Foundations of earthquake statistics in view of non-stationary chaos theory

Yoji AIZAWA

Department of Applied Physics, Advance School of Science and Engineering, Waseda University

This paper is dedicated to the memory of a dear friend of ours, Professor Shuichi Tasaki. He left us many excellent works in the field of nonequilibrium statistical mechanics with his broad interests, and he always encouraged us to challenge to new problems in complex nonlinear systems, which often reveal anomalous fluctuations, characteristic scalings, etc.

In this paper we discuss a simple dynamical model which reproduces a sequence of empirical laws in seismic statistics, and derive a universal relation that connects two important statistical laws; the Gutenberg - Richter distribution and the interoccurrence time distribution. These results imply that the seismic statistics can be well understood in terms of the stationary-nonstationary chaos transition near the critical regime. The results derived from the Data-Catalog in Japan, California, and Taiwan (JMA, SCDEC, TCMA), which support the universal relation, are briefly discussed.

§1. Introduction

Chaotic behaviors appeared in deterministic systems often reveal very clear statistical laws, and the origin of their probabilistic aspects is successfully understood in ergodic-theoretical framework. Earthquakes and their time evolution are also considered to be a kind of statistical phenomena which demonstrates some probabilistic natures, but their statistical laws still remain empirical ones at the present time though they are useful for future prediction and correspondence. We have to admit that it is not so easy to construct the dynamical model which precisely describes the details of the physical changes in earth crusts, however, it is an important subject for us to elucidate the universal aspects in seismological statistics and their dynamical origins from the viewpoint of deterministic chaos theory.

In this paper, we concentrate our discussions to the empirical laws in seismological statistics, that are clearly shown in a sort of scaling relations obtained by Wadati, Omori, Enya, Ishimoto-Iida, and Utsu. It will be shown that all these scaling relations are reproduced by a simple dynamical model based on the socalled Modified Bernoulli Map at least qualitatively, and that these statistical laws are universal in the critical regime which is induced by a big earthquake. The model proposed here is a toy-model, but it is emphasized that the model displays not only the ergodic-theoretical universal aspects observed in a wide class of dynamical systems, but also the common features in the earthquake statistics of the critical regime.[7)]

§2. Chaotic dynamics which reveals the stationary - nonstationary chaos transition

One of the simplest random processes is coin-tossing, which generates a series of symbols, $\{\sigma_k\}$, where k stands for the discrete time at k-th trial, and $\sigma_k = \pm 1$ (for head or tail). The probability for the continuous occurrence of n-heads (or tails) is the exponential distribution, and the mean occurrence rate of head (or tail) approaches to 1/2 when the number of trials goes to large enough. Statistical properties of the coin-tossing is reproduced by a chaotic dynamical system, socalled Bernoulli map, which is defined by

$$x_{k+1} = \begin{cases} 2x_k & (0 \le x < 1/2), \\ 2x_k - 1 & (1/2 \le x \le 1), \end{cases} \tag{2·1}$$

where x_k is the real number and k (integer) the discrete time. The Bernoulli map is ergodic and strong mixing under the invariant distribution $\rho(x) = 1$ $(0 \le x \le 1)$, and that when the symbolic dynamics is defined by $\sigma_k = \pm 1$ for $(0 \le x < 1/2)$, or for $(1/2 \le x \le 1)$ respectively, the sequence $\{\sigma_k\}$ has the same statistical properties as the above mentioned coin-tossing.

Let us generalize the Bernoulli map to include long-term memories in the coin-tossing process. The modified Bernoulli map (MBM) is given by,

$$x_{k+1} = \begin{cases} x_k + 2^{B-1}x_k^B & (0 \le x_k < 1/2), \\ x_k - 2^{B-1}(1 - x_k)^B & (1/2 \le x_k \le 1), \end{cases} \tag{2·2}$$

and the symbolic dynamics $\{\sigma_k\}$ in the same way as the Bernoulli map. When the parameter $B = 1$, the MBM is the original Bernoulli map, but in what follows, we consider the case for $B > 1$. The statistical properties of the MBM are well understood by ergodic-theoretical analysis. For instance, the invariant distribution

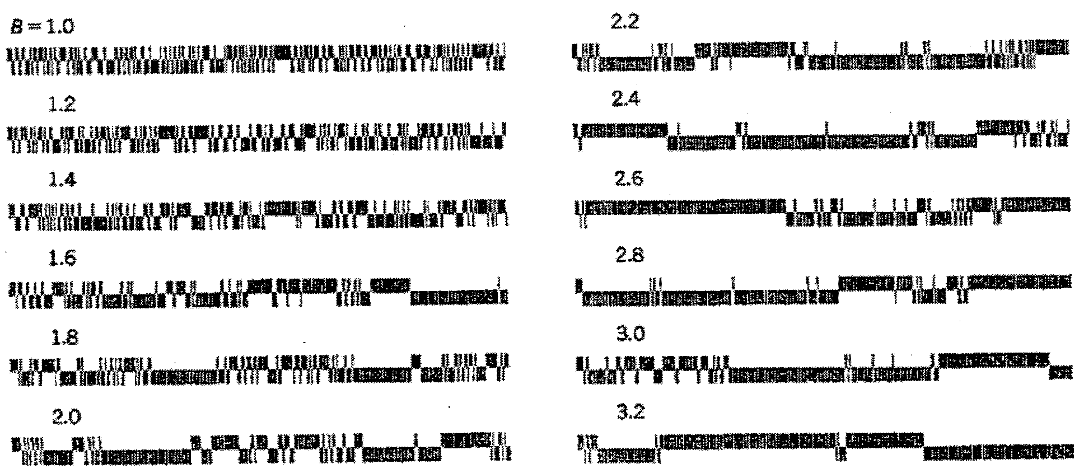

Fig. 1. Symbolic time series of the MBM for various B values.[7)]

$\rho(x)$ is well approximated by,

$$\rho(x) \propto x^{-(B-1)} + (1-x)^{-(B-1)},\ (0 \le x \le 1), \tag{2·3}$$

and the waiting time probability $P(n)$, which describes the probability for the continuous n-occurrence of the state $\sigma = +1$ (or $\sigma = -1$), obeys,

$$P(n) \propto n^{-\beta},\ \left(\beta = \frac{B}{B-1}\right) \tag{2·4}$$

This explains that the strong correlations or the long-term memories are generated in the MBM. Figure 1 shows typical time series for various cases. A striking point is that the sequence $\{\sigma_k\}$ is non-stationary and the mean waiting time $\langle n \rangle$ is divergent for $B \ge 2$, but is stationary with $\langle n \rangle =$ finite for $B < 2$. The critical regime appears near $B \cong 2$ [8-12].

§3. A metaphor model which simulates seismicity statistics

Here we propose a metaphorical model based on the chaotic behaviors in the MBM, which reproduces some scaling aspects appeared in many seismological studies. First, we assume that the symbolic variable $|\sigma_k|$ is proportional to the rate of energy accumulation in the earth crust, and the accumulated energy during the period of $\sigma > 0$ (or $\sigma < 0$) is release at once by an earthquake when the symbolic variable σ_i changes its sign, i.e., $\sigma_{i+1} = -\sigma_i$. This assumption means that the released energy by the earthquake E is related to the waiting time n defined in equation (2·4). Though the mechanism is still unclear, the following form is plausible from the consideration of nonlinear visco-elastic spring energy,

$$E \propto n^{1/\kappa'}, \tag{3·1}$$

where the exponent κ' is a parameter. Therefore, the energy distribution of the earthquake $P(E)$ becomes,

$$P(E) \propto E^{-\kappa},\ \ (\kappa = \kappa'(\beta - 1) + 1). \tag{3·2}$$

The parameter B is considered to express the strength of earth-crusts or the tolerance to the strain in crust. Actually, it is shown in Figure 1 that the number of bigger earthquakes increases when the value B becomes large. The power law distribution in the earthquake energy was first pointed out by Wadati[1)] for several cases, and suggested to be $\kappa \cong 2.1$. Equation (6) is also consistent with the Gutenberg - Richter law [6].

The second assumption of our metaphor model is that the strength parameter B is sharply reduced by the occurrence of a big earthquake (mainshock), namely, the value just before the big earthquake, say B_{pre}, is larger than that after the big earthquake B_{post}, i.e., $B_{pre} > B_{post}$, and that the value B goes back slowly to the original value B_{pre}. If we assume the exponential recovery (relaxation time is τ) of the parameter B, the value B at time k, say $B(k) = B_k$, is given by

$$B_k = B_{pre} - (B_{pre} - B_{post})e^{-k/\tau},\ \ (k = 0, 1, 2, \dots) \tag{3·3}$$

where k stands for the time measured from the moment of the main shock ($k = 0$). From these assumptions, the MBM dynamics after the occurrence of the big earthquake is expressed by,

$$x_{k+1} = \begin{cases} x_k + 2^{B_k-1}x_k^{B_k} & (0 \le x < 1/2), \\ x_k - 2^{B_k-1}(1-x_k)^{B_k} & (1/2 \le x \le 1), \quad (k = 0, 1, 2, \dots). \end{cases} \tag{3·4}$$

Equations (3·3) and (3·4) are the metaphorical chaotic model which is used in what follows.

§4. Empirical statistical laws in seismicity statistics

Numerical results obtained by equations (3·3) and (3·4) are shown in Figures 2, 3, 4, and 5 under the following conditions; $B_{pre} = 5/2$, $B_{post} = 3/2$, and $\tau = 10^3$ [7]. Figure 2 shows two sample paths of $\{\sigma_k\}$ for different initial data of x_0, where we can see that the interoccurrence time between two flip-flop jumps in $\{\sigma_k\}$ is very short when k is small enough. This corresponds to the fact that a series of small aftershocks are induced by a big earthquake. The accumulated number of the aftershocks in the interval $1 \le k \le t$, say $N(t)$, is a random variable depending on the initial data. The average number $\langle N(t)\rangle$ is shown in Figure 3, where the numerical data is very well adjusted by a logarithmic function,

$$\langle N(t)\rangle \propto \log(at + b), \tag{4·1}$$

where a and b are fitting parameters. It is important to point that this is quite consistent with the Omori formula,[2)] and that the formula by Enya[3)] is naively adopted in our metapher model. By changing the values of B_{pre} and B_{post}, we can obtain the generalized Omori formula; $\langle N(t)\rangle \propto (at+b)^\gamma$, but we do not touch this problem here.

Figure 4 shows the energy distribution of aftershocks $P(E)$ at $\kappa' = 1$. The result is well fitted by,

$$P(E) \propto E^{-\kappa}, \quad (\kappa \cong 2.0 \ at \ \ \kappa' = 1) \tag{4·2}$$

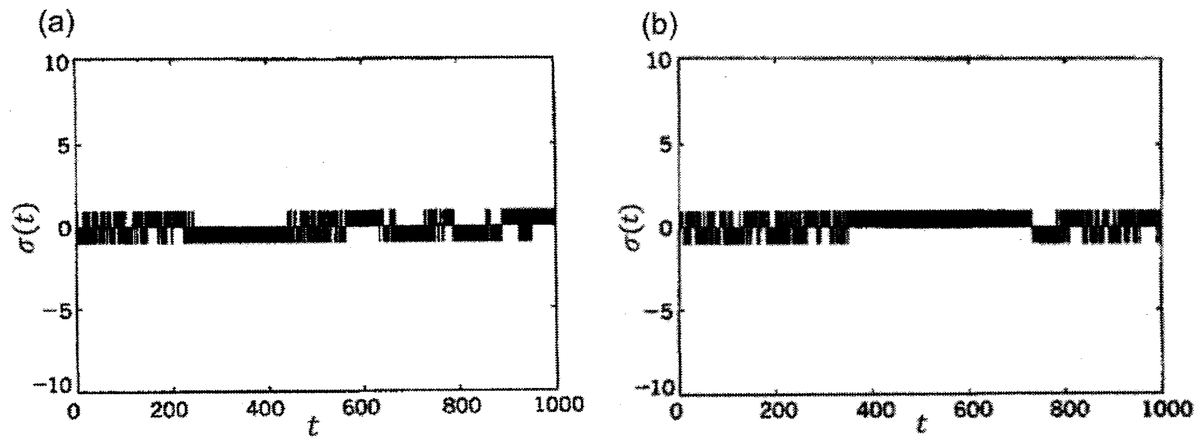

Fig. 2. Symbolic time series after the main shock at $k = 0$ for two different values of x_0.

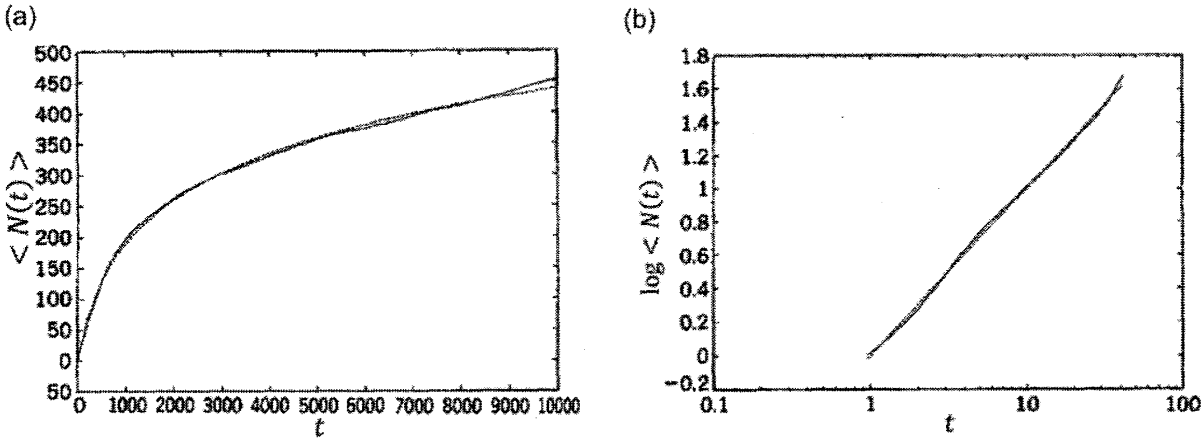

Fig. 3. Cumulative numbers of aftershocks for $1 \le k \le t$. (a) normal plot (log time) and (b) logarithmic plot (short time); Numerical plots are almost completely adjusted by a logarithmic curve (doted line) consistent with the Omori law in long time regime.

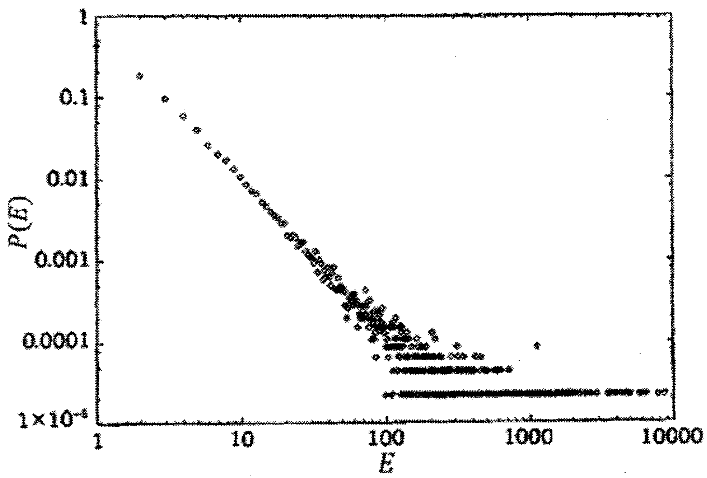

Fig. 4. Energy distribution of aftershocks. ($P(E) \simeq E^{-\kappa}, \kappa \cong 2.0$.)

If we take the effect by κ' into account, the exponent obeys $\kappa = \kappa' + 1$. In these numerical analysis, we used the simulation data for $1 \le k \le T$ $(= 10 \times \tau)$, and the aftershocks occurred in $k > T$ are omitted because they are not the aftershocks induced by the main shock occurred at $k = 0$.

The reason why the exponent κ takes $\kappa \cong 2.0$ (at $\kappa' = 1$) can be analyzed by using the intrinsic properties of the MBM, where the bifurcation parameter B plays an essential roles in the statistical behaviors of the symbolic dynamics $\{\sigma_k\}$. As mentioned in Equation(2·4), there occurs a phase transition between stationary regime ($B < 2$) and nonstationary regime ($B \ge 2$), where the critical fluctuations dominate the statistical behaviors in the recovering process $B(k)$ in equation (3·3), namely, $B(k)$ passes always through the critical value $B = 2$ at a certain time under the condition of $B_{pre} > 2.0$, and $B_{post} < 2.0$. As the result, the waiting time distribution in equation (2·4) becomes $P(n) \propto n^{-2}$, and the energy distribution in a

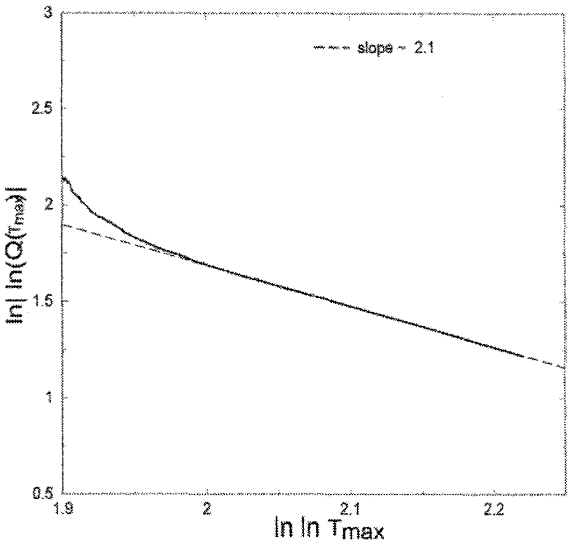

Fig. 5. Waiting time distribution of the maximum aftershock $P(T_{max})$ which obeys a log-Weibull distribution[13),14)].

wide critical regime becomes,

$$P(E) \propto E^{-(\kappa'+1)}. \tag{4·3}$$

Furthermore, by using equation (4·3), the energy distribution of the maximum aftershock E_{max} can be analyzed theoretically; $E_{max} = \mathrm{Max}[E_1, E_2, \cdots E_N]$ where E_j is the j-th aftershock energy and we take N large enough, the distribution $P(E_{max})$ becomes,

$$P(E_{max}) \propto E_{max}^{-(\kappa'+1)} L(E_{max}), \tag{4·4}$$

where $L(x)$ is a slowly varying function $L(x) \propto e^{Cx^{-\kappa'}}$ and C is a constant. Equation (4·4) is the same as the Ishimoto-Iida formula[4)] if we assume that the maximum amplitude a_{max} is related to E_{max} by $E_{max} \propto a_{max}^{\kappa''}$ where $\kappa'' = 2$ in the case of elastic matters, but in general case with nonlinear elasticity κ'' should be a parameter.

An important point in the above consideration is that the earth crust is in a critical state just after a big earthquake happened, and as the result there appear some characteristic scaling laws in statistical properties.

The last problem in our interests is to estimate the onset time of the maximum aftershock T_{max} by use of our metaphor model based on Equations (3·3) and (3·4). As we have not yet succeeded to derive any theoretical results regarding the onset time T_{max}, we only show the numerical results briefly. Figure 5 is the cumulative distribution function $Q(x) = \exp\left[\{\log(x/a) - b\}^{-(c-1)}\right]$ of the onset time $P(T_{max})$, which is adjusted by a log-Weibull distribution,

$$P(T_{max}) \propto \frac{1}{T_{max}} \frac{1}{(\log T_{max})^c} L(\log T_{max}), \tag{4·5}$$

where c is a constant parameter $c > 1$ and $L(x)$ is a slowly varying function and T_{max} is large enough. The dominant scaling term in the r.h.s. in Equation (13) is T_{max}^{-1}, and this is consistent with the formula suggested by Utsu[5] in aftershock statistics. The onset time of the maximum aftershock is one of the most crucial problems in seismology, and must be carefully studied in relation to the Omori formula, especially the fluctuations from the mean behavior $N(t) - < N(t) >$ are strongly correlated to the onset of large aftershocks. Furthermore, the onset time T_{max} depend on the magnitude of the initial mainshock. There are so many difficult problems remained in determining the distribution $P(T_{max})$, but we can expect to obtain the enough information about statistical properties of aftershocks if we find out any useful dynamical models. The metaphor model proposed in the present paper is a challenge toward this goal from the recent development in chaotic dynamical system theory[13-15].

§5. A universal relation and intrinsic meanings of the Gutenberg - Richter parameter

Recent studies show that the interoccurrence time distribution $P(\tau)$ is very well fitted by a Weibull distribution in various data catalog (in Japan, Taiwan, and California)[17-21]. On the other hand, the magnitude distribution $P(m)$ obeys the Gutenberg - Richter law [6]. Then two statistical laws can be unified into a universal relation in the following approach, and it is certified by the above mentioned data-catalog [16].

Here we consider the interrelation between the Gutenberg-Richter law, denoted in this subsection $P(m) \propto e^{-bm}$ and the Weibull distribution for the interoccurrence time $(P(\tau) \propto t^{-\alpha-1} \cdot e^{(\tau/\beta)^{\alpha}})$. We assume that these two statistics are correct over wide ranges, and the parameters (α, β) are depending on the magnitude, i.e., $\alpha(m)$ and $\beta(m)$, then the following relation is easily obtained from the calculation of the mean interoccurrence time between two earthquakes whose magnitude is larger than m ,

$$\beta(m_1)e^{-bm_1}\Gamma\left(1+\frac{1}{\alpha_1}\right) = \beta(m_2)e^{-bm_2}\Gamma\left(1+\frac{1}{\alpha_2}\right), \tag{5·1}$$

where m_1 and m_2 are arbitrary values of m. This implies that the quantity defined by $\beta(m_1)e^{-bm_1}\Gamma\left(1+\frac{1}{\alpha_1}\right)$ is a universal constant when we consider the local earthquakes in a relatively small area.

One of the most important results derived from equation (5·1) is that the GR parameter b is determined by two parameters, in other words, the parameters (α, β) depend on the magnitude m as well as on the GR parameter b,

$$\begin{aligned} \alpha &= f_{\alpha}(m, b) \\ \beta &= f_{\beta}(m, b), \end{aligned} \tag{5·2}$$

where the functional forms of f_{α} and f_{β} characterize the time series of earthquakes under consideration.

It is difficult to determine those forms completely from any seismological relations known so far, but it is possible for us to obtain the universal aspects of f_{α} and

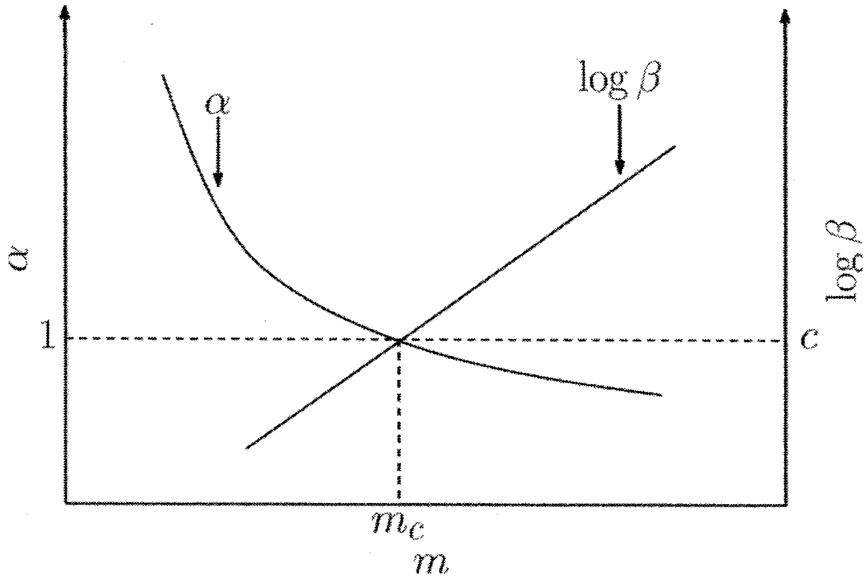

Fig. 6. Schematic picture of universal behavior of $f_\alpha(m,b)$ and $f_\beta(m,b)$ near $m \simeq m_c$.

f_β by a perturbational approach. Here we consider a particular solution of equation (5·2) which satisfies the following conditions; $f_\beta(m,b) = \exp[b(m-m_c)+c]$ and $f_\alpha(m_c,b)=1$, namely, the characteristic time β is a exponentilally increasing function of m, and the interoccurrence time distribution is an exponential one ($\alpha=1$) at $m=m_c$, where b' and c are constant parameters. By use of this simplification, equation (5·1) is rewritten by putting $m_1=m_c$ and $m_2=m$,

$$\begin{aligned}(b'-b)(m-m_c) &= -\log\Gamma\left(1+\frac{1}{\alpha(m)}\right)\\ &\cong \frac{1}{2}\Delta - \frac{3}{4}(\Delta)^2+\cdots,\quad (\Delta=\alpha(m-1)).\end{aligned} \tag{5·3}$$

Here we used the Taylor expansion near $m \cong m_c$ (i.e., $\alpha(m) \cong \alpha(m_c)$). Figure 6 shows the schematic result of equation (5·3). One can see that the universal relation is recognized in many cases [17-21], though the exponential growth of β, $\log\beta(m) \simeq b'(m-m_c)+c$ is a little bit accelerated.

We have to remind that the solution mentioned above is not unique, but many other solutions for equation (5·2) are possible under the universal relation of equation (5·1). Further details will be studied in our forthcoming paper.[16)]

§6. Summary and Discussion

We demonstrated in the present paper that a series of significant empirical laws in seismology are systematically obtained from a simple chaotic dynamics based on the modified Bernoulli map. The essentially important nature of this model is that the stationary-nonstationary chaos transition process is embedded in the model, and that a universal phenomenon is induced in a wide critical regime near the transition point. Actually, the well-known empirical statistical laws in the earthquake statistics ; the formula by Wadati(1932), Omori(1894), Enya(1901), Ishimoto-Iida(1939), and

Utsu(1961), are clearly explained in terms of the critical behaviors of our model. Furthermore, a universal relation between the Gutenberg-Richiter law and the interoccurrence time distribution function is derived theoretically, and it is comfirmed in various earthquake data-Catalog(JMA, SCDEC, and TCMA) as well as in our theoretical model discussed in § 3. The details will be discussed in the following paper[16], and the review article of our studies will appear elsewhere[17,18].

Acknowledgements

I deeply appreciate Prof. Shuichi Tasaki for his sincere encouragement, creative discussions and heartwarming friendship for long time more than twenty years. I pray, with my hands clasped, the soul of my dear friend may rest in peace.

References

1) K. Wadati, Journal of the Meteorological Society of Japan (2) **10** 559 (1932).
2) F. Omori, J. Coll. Sci. Imp. Univ. Tokyo, **7** 111 (1894).
3) M. Enya, Report of the Imperial Earthquake Investigation Committee, **35** 35 (1901).
4) M. Ishimoto and K. Iida, Bull. Earthq. Res. Inst, Univ. Tokyo **17** 443 (1939).
5) T. Utsu, Geophys. Mag., **30** 521 (1961).
6) B. Gutenberg and C. F. Richter, Ann. Geofis. **9** 1 (1956).
7) Y. Aizawa, *Kagaku*, Iwanami publisher, **65** 266 (1995) (in Japanese).
8) Y. Aizawa and T. Kohyama, Prog. Ther. Phys. **71** 847 (1984).
9) Y. Aizawa, Prog. Ther. Phys. **72** 659 (1984).
10) Y. Aizawa, Prog. Ther. Phys. Suppl. **99** 149 (1989).
11) Y. Aizawa, Chaos. Solitons & Fractals **11** 263 (2000).
12) S. Shinkai and Y. Aizawa, Prog. Ther. Phys. **116** 503 (2006).
13) T. Akimoto and Y. Aizawa, Prog. Ther. Phys. **119** 849 (2003).
14) T. Akimoto and Y. Aizawa, Prog. Theor. Phys. **114** 737 (2005).
15) T. Akimoto, T. Hasumi, and Y. Aizawa, Physical Review E, **81** 031133 (2010).
16) Y. Aizawa, T. Hasumi, and S. Tsugawa in preparation (2011).
17) "Earthquake Research and Analysis", (INTECH, 2011).
18) "Earthquakes ; Triggers, Environmental Impact and Potential Hazards", (NOVAPUBLISHERS, 2011).
19) T. Hasumi, Physica A **388** 477 (2009).
20) T. Hasumi, T. Akimoto, and Y. Aizawa, Physica A **388** 483 (2009).
21) T. Hasumi, C. Chen, T. Akimoto, and Y. Aizawa, Techtonophys, **485** 9 (2010).