Is entanglement quantum? [1]

Dipartimento di Matematica and MECENAS, Università di Bari, I-70125 Bari, Italy
INFN, Sezione di Bari, I-70126 Bari, Italy
Paolo Facchi[2]
Dipartimento di Fisica and MECENAS, Università di Bari, I-70126 Bari, Italy
INFN, Sezione di Bari, I-70126 Bari, Italy
Saverio Pascazio[3]

1 Dedication

Shuichi Tasaki was a very talented physicist. He would perform very complicated calculations in very short times. He would work everywhere, even late in the evening, going home. If you happened to meet him in the train, chances were high that he would not "see" you, absorbed as he was in his physics problems.

Shuichi liked all branches of physics, with no exceptions. However, two subjects attracted him most: second-order calculations in the weak coupling limit, leading to markovianity, and the C^* algebraic approach to quantum field theory and statistical mechanics. This article focuses on the latter. It is a pleasure to dedicate it to Shuichi.

2 Introduction

The definition of "quantumness", as opposed to that of classicality, is a complex problem, that can be tackled from different perspectives, both in physics and mathematics. In an algebraic framework, the main focus is on observables, that make up an algebra of (in general non-commuting) operators [1, 2]. The recently introduced notion of "quantumness witness"[3, 4] is based on such an algebraic approach and motivated interesting experiments on qubits [5, 6], testing their quantum features and ruling out (semi)classical descriptions.

Composed quantum systems, made up of two or more subsystems, are more complicated and can be entangled. Both entanglement and quantumness are often investigated by framing them in terms of inequalities: entanglement and separability are discriminated through the Bell

[1]Dedicated to Professor Shuichi Tasaki.
[2]E-mail: paolo.facchi@ba.infn.it
[3]E-mail: saverio.pascazio@ba.infn.it

inequality [7], while quantumness and classicality are discriminated through the Leggett-Garg inequality [8]. We propose here a combined framework and show that any entanglement witness is also a quantumness witness. We focus in particular on the Bell inequality. This work is based on an article written in collaboration with R. Fazio, V. Vedral and K. Yuasa [9].

3 Classicality, quantumness and entanglement

We shall only consider finite dimensional systems. We start off by defining quantumness and entanglement witnesses.

3.1 Classical and quantum systems

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^*$ its dual space (the space of continuous linear complex functionals on $\mathcal{A}$). The set $\mathcal{S} \subset \mathcal{A}^*$ of states of $\mathcal{A}$ consists of positive normalized functionals, i.e. if $\rho \in \mathcal{S}$ then $\rho(A^*A) \geq 0$ for any $A \in \mathcal{A}$ and $\rho(\mathbf{1}) = 1$. We recall that (normal) states $\rho \in \mathcal{S}$ can be *uniquely* realized as traces over density matrices *belonging* to the algebra $\mathcal{A}$:

$$\rho(A) := \mathrm{tr}(\rho A), \qquad \rho \in \mathcal{A}, \quad \rho \geq 0, \quad \mathrm{tr}\rho = 1. \tag{1}$$

See the excellent textbooks [1, 2].

We have the following characterization of commutative (i.e. classical) algebras.

Theorem 1 ([3, 4]). *Given a C^*-algebra $\mathcal{A}$, the following two statements are equivalent:*

1. *$\mathcal{A}$ is commutative. To wit, for any pair $X, Y \in \mathcal{A}$,*

$$[X, Y] := XY - YX = 0. \tag{2}$$

2. *For any pair $X, Y \in \mathcal{A}$ with $X \geq 0$ and $Y \geq 0$,*

$$\{X, Y\} := XY + YX \geq 0. \tag{3}$$

3.2 Quantumness witnesses

As a consequence of Theorem 1, for a quantum system one can always find a pair of positive observables $X, Y \geq 0$ such that the observable

$$Q_{\mathrm{AVR}} = \{X, Y\} \tag{4}$$

is *not* positive. Thus, $Q_{\mathrm{AVR}} \in \mathcal{A}$ is a "witness" of the quantumness (i.e. noncommutativity) of the algebra $\mathcal{A}$ [3, 4].

Definition 1. We say that a state $\rho \in \mathcal{S}(\mathcal{A})$ is *classical* if

$$\rho([X, Y]) = 0, \quad \text{for any pair } X, Y \in \mathcal{A}. \tag{5}$$

A state that is not classical is *quantum*.

Notice that we can have classical states even when the algebra is noncommutative (namely, even when there exist A and B such that $[A, B] \neq 0$). In words, classical states do not "perceive" nonvanishing commutators. Obviously, $\mathcal{A}$ is commutative iff every state $\rho \in \mathcal{S}$ is classical.

Let us now define quantumness witnesses.

Definition 2. We say that an observable $Q \in \mathcal{A}$ is a *quantumness witness* (QW) if

1. for any classical state $\rho \in \mathcal{S}$ one gets $\rho(Q) \geq 0$,

2. there exists a (quantum) state $\sigma \in \mathcal{S}$ such that $\sigma(Q) < 0$.

The fact that the particular observables Q_{AVR} in (4) are QWs follows from the following lemma.

Lemma 1. *For any classical state $\rho \in \mathcal{S}$ and for any pair $X, Y \in \mathcal{A}$ with $X \geq 0$, $Y \geq 0$ it happens that*

$$\rho(\{X, Y\}) \geq 0. \tag{6}$$

Remark. In words, classical states do not even perceive the possible negativity of the anticommutators $\{X, Y\}$: their behavior is fair with respect to (1)–(2) of Theorem 1.

Proof. Since ρ is classical we get

$$\rho(\{X, Y\}) = \rho(2XY - [X, Y]) = 2\rho(XY). \tag{7}$$

Recall that an observable X is nonnegative iff $X = A^*A$ for some $A \in \mathcal{A}$. Therefore,

$$\rho(XY) = \rho(A^*AB^*B) \tag{8}$$

for some $A, B \in \mathcal{A}$. By using again the definition of classicality (5) we conclude

$$\rho(XY) = \rho(BA^*AB^*) = \rho(C^*C) \geq 0, \tag{9}$$

with $C = AB^* \in \mathcal{A}$. □

3.3 Entanglement witnesses

Let our system be made up of two subsystems, that will conventionally be sent to Alice and Bob, whose observations are independent. The notion of independence is reflected in the fact that the total algebra of observables is assumed to factorize

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B}. \tag{10}$$

The two subalgebras $\mathcal{A} \otimes \mathbb{I}$ and $\mathbb{I} \otimes \mathcal{B}$ commute with each other, but each subalgebra can be noncommutative.

Definition 3. A state $\rho \in \mathcal{S}(\mathcal{C})$ is said to be *separable* (with respect to the given bipartition $\mathcal{A} \otimes \mathcal{B}$) if it can be written as a convex combination of product states, namely,

$$\rho = \sum_k p_k \rho_k \otimes \sigma_k, \qquad p_k > 0, \qquad \sum_k p_k = 1, \tag{11}$$

where $\rho_k \in \mathcal{S}(\mathcal{A})$ and $\sigma_k \in \mathcal{S}(\mathcal{B})$ are states of $\mathcal{A}$ and $\mathcal{B}$, respectively. A state that is not separable is said to be *entangled* (with respect to the given bipartition).

Definition 4 ([10, 11]). We say that an observable $E \in \mathcal{C}$ is an *entanglement witness* (EW) if

1. for any separable state $\rho \in \mathcal{S}(\mathcal{C})$ one gets $\rho(E) \geq 0$,
2. there exists a (entangled) state $\sigma \in \mathcal{S}(\mathcal{C})$ such that $\sigma(E) < 0$.

4 All EWs are QWs

We now show that every EW is also a QW. We first consider a preliminary lemma.

Lemma 2. *Any classical state is separable.*

Proof. Notice first that if the algebra $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ is the full algebra of operators

$$\mathcal{C} = B(\mathbb{C}^n) \otimes B(\mathbb{C}^m), \tag{12}$$

then the only classical state is the totally mixed state,

$$\rho = \mathbb{I}_{nm}/nm = \mathbb{I}_n/n \otimes \mathbb{I}_m/m, \tag{13}$$

which is obviously separable. In general, however, the (sub)algebras $\mathcal{A}$ and $\mathcal{B}$ are reducible (i.e. they are proper subalgebras of the full matrix algebra) and one has

$$\mathcal{C} = \Big(\bigoplus_k B(\mathbb{C}^{n_k}) \Big) \otimes \Big(\bigoplus_l B(\mathbb{C}^{m_l}) \Big) = \bigoplus_{k,l} B(\mathbb{C}^{n_k}) \otimes B(\mathbb{C}^{m_l}) =: \bigoplus_{k,l} \mathcal{C}_{kl}, \tag{14}$$

where each $\mathcal{C}_{kl}$ is an irreducible algebra of dimension $n_k m_l$. All observables are block-diagonal and the classical states have the form

$$\rho = \bigoplus_{k,l} p_{kl} \mathbb{I}_{n_k}/n_k \otimes \mathbb{I}_{m_l}/m_l, \tag{15}$$

with $p_{kl} \geq 0$ and $\sum_{kl} p_{kl} = 1$, i.e. they are separable. □

Remark. Notice that if the two subalgebras are reducible, states (thought as density matrices) inherit their block-diagonal structure.

Our main theorem is now an easy consequence of the lemma just proved.

Proposition 1. *Any EW is a QW.*

Proof. Consider an EW $E \in \mathcal{C}$. By definition $\rho(E) \geq 0$ for any separable $\rho \in \mathcal{S}$. But by the previous lemma all classical states are separable. It follows that $\rho(E) \geq 0$ for any classical state ρ. Moreover, by definition, $\sigma(E) < 0$ for some entangled state σ, which by the previous lemma must be a quantum state. Thus, E is a QW. □

Remark. The converse is, of course, not true. If the algebra $\mathcal{A}$ is noncommutative, and $Q \in \mathcal{A}$ is a QW of the quantum state $\sigma \in \mathcal{S}(\mathcal{A})$, then

$$\tilde{Q} = Q \otimes \mathbb{I} \in \mathcal{C} \tag{16}$$

is also a QW (of the total algebra), but it is *not* an EW. Indeed, it is negative on separable states of the form $\sigma \otimes \omega$ (for any $\omega \in \mathcal{S}(\mathcal{B})$), namely,

$$(\sigma \otimes \omega)(\tilde{Q}) < 0. \tag{17}$$

5 Bell inequality

An interesting example of EW is the Bell-CHSH observable on $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$

$$E_{\text{Bell}} = 2 + (A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2), \tag{18}$$

where $A_k \in \mathcal{A}$ and $B_k \in \mathcal{B}$ are dichotomic observables (with eigenvalues ± 1) of Alice and Bob, respectively. If $\rho(E_{\text{Bell}}) < 0$, E_{Bell} witnesses the violation of the Bell-CHSH inequality in the entangled state ρ.

For instance, if we take

$$A_1 = \sigma_x, \qquad B_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \qquad A_2 = \sigma_y, \qquad B_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), \tag{19}$$

where $\sigma_{x,y,z}$ are Pauli operators, then

$$E_{\text{Bell}} = 2 + \sqrt{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y). \tag{20}$$

We now prove that the Bell-CHSH observable is also an anticommutator QW. Let

$$\begin{aligned} X &= 1 + (A_1 \otimes B_1 - A_2 \otimes B_2)/2 \geq 0, \\ Y &= 1 + (A_1 \otimes B_2 + A_2 \otimes B_1)/2 \geq 0, \end{aligned} \tag{21}$$

Then

$$\begin{aligned} XY &= E_{\text{Bell}}/2 + ([A_1, A_2] \otimes \mathbb{I} + \mathbb{I} \otimes [B_1, B_2])/4, \\ YX &= E_{\text{Bell}}/2 - ([A_1, A_2] \otimes \mathbb{I} - \mathbb{I} \otimes [B_1, B_2])/4, \end{aligned} \tag{22}$$

so that

$$Q_{\text{AVR}} = \{X, Y\} = E_{\text{Bell}}. \tag{23}$$

This shows that the EW E_{Bell} is an anticommutator QW: if the Bell-CHSH inequality is violated by an entangled state ρ, then $\rho(\{X, Y\}) < 0$.

Remark. An interesting remark is the following one: consider two particles, on which Alice and Bob measure dichotomic observables. They put together their results and find that a state ρ exists such that $\rho(E_{\text{Bell}}) < 0$. Then they can conclude that their local observables do not commute. In this sense, one can say that the Bell inequality is testing quantumness: by looking only at the correlations of the two subsystems, one can check whether the two local algebras are noncommutative. (Incidentally, it is easy to prove that in this case *both* algebras $\mathcal{A}$ and $\mathcal{B}$ are noncommutative. Indeed, if one of the two algebras were classical, then any state ρ of the composed system would necessarily be separable. See e.g. Prop. 2.5 in [12].)

6 Conclusions and perspectives

We have discussed the notions of quantumness and entanglement, showing that every entanglement witness is also a quantumness witness. The answer to the question posed in the title of this note is therefore affirmative. This conclusion may appear obvious. However, we would like to emphasize that our analysis makes use of strict mathematical definitions of witnesses. This enables us to put (physical) intuition on firm mathematical grounds. In turn, theorems and their derivations disclose alternative viewpoints: we observed in Sec. 5 that the Bell inequality, written as a QW, tests the quantumness of the composed system. This enables one to look at the Bell inequality from a novel perspective.

An interesting aspect that could be investigated in the future is whether the combined notions of quantumness and entanglement witnesses could shed light on the elusive notion of bound entanglement [13, 14], for which the PT criterion does not apply.

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