \mathcal{H}', \Lambda^*)$ the generalized eigenvalues of U are the eigenvalues of W.*

This result is related with the second approach to the problem of irreversibility proposed by I. Prigogine and his collaborators, the *complex spectral theory*, where the reversible evolution restricted to certain classes of initial densities and observables is represented as a superposition of decaying eigenmodes (see [26] and references therein). A comparative study of both theories is a difficult task and can be done in some particular cases using spectral analysis of evolution semigroups on vector spaces. In Ref. [26] S. Tasaki has performed such comparative studies of both theories of irreversibility in the case of the evolution operators associated with the baker map. He applied the Λ-transformation theory to the baker map and study the spectral property of the transformed evolution operator. Then he characterized the properties of the transformed operator as those of the original operator restricted to a subspace of the Hilbert space giving a new interpretation of the Λ-transformation. Finally he derived explicitly the first two generalized

decaying eigenmodes and decomposed (Pollicott-Ruelle decomposition) the expectation values of a certain class of observables with respect to a certain class of initial densities into a sum of the decaying eigenmodes and a residual faster decaying term. In result he has shown a common feature of the two approaches and non equivalence of both theories.

3 A case study by the baker map

In this section we will focus our attention on the baker map. In particular we apply obtained recently general approach to spectral representations of evolution operators and derivation of generalized eigenvalues in this case.

The baker map is one of the first examples of reversible mixing transformations and was introduced by Hopf [14]. It is defined on the unit square $[0,1)^2$ as a two-step operation: (1) squeeze the unit square to a $2\times 1/2$-rectangle and (2) cut the rectangle into two $1\times 1/2$-rectangles and pile them up to recover the unit square:

$$B(x,y) = \begin{cases} \left(2x, \dfrac{y}{2}\right), & 0 \le x < \dfrac{1}{2}, \\ \left(2x-1, \dfrac{y+1}{2}\right), & \dfrac{1}{2} \le x < 1. \end{cases}$$

It is a typical Kolmogorov system (K-system) with the Lebesgue measure as an ergodic invariant measure. The time evolution of the probability densities $\rho(x,y)$ is governed by the Frobenius-Perron operator (we drop here the prime of previous sections)

$$U\,\rho(x,y) \equiv \rho\left(B^{-1}(x,y)\right) = \begin{cases} \rho\left(\dfrac{x}{2}, 2y\right), & 0 \le y < \dfrac{1}{2}, \\ \rho\left(\dfrac{x+1}{2}, 2y-1\right), & \dfrac{1}{2} \le y < 1. \end{cases} \tag{11}$$

The operator U is unitary on the Hilbert space $L^2([0,1)^2)$ of square integrable functions with respect to the Lebesgue measure. It is well known [5] that for K-systems, U has a Lebesgue spectrum: an infinitely degenerate continuous spectrum on the unit circle plus a point eigenvalue at $z=1$ associated with the equilibrium.

3.1 Λ-transformation

For the baker map a Λ-transformation is constructed as follows [23]. Let χ_0 be the function

$$\chi_0(x,y) \equiv \begin{cases} -1, & 0 \le x < 1/2, \\ 1, & 1/2 \le x < 1, \end{cases}$$

and, for each finite set $S = \{n_1, \ldots, n_r\}$ of integers, ($n_j \neq n_k$ if $j \neq k$), set

$$\chi_S(x,y) \equiv U^{n_1}\chi_0(x,y)\, U^{n_2}\chi_0(x,y) \cdots U^{n_r}\chi_0(x,y).$$

Then the family of functions $\{\chi_S\}$ together with the unit function 1 form a complete orthonormal set of $L^2([0,1)^2)$. Note that

$$U\chi_S = \chi_{S+1}\,,$$

where $S+1 = \{n_1+1,\ldots,n_r+1\}$ if $S = \{n_1,\ldots,n_r\}$. Now, for each integer n, define the operator E_n to be the orthogonal projection operator onto the subspace spanned by χ_S such that $n_S \equiv \max\{n_j \in S\} = n$. The Λ-transformation is defined by

$$\Lambda \equiv \sum_{n=-\infty}^{\infty} \lambda_n E_n + P_0\,,$$

where P_0 is the one-dimensional orthogonal projection onto the subspace of constant functions and $\{\lambda_n\}_{-\infty<n<\infty}$ is a positive monotonically decreasing sequence bounded by 1 such that λ_{n+1}/λ_n also decreases monotonically as n increases.[2] This leads to

$$W \equiv \Lambda U \Lambda^{-1} = \sum_{n=-\infty}^{\infty} \frac{\lambda_{n+1}}{\lambda_n} U E_n + P_0\,. \tag{12}$$

The operator W is a contraction such that $W1 = 1$ and $||W^n(\rho-1)||$ decreases strictly monotonically to 0 as $n \to \infty$.

3.2 Spectral representations

The connections between Λ transformation and Sz.-Nagy-Foiaş dilation theories lead to spectral representations for the Frobenius-Perron operator U of the baker map given in (11) and the associated contraction W given in (12). These spectral representations derive from the Sz.-Nagy-Foiaş functional models and their interest lies in the structure of the Hilbert spaces where they are defined.

Let $\partial D = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ be the unit circle in the complex plane, and consider the Hilbert space $L^2 \ominus \mathcal{C} \equiv L^2([0,1)^2) \ominus \mathcal{C}$, the orthogonal complement of the subspace $\mathcal{C}$ of constant functions in $L^2([0,1)^2)$. In what follows, $L^2(\partial D; L^2 \ominus \mathcal{C})$ shall denote the set of all measurable functions $v : \partial D \to L^2 \ominus \mathcal{C}$ such that $\frac{1}{2\pi}\int_0^{2\pi} ||v(e^{i\theta})||^2\, d\theta < \infty$ (modulo sets of measure zero); measurability here can be interpreted either strongly or weakly, which amounts to the same due to the separability of $L^2 \ominus \mathcal{C}$. The functions in $L^2(\partial D; L^2 \ominus \mathcal{C})$ constitute a Hilbert space with pointwise definition of linear operations and inner product given by $(u,v) \equiv \frac{1}{2\pi}\int_0^{2\pi} (u(e^{i\theta}), v(e^{i\theta}))\, d\theta$, $(u, v \in L^2(\partial D; L^2 \ominus \mathcal{C}))$.

Spectral models of the operators associated to the baker map are given in the following terms [9]:

[2] An example of such sequence $\{\lambda_n\}_{-\infty<n<\infty}$ is given by

$$\lambda_n = \frac{1}{1+e^{n/\tau}}\,, \quad \tau > 0\,.$$

Theorem 5 *(a) The Frobenius-Perron operator U of the baker map given in (11) has the following spectral representation:*

$$\begin{aligned} \hat{U} : \hat{H} &\to \hat{\mathcal{H}} \\ v &\mapsto e^{i\theta} v(e^{i\theta}), \end{aligned}$$

where

$$\hat{\mathcal{H}} = \Big\{ v \in L^2(\partial D; L^2 \ominus \mathcal{C}) : \\ v(e^{i\theta}) \in \overline{span}\Big\{ \sum_{k=-\infty}^{\infty} e^{ik\theta}(\beta_{n_S+k-1}\chi_{S+k-1} - \beta_{n_S-k-1}\chi_{S-k-1}) : \forall S \Big\}\Big\}$$

and, for each $n \in \mathbb{Z}$, $\beta_n = (\lambda_n^2 - \lambda_{n+1}^2)^{1/2}$.

(b) The associated contraction W given in (12) has the spectral representation $\hat{W} : \hat{\mathcal{H}}' \to \hat{\mathcal{H}}'$ given by

$$\hat{W}\Big(\sum_S a_S(e^{i\theta})\chi_S\Big) = \sum_S e^{-i\theta}(a_S(e^{i\theta}) - a_S^0)\chi_S\,,$$

where

$$\hat{\mathcal{H}}' = \Big\{ \sum_S a_S(e^{i\theta})\chi_S \in L^2(\partial D; L^2 \ominus \mathcal{C}) : a_S(e^{i\theta}) = \sum_{l=0}^{\infty} e^{il\theta} \frac{\beta_{n_S}}{\beta_{n_S-l}} a_{S-l}^0\,, \forall S \Big\}$$

and $a_S^0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} a_S(e^{i\theta})\, d\theta.$

3.3 Grossmann generalized eigenvalues

By virtue of Proposition 2, the generalized eigenvalues of U in the nested Hilbert space $(\mathcal{H}, \mathcal{H}', \Lambda^*)$ are the eigenvalues of W. For each λ in the unit disc $D = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ of the complex plane let $\Theta_W(\lambda)$ be the *characteristic function of W at λ* given by [9]

$$\Theta_W(\lambda) = \sum_{n=-\infty}^{\infty} \Big[-\frac{1}{\gamma_{n+1}} + \beta_n \Big(\sum_{k=0}^{\infty} \lambda^k \alpha_{n-k+1} U^{*k} \Big) \Big] U E_n\,,$$

where, for each $n \in \mathbb{Z}$,

$$\alpha_n = \left(\frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n-1}^2} \right)^{1/2}, \quad \beta_n = (\lambda_n^2 - \lambda_{n+1}^2)^{1/2}, \quad \gamma_n = \frac{\lambda_n}{\lambda_{n-1}}\,.$$

It is well known [17, Th.VI.4.1] that the point spectrum of W restricted to $L^2 \ominus \mathcal{C}$ is the set of points $\lambda \in D$ for which $\Theta_W(\lambda)$ is not one-to-one .

To obtain $\operatorname{Ker}\Theta_W(\lambda)$ we must solve the equation $[\Theta_W(\lambda)](\sum_S a_S(\lambda)\chi_S) = 0$, which amounts to

$$a_S(\lambda) = \gamma_{n_S+1}\alpha_{n_S+1} \sum_{k=0}^{\infty} \lambda^k \beta_{n_S+k} a_{S+k}(\lambda)\,, \quad \forall S\,.$$

In particular, subtracting to this equation for S the equation for $S+1$ multiplied by λ we get the recurrence relation

$$a_{S+1}(\lambda) = \lambda^{-1} \frac{\beta_{n_S+1}}{\beta_{n_S}} a_S(\lambda)\,,$$

so that

$$a_{S+k}(\lambda) = \lambda^{-k}\frac{\beta_{n_S+k}}{\beta_{n_S}}a_S(\lambda)\,, \quad \forall S\,, \forall k \in \mathbb{Z}\,. \tag{13}$$

Let

$$c_{\pm} = \lim_{k\to\pm\infty}\frac{\lambda_{k\pm1}}{\lambda_k}\,.$$

It is immediate to see that $\sum_S |a_S(\lambda)|^2$ converges when $c_+ < |\lambda| < c_-$ and that $\sum_S |a_S(\lambda)|^2$ diverges if $c_- < |\lambda|$ or $|\lambda| < c_+$. Therefore:

Proposition 3 *In the nested Hilbert space $(\mathcal{H}', \mathcal{H}, \Lambda^*)$ the set of Grossmann generalized eigenvalues of U contains the annulus $\{\lambda \in \mathbb{C} : c_+ < |\lambda| < 1\}$ and is contained in the annulus $\{\lambda \in \mathbb{C} : c_+ \leq |\lambda| < 1\}$. The corresponding eigenvectors $\sum_S a_S(\lambda)\chi_S$ must satisfy (13).*

See Antoniou-Tasaki [4] and Tasaki [26] for generalized spectral decompositions including the Pollicot-Ruelle resonances.

4 Concluding remarks

The above presented Λ transformation theory was only one of many fields of research activity of S. Tasaki. Unfortunately we have not been able to give a comprehensive overview of his achievements in other fields. However, concluding this paper we would like to mention briefly about some other results contained in the papers written commonly with one of us (Z.S.) and other colleagues. One of them is the article about rigged Hilbert spaces for chaotic dynamical systems [24] where we studied the spectral properties of the Frobenius-Perron and the Koopman operators of some chaotic maps, namely the Renyi map and the baker map. The article concerns generalized spectral decompositions of evolution operators which acquire meaning on dual pairs of locally convex spaces. The main result is a partially resolved problem of the tightest rigging, i.e. such choice of a dual pair that the test function space is the largest possible among admissible families of test function spaces. We have shown that the rigged Hilbert space for the Renyi map has some of the properties of a strict inductive limit and give a detailed description of the rigged Hilbert space for the baker map.

In the article on intrinsic irreversibility of quantum systems with diagonal singularity [2], which is related to the work of the Brussels-Austin group on irreversibility, we have shown that quantum large Poincare systems with diagonal singularity lead to an extension of quantum theory beyond the conventional Hilbert space framework. We characterized the algebra of observables, the states and the logic of the extended quantum theory of intrinsically irreversible systems with diagonal singularity. The article contains a thorough study of the properties of the corresponding Banach algebra of observables and the general ideas are illustrated for the Friedrichs model.

The three articles Ref. [27, 29, 28] are devoted to study of one-dimensional piecewise linear maps. We have determined invariant measures and given the ergodic characterization of families of piecewise linear maps corresponding to the dynamics on fractal repellers and studied the spectral properties of the Frobenius-Perron operator. In particular, we have obtained the right and left eigenvectors corresponding to the Pollicott-Ruelle resonances. The Frobenius-Perron operator is also shown to admit a generalized spectral decomposition consisting of only isolated point spectra on suitable test function spaces. We have also constructed a class of one-dimensional piecewise linear maps admitting fractal invariant sets and uncountably many invariant measures and shown that they are ergodic. An interested reader can find more more about the latter resuls in another articles contained in this volume

Finally, let us mention about the article Ref. [3] where the concept of complex delta function has been applied to the perturbation analysis of unstable dynamical systems.

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