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Periodic solutions of the model equation
describing electrodynamics of the Josephson junction

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Abstract
A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are obtained in the form of parametric representation. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions. Last, the exact method is applied to the sine-Hilbert equation to obtain periodic solutions. The detail of this report has been published in J. Phys. A: Math. Theor. 42 (2009) 025401.

1. Model equation
1.1 Nonlocal model equation
Consider a Josephson junction with a thin layer between two superconductors. The phase difference $\phi(x, t)$ across the Josephson junction is described by the following model equation:

$$\omega_J^{-2}\phi_{tt} + \omega_J^{-2}\eta\phi_t = -\sin \phi + \frac{\lambda_J^2}{\pi \lambda_L} \int_{-\infty}^{\infty} K_0 \left( \frac{|x - x'|}{\lambda_L} \right) \phi_{xx}(x', t) dx' + \gamma. \quad (1)$$

$K_0$: modified Bessel function of order zero, $\omega_J$: Josephson plasma frequency, $\lambda_L$: London penetration depth, $\lambda_J$: Josephson penetration depth, $\gamma$: bias current density across the junction, $\eta$: positive parameter characterizing the resistance of a unit area of the tunneling junction.

Let $l$ be the characteristic space scale of $\phi$. When $\lambda_L << l$, then $K_0(x) \sim \pi \delta(x)$ and Eq. (1) reduces to the perturbed sine-Gordon equation

$$\omega_J^{-2}\phi_{tt} + \omega_J^{-2}\eta\phi_t = -\sin \phi + \frac{\lambda_J^2}{\lambda_L} \phi_{xx} + \gamma. \quad (2)$$
If $l \ll \lambda_L$, then $K_0(|x|) \sim -\ln |x|$ and Eq. (1) becomes
\[
\omega_j^{-2}\phi_{tt} + \omega_j^{-2}\eta\phi_t = -\sin \phi + \frac{\lambda_L^2}{\pi \lambda_L} \int_{-\infty}^{\infty} \frac{\phi_x(x', t)}{x' - x} \, dx' + \gamma. \tag{3}
\]
In the following, we consider the overdamped case $\eta \gg 1$ and the zero bias current $\gamma = 0$. Eq. (3) can then be written in an appropriate dimensionless form as
\[
\phi_t = -\sin \phi + H\phi_x, \quad H\phi_x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi_x(x', t)}{x' - x} \, dx'. \tag{4}
\]

1.2 Remarks

- Equation (1) is derived from Maxwell’s equations combined with the London equation and the Josephson equation:
- Equation (4) has been proposed for the first time in a purely mathematical context:
- As for a review on nonlocal Josephson electrodynamics:
  - A.A. Abdumalikov et al, Superconductor Science and Technology, 22 (2009) 023001

2. Exact method of solution

2.1 A nonlinear dynamical system

- Dependent variable transformation

  We seek periodic solution of (4) of the form
  \[
  \phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j), \tag{5}
  \]
  where $x_j = x_j(t)$ are complex functions of $t$ with $\text{Im} x_j(t) > 0$, $\beta$ is a positive parameter, $N$ is an arbitrary positive integer and $f^*$ denotes the complex conjugate expression of $f$. Using a formula for the Hilbert transform, one has $H\phi_x = -(\ln f^* f)$. Substitution of this expression and (5) into (4) gives the following bilinear equation for $f$ and $f^*$
  \[
  i(f_t^* f - f^* f_t) = \frac{i}{2} (f^2 - f^{*2}) - f_x^* f - f^* f_x. \tag{6}
  \]
A system of nonlinear ODEs for $x_j$

We divide (6) by $f^*f$, substitute $f$ from (5) and then evaluate the residue at $x = x_j$ on both sides. This gives a system of nonlinear ODEs for $x_j$

$$
\dot{x}_j = -\frac{1}{2\beta} \prod_{l=1}^{N} \sin \beta(x_j - x_l^*) + i, \quad j = 1, 2, ..., N,
$$

(7)

where an overdot denotes differentiation with respect to $t$.

We introduce the following notations:

$$
z = e^{2i\beta x}, \quad \xi_j = e^{2i\beta x_j}, \quad \eta_j = e^{2i\beta x_j^*}, \quad j = 1, 2, ..., N,
$$

(8a)

$$
s_1 = \sum_{j=1}^{N} x_j, \quad s_2 = \sum_{j<l}^{N} x_j x_l, \quad ..., \quad s_N = \prod_{j=1}^{N} x_j,
$$

(8b)

$$
u_1 = \sum_{j=1}^{N} \xi_j, \quad u_2 = \sum_{j<l}^{N} \xi_j \xi_l, \quad ..., \quad u_N = \prod_{j=1}^{N} \xi_j,
$$

(8c)

$$
v_1 = \sum_{j=1}^{N} \eta_j, \quad v_2 = \sum_{j<l}^{N} \eta_j \eta_l, \quad ..., \quad v_N = \prod_{j=1}^{N} \eta_j,
$$

(8d)

$$
t_j = \sum_{l=1}^{N} \xi_l^j, \quad j = 1, 2, ..., N.
$$

(8e)

In terms of $u_j (j = 1, 2, ..., N)$ and $s_1$, $f$ can be written as

$$
f = \frac{e^{-i\beta(Nz-s_1)}}{(2\beta i)^N} \left(z^N - u_1 z^{N-1} + u_2 z^{N-2} + ... + (-1)^N u_N \right).
$$

(9)

Thus, $u_j (j = 1, 2, ..., N)$ and $s_1$ determine the function $f$ completely.

Let us derive a system of equations for $u_j$. To this end, We rewrite (7) in terms of $\xi_j$ and $\eta_j$ as

$$
\dot{\xi}_j = -\frac{1}{2\alpha u_N} \frac{\prod_{l=1}^{N} (\xi_j - \eta_l)}{\prod_{l \neq j}^{N} (\xi_j - \xi_l)} - 2\beta \xi_j, \quad j = 1, 2, ..., N,
$$

(10a)

where

$$
\alpha = \prod_{j=1}^{N} (\xi_j \eta_j)^{-1/2} = e^{-i\beta (s_1 + s_1^*)}, \quad u_N = \prod_{j=1}^{N} \xi_j = e^{2i\beta s_1}.
$$

(10b)

Later, we show that $\alpha$ is a constant independent of $t$ and $u_N$ obeys a single nonlinear ODE.
2.2 Linearization

The system of nonlinear ODEs (10) can be linearized in terms of the variables \( u_j \) defined by (8c). We multiply \( \xi_j^{n-1} \) on both sides of (10a) and sum up with respect to \( j \) from 1 to \( N \) to obtain

\[
\frac{1}{n} \dot{t}_n = -\frac{\alpha}{2} u_N \sum_{s=0}^{n} (-1)^s v_s I_{n-s} - 2\beta t_n, \quad n = 1, 2, \ldots, N, \tag{11a}
\]

where \( I_{n-s} \) is defined by

\[
I_{n-s} = \sum_{j=1}^{N} \frac{\xi_j^{N-n-s-1}}{\prod_{i=1}^{N} (\xi_j - \xi_i)}. \tag{11b}
\]

In deriving (11), we have used the identity

\[
I_n = 0, \quad -N + 1 \leq n \leq -1. \tag{11c}
\]

- Time evolution of \( u_j \)

The time evolution of \( u_n \) follows from (11a) with the help of the formulas

\[
u_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j u_j t_{n-j}, \quad 1 \leq n \leq N, \quad \sum_{j=0}^{n} (-1)^j u_j I_{n-j} = 0, \quad n \geq 1, \tag{12}
\]

where \( u_0 = 1 \) and \( I_0 = 1 \). In fact, differentiating the first formula in (12) by \( t \) and substituting (11a) for \( \dot{t}_{n-j} \), we can show that the quantity \( h_n \) defined by

\[
h_n = \dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{N-n} + 2\beta nu_n, \quad n = 1, 2, \ldots, N, \tag{13}
\]

satisfies the relation

\[
h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j} + \frac{(-1)^{n+1} r_n}{2n\alpha}, \tag{14a}
\]

where

\[
r_n = \sum_{j=1}^{n} u_{N-j+n} \left[ -\sum_{s=1}^{j} (-1)^{n-s} s I_{j-s} + (-1)^{n-j} t_j \right]. \tag{14b}
\]

The quantity in the brackets on the right-hand side of (14b) can be shown to vanish identically so that \( r_n \equiv 0 \). It follows from this and (14a) that

\[
h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j}, \quad n = 1, 2, \ldots, N. \tag{15}
\]
Solving (15) with the initial condition $h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0$, we obtain the relations $h_n \equiv 0 \ (n = 1, 2, \ldots, N)$. Thus, we see that $u_n$ evolves according to the following system of ODEs

$$
\dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{n-n}^* + 2\beta_n u_n = 0, \quad n = 1, 2, \ldots, N. \tag{16}
$$

It is remarkable that $u_N$ obeys a single nonlinear ODE of the form

$$
\dot{u}_N + \frac{\alpha}{2} u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \quad u_N = e^{2i\beta s_1}, \quad \alpha = \sqrt{\frac{u_N^*}{u_N}}, \tag{17}
$$

and other $N - 1$ variables $u_1, u_2, \ldots, u_{N-1}$ constitute a system of linear ODEs. Rewriting (17) in terms of $s_1$, we can put it into a nonlinear ODE for $s_1$

$$
\dot{s}_1 = \frac{1}{2i\beta} \sinh(2\beta \text{Im} s_1) + iN, \tag{18}
$$

where $\text{Im} s_1$ implies the imaginary part of $s_1$.

3. Periodic solutions

3.1 Construction of periodic solutions

The first step for constructing periodic solutions is to integrate (18). It follows from the real and imaginary parts of (18) that

$$
\text{Re} \dot{s}_1 = 0, \quad \text{Im} \dot{s}_1 = -\frac{1}{2\beta} \sinh(2\beta \text{Im} s_1) + N. \tag{19}
$$

Thus, the real part of $s_1$ becomes a constant $\text{Re} s_1(t) = \text{Re} s_1(0) \equiv b$ whereas integration of the equation for $\text{Im} s_1$ yields an explicit expression. In terms of a new variable $y = 2\beta \text{Im} s_1$, it is given by

$$
e^{-y} = \frac{2\nu_N \left(-\tanh \frac{y_0}{2} + 1\right) \cosh \nu_N t + \left\{(2\beta N + 1) \tanh \frac{y_0}{2} - 2\beta N + 1\right\} \sinh \nu_N t}{2\nu_N \left(\tanh \frac{y_0}{2} + 1\right) \cosh \nu_N t + \left\{(2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1\right\} \sinh \nu_N t}, \tag{20}
$$

where $\nu_N = \sqrt{(\beta N)^2 + (1/4)}$ and $y_0 = y(0) = 2\beta \text{Im} s_1(0)$. For $n = 1, 2, \ldots, N - 1$, on the other hand, (16) can be written in the form

$$
\dot{u}_n = -\left(\frac{1}{2} e^{-2\beta \text{Im} s_1} + 2\beta n\right) u_n + \frac{\alpha^{-1}}{2} u_{n-n}^*. \tag{21}
$$

Note from (10b) and $\text{Re} s_1 = b$ that $\alpha = e^{-2i\beta b}$ becomes a constant. The solution of the initial value problem for (21) can be put into the form of a rational function

$$
u_n(t) = \frac{G_n}{F}, \quad n = 1, 2, \ldots, N - 1, \tag{22a}$$
with
\[ F = 2\nu_N \left( \tanh \frac{y_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1 \right\} \sinh \nu_N t, \]
\hspace{1cm} (22b)

\[ G_n = 2\nu_N \left( \tanh \frac{y_0}{2} + 1 \right) \left[ u_n(0) \cosh \nu_n t + \frac{1}{\nu_n} \left\{ \beta(N - 2n)u_n(0) + \frac{1}{2} u_{N-n}^*(0) \right\} \sinh \nu_n t \right], \]
\hspace{1cm} (22c)

where \( \nu_n = \sqrt{\beta^2(N - 2n)^2 + (1/4)} \). We see that the expression (22) with \( n = N \) produces (20) and hence it can be used for all \( u_n \).

3.2 Properties of solutions

- Asymptotic form of the solution as \( t \to \infty \)

\[ u_n \to 0, \quad n = 1, 2, \ldots, N - 1, \quad u_N \to e^{2i\beta b} \left( \sqrt{4(\beta N)^2 + 1} - 2\beta N \right), \]
\hspace{1cm} (23)

\[ \phi \sim 2 \tan^{-1} \left[ \frac{\sqrt{4(\beta N)^2 + 1} - 1}{2\beta N} \tan \beta \left( N x - b - \frac{N\pi}{2\beta} \right) \right], \]
\hspace{1cm} (24)

\[ u \equiv \phi_x \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2 + 1} + (-1)^N \cos 2\beta(N x - b)}. \]
\hspace{1cm} (25)

- Novel features of solutions

1) The asymptotic form of \( u \) does not depend on initial conditions except for a phase constant \( b \). It represents a train of nonlinear periodic standing waves.

2) The initial profile of \( u \) with a spatial period \( \pi/\beta \) evolves into a periodic wave with a period \( \pi/N\beta \).

3) The amplitude of the wave \( A(=u_{\text{max}} - u_{\text{min}}) \) is a constant independent of the wavenumber. Indeed, \( u_{\text{max}} = \sqrt{4(\beta N)^2 + 1} + 1 \), \( u_{\text{min}} = \sqrt{4(\beta N)^2 + 1} - 1 \) and hence \( A = 2 \).

4) The steady profile (25) satisfies the Peierls equation \( H\phi_x = \sin \phi \) in the theory of dislocation

Example 1: $N = 1, \ x_1(0) = 3i, \ \beta = 0.2$

![Graph](image1.png)

**Figure 1.** Time evolution of $u$ for $N = 1$ (periodic case).

Example 2: $N = 2, \ x_1(0) = 4i, \ x_2(0) = 2i, \ \beta = 0.4$

![Graph](image2.png)

**Figure 2.** Time evolution of $u$ for $N = 2$ (periodic case).

### 3.3 Long-wave limit $\beta \to 0$

The long-wave limit $\beta \to 0$ of the periodic solutions can be derived easily. We quote the results:

$$
\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} (x - x_j) = \sum_{j=0}^{N} s_j(t)x^{N-j}, \ (s_0 = 1), \quad (26)
$$

$$
\delta_j = -i \text{Im} \ s_j + i(N - j + 1)s_{j-1}, \quad j = 1, 2, ..., N. \quad (27)
$$
For $N = 2$, the solution reads as follows:

\[ f = x^2 - s_1 x + s_2, \quad (28a) \]
\[ s_1 = b_1 + i[-(a_1 - 2)(1 - e^{-t}) + a_1], \quad (28b) \]
\[ s_2 = -2t - (a_1 - 2)(1 - e^{-t}) + b_2 + i[-(a_2 - b_1)(1 - e^{-t}) + a_2]. \quad (28c) \]

The large time asymptotic of the solution $u \equiv \phi_x$ is given by a superposition of $N$ Lorentzian pulses

\[ u \sim \sum_{j=1}^{N} \frac{2}{(x - \sqrt{2t \cdot x_{j,N}})^2 + 1}, \quad (29) \]

where $x_{j,N}$ is the $n$th root of the Hermite polynomial of order $N$. These results have been detailed in Matsuno (1992).

**Example 1**: Nonperiodic case $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$

![Figure 3. Time evolution of $u$ for $N = 2$ (nonperiodic case).](image)

4. **Application**

The exact method of solution developed so far can be applied to obtain periodic solutions of the sine-Hilbert (sH) equation

\[ H \theta_t = -\sin \theta, \quad \theta = \theta(x,t). \quad (30) \]

4.1 **Remark**

- The sH equation was introduced by Degasperis and Santini in a purely mathematical context:
  
The reduction to a Riemann-Hilbert scattering problem was given by Degasperis et al:

An exact method of solution by means of bilinear transformation method was developed by Matsuno:

4.2 Periodic solutions
Here, we summarize the procedure for constructing periodic solutions of the sH equation. We seek periodic solutions of the form (5)

$$\theta = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin(\beta(x - x_j)), \quad (31)$$

The corresponding bilinear equation for $f$ is given by

$$(f^* f)_t = \frac{1}{2}(f^2 - f^{*2}). \quad (32)$$

The system of equations for $x_j$ becomes

$$\dot{x}_j = \frac{1}{2i\beta} \frac{1}{\prod_{l=1}^{N} \sin(\beta(x_j - x_l))} \frac{\prod_{l=1}^{N} \sin(\beta(x_j - x_l))}{\prod_{l=1}^{N} \sin(\beta(x_j - x_l))}, \quad j = 1, 2, ..., N, \quad (33)$$

and $u_j$ satisfies the system of equations

$$\dot{u}_j = i \left( -\frac{c}{2} u_N u_j + \frac{1}{2c} u_{N-j}^* \right), \quad c = \sqrt{\frac{u_N^*}{u_N}}, \quad j = 1, 2, ..., N. \quad (34)$$

The above system can be solved analytically and solutions are given explicitly.

**Example:** $N = 1$
Substituting $u_1 = e^{2i\beta s_1}$ into (34)

$$\dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1), \quad (35a)$$

$$\text{Re} \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1), \quad \text{Im} \dot{s}_1 = 0, \quad (35b)$$

$$x_1 = s_1 = at + b + i \frac{1}{2\beta} \sinh^{-1}(2\beta a), \quad a = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1) \quad b = \text{Res}_1(0), \quad (35c)$$

$$u \equiv \theta_x = \frac{4\beta^2 a}{\sqrt{1 + 4\beta^2 a^2} - \cos 2\beta(x - at - b)}. \quad (36)$$

Note that the solution is not a standing wave but a traveling wave.
5. Summary

- We have constructed periodic solutions of a resistive model describing Josephson electrodynamics by means of a novel linearization procedure.

- The large time asymptotic of the periodic solution has a steady profile which is formed by a balance between nonlinearity and dissipation. This feature is in striking contrast to periodic solutions of nonlinear dispersive wave equations.

- The exact method of solution developed here was applied to the sine-Hilbert equation to obtain periodic solutions.