<table>
<thead>
<tr>
<th>Title</th>
<th>Periodic solutions of the model equation describing electrodynamics of the Josephson junction (Mathematical Physics and Application of Nonlinear Wave Phenomena)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Matsuno, Yoshimasa</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2010), 1701: 25-34</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/169995">http://hdl.handle.net/2433/169995</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Periodic solutions of the model equation

describing electrodynamics of the Josephson junction

Yoshimasa Matsuno
Division of Applied Mathematical Science
Graduate School of Science and Engineering
Yamaguchi University

Abstract

A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are obtained in the form of parametric representation. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions. Last, the exact method is applied to the sine-Hilbert equation to obtain periodic solutions. The detail of this report has been published in J. Phys. A: Math. Theor. 42 (2009) 025401.

1. Model equation
1.1 Nonlocal model equation

Consider a Josephson junction with a thin layer between two superconductors. The phase difference $\phi(x, t)$ across the Josephson junction is described by the following model equation:

$$\omega J^2 \phi_{tt} + \omega J^2 \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\pi \lambda_L} \int_{-\infty}^{\infty} K_0 \left( \frac{|x - x'|}{\lambda_L} \right) \phi_{x'x'}(x', t) dx' + \gamma.$$  \hspace{1cm} (1)

$K_0$: modified Bessel function of order zero, $\omega J$: Josephson plasma frequency, $\lambda_L$: London penetration depth, $\lambda_J$: Josephson penetration depth, $\gamma$: bias current density across the junction, $\eta$: positive parameter characterizing the resistance of a unit area of the tunneling junction.

Let $l$ be the characteristic space scale of $\phi$. When $\lambda_L << l$, then $K_0(x) \sim \pi \delta(x)$ and Eq. (1) reduces to the perturbed sine-Gordon equation

$$\omega J^2 \phi_{tt} + \omega J^2 \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\lambda_L} \phi_{xx} + \gamma.$$  \hspace{1cm} (2)
If $l \ll \lambda_L$, then $K_0(|x|) \sim -\ln |x|$ and Eq. (1) becomes
\[
\omega_j^2 \phi_{tt} + \omega_j^2 \eta \phi_t = -\sin \phi + \frac{\lambda_j^2}{\pi \lambda_L} \int_{-\infty}^{\infty} \frac{\phi_{x'}(x',t)}{x' - x} \, dx' + \gamma. \tag{3}
\]

In the following, we consider the overdamped case $\eta \gg 1$ and the zero bias current $\gamma = 0$. Eq. (3) can then be written in an appropriate dimensionless form as

\[
\phi_t = -\sin \phi + H \phi_x, \quad H \phi_x = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_{x'}(x',t)}{x' - x} \, dx'. \tag{4}
\]

1.2 Remarks
- Equation (1) is derived from Maxwell’s equations combined with the London equation and the Josephson equation:
- Equation (4) has been proposed for the first time in a purely mathematical context:
- As for a review on nonlocal Josephson electrodynamics:
  A.A. Abdumalikov et al, Superconductor Science and Technology, 22 (2009) 023001

2. Exact method of solution

2.1 A nonlinear dynamical system
- Dependent variable transformation

We seek periodic solution of (4) of the form
\[
\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j), \tag{5}
\]
where $x_j = x_j(t)$ are complex functions of $t$ with $\text{Im } x_j(t) > 0$, $\beta$ is a positive parameter, $N$ is an arbitrary positive integer and $f^*$ denotes the complex conjugate expression of $f$. Using a formula for the Hilbert transform, one has $H \phi_x = -(\ln f^*f)_x$. Substitution of this expression and (5) into (4) gives the following bilinear equation for $f$ and $f^*$
\[
i(f_t f - f^* f_t) = \frac{i}{2} (f^2 - f^{*2}) - f^*_x f - f^* f_x. \tag{6}
\]
• A system of nonlinear ODEs for $x_j$

We divide (6) by $f^*f$, substitute $f$ from (5) and then evaluate the residue at $x = x_j$ on both sides. This gives a system of nonlinear ODEs for $x_j$

$$
\dot{x}_j = -\frac{1}{2\beta} \prod_{l=1}^{N} \sin \beta(x_j - x_l^*) + i, \quad j = 1, 2, ..., N,
$$

(7)

where an overdot denotes differentiation with respect to $t$.

We introduce the following notations:

$$
z = e^{2i\beta x}, \quad \xi_j = e^{2i\beta x_j}, \quad \eta_j = e^{2i\beta x_j^*}, \quad j = 1, 2, ..., N,
$$

(8a)

$$
s_1 = \sum_{j=1}^{N} x_j, \quad s_2 = \sum_{j<l}^{N} x_j x_l, \quad ..., \quad s_N = \prod_{j=1}^{N} x_j,
$$

(8b)

$$
u_1 = \sum_{j=1}^{N} \xi_j, \quad u_2 = \sum_{j<l}^{N} \xi_j \xi_l, \quad ..., \quad u_N = \prod_{j=1}^{N} \xi_j,
$$

(8c)

$$
v_1 = \sum_{j=1}^{N} \eta_j, \quad v_2 = \sum_{j<l}^{N} \eta_j \eta_l, \quad ..., \quad v_N = \prod_{j=1}^{N} \eta_j,
$$

(8d)

$$
t_j = \sum_{i=1}^{N} \xi_i^j, \quad j = 1, 2, ..., N.
$$

(8e)

In terms of $u_j (j = 1, 2, ..., N)$ and $s_1$, $f$ can be written as

$$
f = \frac{e^{-i\beta(Nz - s_1)}}{(2\beta i)^N} \left(z^N - u_1 z^{N-1} + u_2 z^{N-2} + ... + (-1)^N u_N \right).
$$

(9)

Thus, $u_j (j = 1, 2, ..., N)$ and $s_1$ determine the function $f$ completely.

Let us derive a system of equations for $u_j$. To this end, We rewrite (7) in terms of $\xi_j$ and $\eta_j$ as

$$
\dot{\xi}_j = -\frac{1}{2\alpha u_N} \prod_{l=1}^{N} (\xi_j - \eta_l) \prod_{(i \neq j)}^{N} (\xi_j - \xi_l) - 2\beta \xi_j, \quad j = 1, 2, ..., N,
$$

(10a)

where

$$
\alpha = \prod_{j=1}^{N} (\xi_j \eta_j)^{-1/2} = e^{-i\beta(s_1 + s_1^*)}, \quad u_N = \prod_{j=1}^{N} \xi_j = e^{2i\beta s_1}.
$$

(10b)

Later, we show that $\alpha$ is a constant independent of $t$ and $u_N$ obeys a single nonlinear ODE.
2.2 Linearization

The system of nonlinear ODEs (10) can be linearized in terms of the variables \(u_j\) defined by (8c). We multiply \(\xi_j^{n-1}\) on both sides of (10a) and sum up with respect to \(j\) from 1 to \(N\) to obtain

\[
\frac{1}{n} \dot{t}_n = -\frac{\alpha}{2} u_N \sum_{s=0}^{n} (-1)^s v_s I_{n-s} - 2\beta t_n, \quad n = 1, 2, \ldots, N, \tag{11a}
\]

where \(I_{n-s}\) is defined by

\[
I_{n-s} = \sum_{j=1}^{N} \frac{\xi_j^{N+n-s-1}}{\prod_{(i\neq j)} (\xi_j - \xi_i)}. \tag{11b}
\]

In deriving (11), we have used the identity

\[
I_n = 0, \quad -N + 1 \leq n \leq -1. \tag{11c}
\]

- Time evolution of \(u_j\)

The time evolution of \(u_n\) follows from (11a) with the help of the formulas

\[
u_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j u_j t_{n-j}, \quad 1 \leq n \leq N, \quad \sum_{j=0}^{n} (-1)^j u_j I_{n-j} = 0, \quad n \geq 1, \tag{12}\]

where \(u_0 = 1\) and \(I_0 = 1\). In fact, differentiating the first formula in (12) by \(t\) and substituting (11a) for \(\dot{t}_{n-j}\), we can show that the quantity \(h_n\) defined by

\[
h_n = \dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u^2_{N-n} + 2\beta n u_n, \quad n = 1, 2, \ldots, N, \tag{13}\]

satisfies the relation

\[
h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j} + \frac{(-1)^{n+1} r_n}{2n\alpha}, \tag{14a}
\]

where

\[
r_n = \sum_{j=1}^{n} u_{N-j+n} \left[ - \sum_{s=1}^{j} (-1)^{n-s} s I_{j-s} + (-1)^{n-j} t_j \right]. \tag{14b}\]

The quantity in the brackets on the right-hand side of (14b) can be shown to vanish identically so that \(r_n \equiv 0\). It follows from this and (14a) that

\[
h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j}, \quad n = 1, 2, \ldots, N. \tag{15}\]
Solving (15) with the initial condition $h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0$, we obtain the relations $h_n \equiv 0 \ (n = 1, 2, ..., N)$. Thus, we see that $u_n$ evolves according to the following system of ODEs

$$\dot{u}_n + \frac{\alpha}{2} u_n u_n - \frac{\alpha^{-1}}{2} u_{n-n}^- + 2\beta n u_n = 0, \quad n = 1, 2, ..., N. \quad (16)$$

It is remarkable that $u_N$ obeys a single nonlinear ODE of the form

$$\dot{u}_N + \frac{\alpha}{2} u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \quad u_N = e^{2i\beta s_1}, \quad \alpha = \frac{u_N^*}{u_N}, \quad (17)$$

and other $N - 1$ variables $u_1, u_2, ..., u_{N-1}$ constitute a system of linear ODEs. Rewriting (17) in terms of $s_1$, we can put it into a nonlinear ODE for $s_1$

$$\dot{s}_1 = \frac{1}{2i\beta} \sinh(2\beta \text{Im} s_1) + iN, \quad (18)$$

where $\text{Im} s_1$ implies the imaginary part of $s_1$.

3. Periodic solutions

3.1 Construction of periodic solutions

The first step for constructing periodic solutions is to integrate (18). It follows from the real and imaginary parts of (18) that

$$\text{Re} \dot{s}_1 = 0, \quad \text{Im} \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1) + N. \quad (19)$$

Thus, the real part of $s_1$ becomes a constant $\text{Re} s_1(t) = \text{Re} s_1(0) \equiv b$ whereas integration of the equation for $\text{Im} s_1$ yields an explicit expression. In terms of a new variable $y = 2\beta \text{Im} s_1$, it is given by

$$e^{-y} = \frac{2\nu_N \left( - \tanh \frac{\nu_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N + 1) \tanh \frac{\nu_0}{2} - 2\beta N + 1 \right\} \sinh \nu_N t}{2\nu_N \left( \tanh \frac{\nu_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{\nu_0}{2} + 2\beta N + 1 \right\} \sinh \nu_N t}, \quad (20)$$

where $\nu_N = \sqrt{(\beta N)^2 + (1/4)}$ and $y_0 = y(0) = 2\beta \text{Im} s_1(0)$. For $n = 1, 2, ..., N - 1$, on the other hand, (16) can be written in the form

$$\dot{u}_n = -\left( \frac{1}{2} e^{-2\beta \text{Im} s_1} + 2\beta n \right) u_n + \frac{\alpha^{-1}}{2} u_{n-n}^*. \quad (21)$$

Note from (10b) and $\text{Re} s_1 = b$ that $\alpha = e^{-2i\beta b}$ becomes a constant. The solution of the initial value problem for (21) can be put into the form of a rational function

$$u_n(t) = \frac{G_n}{F}, \quad n = 1, 2, ..., N - 1, \quad (22a)$$
with
\[ F = 2\nu_N \left( \tanh \frac{y}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{y}{2} + 2\beta N + 1 \right\} \sinh \nu_N t, \]

(22b)

\[ G_n = 2\nu_N \left( \tanh \frac{y}{2} + 1 \right) \left[ u_n(0) \cosh \nu_n t + \frac{1}{\nu_n} \left\{ \beta(N - 2n)u_n(0) + \frac{\alpha^{-1}}{2} u_{n-n}^*(0) \right\} \sinh \nu_n t \right], \]

(22c)

where \( \nu_n = \sqrt{\beta^2(N - 2n)^2 + (1/4)} \). We see that the expression (22) with \( n = N \) produces (20) and hence it can be used for all \( u_n \).

3.2 Properties of solutions

• Asymptotic form of the solution as \( t \to \infty \)

\[ u_n \to 0, \quad n = 1, 2, ..., N - 1, \quad u_N \to e^{2i\beta b}(\sqrt{4(\beta N)^2 + 1} - 2\beta N), \]

(23)

\[ \phi \sim 2 \tan^{-1} \left[ \frac{\sqrt{4(\beta N)^2 + 1} - 1}{2\beta N} \tan \beta \left( N x - b - \frac{N\pi}{2\beta} \right) \right], \]

(24)

\[ u \equiv \phi_x \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2 + 1} + (-1)^N \cos 2\beta(N x - b)}. \]

(25)

• Novel features of solutions

1) The asymptotic form of \( u \) does not depend on initial conditions except for a phase constant \( b \). It represents a train of nonlinear periodic standing waves.

2) The initial profile of \( u \) with a spatial period \( \pi/\beta \) evolves into a periodic wave with a period \( \pi/N\beta \).

3) The amplitude of the wave \( A(= u_{\text{max}} - u_{\text{min}}) \) is a constant independent of the wavenumber. Indeed, \( u_{\text{max}} = \sqrt{4(\beta N)^2 + 1} + 1, u_{\text{min}} = \sqrt{4(\beta N)^2 + 1} - 1 \) and hence \( A = 2 \).

4) The steady profile (25) satisfies the Peierls equation \( H \phi_x = \sin \phi \) in the theory of dislocation.

Example 1: $N = 1$, $x_1(0) = 3i$, $\beta = 0.2$

Figure 1. Time evolution of $u$ for $N = 1$ (periodic case).

Example 2: $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$, $\beta = 0.4$

Figure 2. Time evolution of $u$ for $N = 2$ (periodic case).

3.3 Long-wave limit $\beta \to 0$

The long-wave limit $\beta \to 0$ of the periodic solutions can be derived easily. We quote the results:

$$
\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} (x - x_j) = \sum_{j=0}^{N} s_j(t)x^{N-j}, \quad (s_0 = 1), \quad (26)
$$

$$
\dot{s}_j = -i \text{Im} s_j + i(N - j + 1)s_{j-1}, \quad j = 1, 2, \ldots, N. \quad (27)
$$
For $N = 2$, the solution reads as follows:

\[ f = x^2 - s_1 x + s_2, \]  
\[ s_1 = b_1 + i[-(a_1 - 2)(1 - e^{-t}) + a_1], \]  
\[ s_2 = -2t - (a_1 - 2)(1 - e^{-t}) + b_2 + i[-(a_2 - b_1)(1 - e^{-t}) + a_2]. \]

The large time asymptotic of the solution $u \equiv \phi_x$ is given by a superposition of $N$ Lorentzian pulses

\[ u \sim \sum_{j=1}^{N} \frac{2}{(x - \sqrt{2t \ x_{j,N}})^2 + 1}, \]  

where $x_{j,N}$ is the $n$th root of the Hermite polynomial of order $N$. These results have been detailed in Matsuno (1992).

**Example 1:** Nonperiodic case $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$

![Figure 3. Time evolution of $u$ for $N = 2$ (nonperiodic case).](image)

4. Application

The exact method of solution developed so far can be applied to obtain periodic solutions of the sine-Hilbert ($sH$) equation

\[ H \theta_t = -\sin \theta, \quad \theta = \theta(x, t). \]  

4.1 Remark

- The $sH$ equation was introduced by Degasperis and Santini in a purely mathematical context:
  
• The reduction to a Riemann-Hilbert scattering problem was given by Degasperis et al:
• An exact method of solution by means of biliinear transformation method was developed by Matsuno:

4.2 Periodic solutions

Here, we summarize the procedure for constructing periodic solutions of the sH equation. We seek periodic solutions of the form (5)

$$\theta = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j),$$  \hspace{1cm} (31)

The corresponding bilinear equation for $f$ is given by

$$(f^*f)_t = \frac{1}{2}(f^2 - f^{*2}).$$  \hspace{1cm} (32)

The system of equations for $x_j$ becomes

$$\dot{x}_j = \frac{1}{2i\beta} \frac{\prod_{l=1}^{N} \sin \beta(x_j - x_j^*)}{\prod_{(l\neq j)}^{N} \sin \beta(x_j - x_l)}, \quad j = 1, 2, \ldots, N,$$  \hspace{1cm} (33)

and $u_j$ satisfies the system of equations

$$\dot{u}_j = i \left( -\frac{c}{2} u_N u_j + \frac{1}{2c} u_{N-j}^* \right), \quad c = \sqrt{\frac{u_N^*}{u_N}}, \quad j = 1, 2, \ldots, N.$$  \hspace{1cm} (34)

The above system can be solved analytically and solutions are given explicitly.

**Example:** $N = 1$

Substituting $u_1 = e^{2i\beta s_1}$ into (34)

$$\dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1),$$  \hspace{1cm} (35a)

$$\text{Re} \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1), \quad \text{Im} \dot{s}_1 = 0,$$  \hspace{1cm} (35b)

$$x_1 = s_1 = at + b + \frac{1}{2\beta} \sinh^{-1}(2\beta a), \quad a = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1) \quad b = \text{Res}_1(0),$$  \hspace{1cm} (35c)

$$u \equiv \theta_x = \frac{4\beta^2 a}{\sqrt{1 + 4\beta^2 a^2} - \cos 2\beta(x - at - b)}.$$  \hspace{1cm} (36)

Note that the solution is not a standing wave but a traveling wave.
5. Summary

• We have constructed periodic solutions of a resistive model describing Josephson electrodynamics by means of a novel linearization procedure.

• The large time asymptotic of the periodic solution has a steady profile which is formed by a balance between nonlinearity and dissipation. This feature is in striking contrast to periodic solutions of nonlinear dispersive wave equations.

• The exact method of solution developed here was applied to the sine-Hilbert equation to obtain periodic solutions.