Title
Periodic solutions of the model equation describing electrodynamics of the Josephson junction (Mathematical Physics and Application of Nonlinear Wave Phenomena)

Author(s)
Matsuno, Yoshimasa

Citation
数理解析研究所講究録 数理科学 2010, 1701: 25-34

URL
http://hdl.handle.net/2433/169995

Right

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Abstract

A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are obtained in the form of parametric representation. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions. Last, the exact method is applied to the sine-Hilbert equation to obtain periodic solutions. The detail of this report has been published in J. Phys. A: Math. Theor. 42 (2009) 025401.

1. Model equation

1.1 Nonlocal model equation

Consider a Josephson junction with a thin layer between two superconductors. The phase difference $\phi(x, t)$ across the Josephson junction is described by the following model equation:

$$\omega_j^2 \phi_{tt} + \omega_j^2 \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\pi \lambda_L} \int_{-\infty}^{\infty} K_0 \left( \frac{|x-x'|}{\lambda_L} \right) \phi_{x'x'}(x', t) dx' + \gamma. \quad (1)$$

$K_0$: modified Bessel function of order zero, $\omega_j$: Josephson plasma frequency, $\lambda_L$: London penetration depth, $\lambda_J$: Josephson penetration depth, $\gamma$: bias current density across the junction, $\eta$: positive parameter characterizing the resistance of a unit area of the tunneling junction.

Let $l$ be the characteristic space scale of $\phi$. When $\lambda_L << l$, then $K_0(x) \sim \pi \delta(x)$ and Eq. (1) reduces to the perturbed sine-Gordon equation

$$\omega_j^2 \phi_{tt} + \omega_j^2 \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\lambda_L} \phi_{xx} + \gamma. \quad (2)$$
If \( l << \lambda_L \), then \( K(|x|) \sim -\ln |x| \) and Eq. (1) becomes
\[
\omega_J^{-2} \phi_{tt} + \omega_J^{-2} \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\pi \lambda_L} \int_{-\infty}^{\infty} \frac{\phi(x',t)}{x' - x} dx' + \gamma.
\] (3)

In the following, we consider the overdamped case \( \eta >> 1 \) and the zero bias current \( \gamma = 0 \). Eq. (3) can then be written in an appropriate dimensionless form as
\[
\phi_t = -\sin \phi + H\phi_x, \quad H\phi_x = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi(x',t)}{x' - x} dx'.
\] (4)

1.2 Remarks
- Equation (1) is derived from Maxwell's equations combined with the London equation and the Josephson equation:
- Equation (4) has been proposed for the first time in a purely mathematical context:
- As for a review on nonlocal Josephson electrodynamics:
  - A.A. Abdumalikov et al, Superconductor Science and Technology, 22 (2009) 023001

2. Exact method of solution
2.1 A nonlinear dynamical system
- Dependent variable transformation
  - We seek periodic solution of (4) of the form
  \[
  \phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j),
  \] (5)
  where \( x_j = x_j(t) \) are complex functions of \( t \) with \( \text{Im} x_j(t) > 0 \), \( \beta \) is a positive parameter, \( N \) is an arbitrary positive integer and \( f^* \) denotes the complex conjugate expression of \( f \). Using a formula for the Hilbert transform, one has \( H\phi_x = -(\ln f^* f)_x \). Substitution of this expression and (5) into (4) gives the following bilinear equation for \( f \) and \( f^* \)
  \[
i(f_t^* f - f^* f_t) = \frac{i}{2} (f^2 - f^{*2}) - f_x^* f - f^* f_x.
  \] (6)
A system of nonlinear ODEs for $x_j$

We divide (6) by $f^*f$, substitute $f$ from (5) and then evaluate the residue at $x = x_j$ on both sides. This gives a system of nonlinear ODEs for $x_j$

$$\dot{x}_j = -\frac{1}{2\beta} \prod_{l=1}^{N} \sin \beta(x_j - x_l^*) + i, \quad j = 1, 2, ..., N, \quad (7)$$

where an overdot denotes differentiation with respect to $t$.

We introduce the following notations:

$$z = e^{2i\beta}, \quad \xi_j = e^{2i\beta x_j}, \quad \eta_j = e^{2i\beta x_j^*}, \quad j = 1, 2, ..., N, \quad (8a)$$

$$s_1 = \sum_{j=1}^{N} x_j, \quad s_2 = \sum_{j < l} x_j x_l, \quad ..., \quad s_N = \prod_{j=1}^{N} x_j, \quad (8b)$$

$$u_1 = \sum_{j=1}^{N} \xi_j, \quad u_2 = \sum_{j < l} \xi_j \xi_l, \quad ..., \quad u_N = \prod_{j=1}^{N} \xi_j, \quad (8c)$$

$$v_1 = \sum_{j=1}^{N} \eta_j, \quad v_2 = \sum_{j < l} \eta_j \eta_l, \quad ..., \quad v_N = \prod_{j=1}^{N} \eta_j, \quad (8d)$$

$$t_j = \sum_{l=1}^{N} \xi_l^j, \quad j = 1, 2, ..., N. \quad (8e)$$

In terms of $u_j (j = 1, 2, ..., N)$ and $s_1$, $f$ can be written as

$$f = \frac{e^{-i\beta(Nx - s_1)}}{(2\beta i)^N} \left( z^N - u_1 z^{N-1} + u_2 z^{N-2} + ... + (-1)^N u_N \right). \quad (9)$$

Thus, $u_j \ (j = 1, 2, ..., N)$ and $s_1$ determine the function $f$ completely.

Let us derive a system of equations for $u_j$. To this end, We rewrite (7) in terms of $\xi_j$ and $\eta_j$ as

$$\dot{\xi}_j = -\frac{1}{2} \alpha u_N \prod_{l=1}^{N} (\xi_j - \xi_l) \prod_{(i \neq j)}^{N} (\xi_j - \xi_i) - 2\beta \xi_j, \quad j = 1, 2, ..., N, \quad (10a)$$

where

$$\alpha = \prod_{j=1}^{N} (\xi_j \eta_j)^{-1/2} = e^{-i\beta(s_1 + s_1^*)}, \quad u_N = \prod_{j=1}^{N} \xi_j = e^{2i\beta s_1}. \quad (10b)$$

Later, we show that $\alpha$ is a constant independent of $t$ and $u_N$ obeys a single nonlinear ODE.
2.2 Linearization

The system of nonlinear ODEs (10) can be linearized in terms of the variables $u_j$ defined by (8c). We multiply $\xi_j^{n-1}$ on both sides of (10a) and sum up with respect to $j$ from 1 to $N$ to obtain

$$\frac{1}{n} \dot{i}_n = -\frac{\alpha}{2} u_N \sum_{s=0}^{n} (-1)^{s} v_s I_{n-s} - 2\beta t_n, \quad n = 1, 2, \ldots, N,$$  \hspace{1cm} (11a)

where $I_{n-s}$ is defined by

$$I_{n-s} = \sum_{j=1}^{N} \frac{\xi_j^{N+n-s-1}}{\prod_{i=1}^{N} (\xi_j - \xi_i)}. \hspace{1cm} (11b)$$

In deriving (11), we have used the identity

$$I_n = 0, \quad -N + 1 \leq n \leq -1. \hspace{1cm} (11c)$$

- Time evolution of $u_j$

The time evolution of $u_n$ follows from (11a) with the help of the formulas

$$u_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^{j} u_j t_{n-j}, \quad 1 \leq n \leq N, \quad \sum_{j=0}^{n} (-1)^{j} u_j I_{n-j} = 0, \quad n \geq 1,$$  \hspace{1cm} (12)

where $u_0 = 1$ and $I_0 = 1$. In fact, differentiating the first formula in (12) by $t$ and substituting (11a) for $\dot{i}_{n-j}$, we can show that the quantity $h_n$ defined by

$$h_n = \dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{N-n} + 2\beta u_n, \quad n = 1, 2, \ldots, N,$$  \hspace{1cm} (13)

satisfies the relation

$$h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^{j} h_j t_{n-j} + \frac{(-1)^{n+1} r_n}{2n\alpha}, \hspace{1cm} (14a)$$

where

$$r_n = \sum_{j=1}^{n} u_{N-j+n} \left[ -\sum_{s=1}^{j} (-1)^{n-s} s I_{j-s} + (-1)^{n-j} t_j \right]. \hspace{1cm} (14b)$$

The quantity in the brackets on the right-hand side of (14b) can be shown to vanish identically so that $r_n \equiv 0$. It follows from this and (14a) that

$$h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^{j} h_j t_{n-j}, \quad n = 1, 2, \ldots, N.$$  \hspace{1cm} (15)
Solving (15) with the initial condition $h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0$, we obtain the relations $h_n \equiv 0 \ (n = 1, 2, \ldots, N)$. Thus, we see that $u_n$ evolves according to the following system of ODEs

$$\dot{u}_n + \frac{\alpha}{2} u_n u_n - \frac{\alpha^{-1}}{2} u_{n-n}^* + 2\beta n u_n = 0, \quad n = 1, 2, \ldots, N.$$  \hspace{1cm} \text{(16)}

It is remarkable that $u_N$ obeys a single nonlinear ODE of the form

$$\dot{u}_N + \frac{\alpha}{2} u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \quad u_N = e^{2i\beta s_1}, \quad \alpha = \sqrt{\frac{u_N^*}{u_N}},$$  \hspace{1cm} \text{(17)}

and other $N - 1$ variables $u_1, u_2, \ldots, u_{N-1}$ constitute a system of linear ODEs. Rewriting (17) in terms of $s_1$, we can put it into a nonlinear ODE for $s_1$

$$\dot{s}_1 = \frac{1}{2i\beta} \sinh(2\beta \text{Im} s_1) + iN,$$  \hspace{1cm} \text{(18)}

where $\text{Im} s_1$ implies the imaginary part of $s_1$.

### 3. Periodic solutions

#### 3.1 Construction of periodic solutions

The first step for constructing periodic solutions is to integrate (18). It follows from the real and imaginary parts of (18) that

$$\text{Re} \dot{s}_1 = 0, \quad \text{Im} \dot{s}_1 = -\frac{1}{2\beta} \sinh(2\beta \text{Im} s_1) + N.$$  \hspace{1cm} \text{(19)}

Thus, the real part of $s_1$ becomes a constant $\text{Re} s_1(t) = \text{Re} s_1(0) \equiv b$ whereas integration of the equation for $\text{Im} s_1$ yields an explicit expression. In terms of a new variable $y = 2\beta \text{Im} s_1$, it is given by

$$e^{-y} = \frac{2\nu_N}{2\nu_N} \left(-\tanh \frac{y_0}{2} + 1\right) \cosh \nu_N t + \left\{(2\beta N + 1) \tanh \frac{y_0}{2} - 2\beta N + 1\right\} \sinh \nu_N t,$$

where $\nu_N = \sqrt{(\beta N)^2 + (1/4)}$ and $y_0 = y(0) = 2\beta \text{Im} s_1(0)$. For $n = 1, 2, \ldots, N - 1$, on the other hand, (16) can be written in the form

$$\dot{u}_n = -\left(\frac{1}{2} e^{-2\beta \text{Im} s_1} + 2\beta n\right) u_n + \frac{\alpha^{-1}}{2} u_{n-n}^*.$$  \hspace{1cm} \text{(21)}

Note from (10b) and $\text{Re} s_1 = b$ that $\alpha = e^{-2i\beta b}$ becomes a constant. The solution of the initial value problem for (21) can be put into the form of a rational function

$$u_n(t) = \frac{G_n}{F}, \quad n = 1, 2, \ldots, N - 1,$$  \hspace{1cm} \text{(22a)}
with
\[ F = 2\nu_N \left( \tanh \frac{\nu}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{\nu}{2} + 2\beta N + 1 \right\} \sinh \nu_N t, \]  
\tag{22b}

\[ G_n = 2\nu_N \left( \tanh \frac{\nu}{2} + 1 \right) \left[ u_n(0) \cosh \nu_n t 
+ \frac{1}{\nu_n} \left\{ \beta(N - 2n)u_n(0) + \frac{\alpha^{-1}}{2} u_{n-n}^*(0) \right\} \sinh \nu_n t \right], \]  
\tag{22c}

where \( \nu_n = \sqrt{\beta^2(N - 2n)^2 + (1/4)} \). We see that the expression (22) with \( n = N \) produces (20) and hence it can be used for all \( u_n \).

3.2 Properties of solutions

- Asymptotic form of the solution as \( t \to \infty \)

\[ u_n \to 0, \quad n = 1, 2, ..., N - 1, \quad u_N \to e^{2\beta b} \left( \sqrt{4(\beta N)^2 + 1} - 2\beta N \right), \]  
\tag{23}

\[ \phi \sim 2 \tan^{-1} \left[ \frac{\sqrt{4(\beta N)^2 + 1} - 1}{2\beta N} \tan \beta \left( N x - b - \frac{N \pi}{2\beta} \right) \right], \]  
\tag{24}

\[ u \equiv \phi_x \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2 + 1} + (-1)^N \cos 2\beta (N x - b)}. \]  
\tag{25}

- Novel features of solutions

1) The asymptotic form of \( u \) does not depend on initial conditions except for a phase constant \( b \). It represents a train of nonlinear periodic standing waves.

2) The initial profile of \( u \) with a spatial period \( \pi/\beta \) evolves into a periodic wave with a period \( \pi/N\beta \).

3) The amplitude of the wave \( A(=u_{\text{max}} - u_{\text{min}}) \) is a constant independent of the wavenumber. Indeed, \( u_{\text{max}} = \sqrt{4(\beta N)^2 + 1} + 1, u_{\text{min}} = \sqrt{4(\beta N)^2 + 1} - 1 \) and hence \( A = 2 \).

4) The steady profile (25) satisfies the Peierls equation \( H\phi_x = \sin \phi \) in the theory of dislocation

Example 1: $N = 1$, $x_1(0) = 3i$, $\beta = 0.2$

![Figure 1. Time evolution of $u$ for $N = 1$ (periodic case).](image1)

Example 2: $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$, $\beta = 0.4$

![Figure 2. Time evolution of $u$ for $N = 2$ (periodic case).](image2)

3.3 Long-wave limit $\beta \to 0$

The long-wave limit $\beta \to 0$ of the periodic solutions can be derived easily. We quote the results:

\[ \phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} (x - x_j) = \sum_{j=0}^{N} s_j(t)x^{N-j}, \quad (s_0 = 1), \]  
\[ \dot{s}_j = -i \text{Im} s_j + i(N - j + 1)s_{j-1}, \quad j = 1, 2, \ldots, N. \]
For $N = 2$, the solution reads as follows:

$$f = x^2 - s_1 x + s_2,$$

$(28a)$

$$s_1 = b_1 + i[-(a_1 - 2)(1 - e^{-t}) + a_1],$$

$(28b)$

$$s_2 = -2t - (a_1 - 2)(1 - e^{-t}) + b_2 + i[-(a_2 - b_1)(1 - e^{-t}) + a_2].$$

$(28c)$

The large time asymptotic of the solution $u \equiv \phi_x$ is given by a superposition of $N$ Lorentzian pulses

$$u \sim \sum_{j=1}^{N} \frac{2}{(x - \sqrt{2t} x_{j,N})^2 + 1},$$

$(29)$

where $x_{j,N}$ is the $n$th root of the Hermite polynomial of order $N$. These results have been detailed in Matsuno (1992).

**Example 1:** Nonperiodic case $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$

![Figure 3. Time evolution of $u$ for $N = 2$ (nonperiodic case).](image)

**4. Application**

The exact method of solution developed so far can be applied to obtain periodic solutions of the sine-Hilbert (sH) equation

$$H \theta_t = -\sin \theta, \quad \theta = \theta(x,t).$$

$(30)$

**4.1 Remark**

- The sH equation was introduced by Degasperis and Santini in a purely mathematical context:
  
• The reduction to a Riemann-Hilbert scattering problem was given by Degasperis et al:
• An exact method of solution by means of bilinear transformation method was developed by Matsuno:

4.2 Periodic solutions

Here, we summarize the procedure for constructing periodic solutions of the sH equation. We seek periodic solutions of the form (5)

\[ \theta = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin(\beta(x - x_j)), \tag{31} \]

The corresponding bilinear equation for \( f \) is given by

\[ (f^* f)_t = \frac{1}{2}(f^2 - f'^2). \tag{32} \]

The system of equations for \( x_j \) becomes

\[ \dot{x}_j = \frac{1}{2i\beta} \prod_{l=1}^{N} \sin \beta(x_j - x^*_l), \quad j = 1, 2, \ldots, N, \tag{33} \]

and \( u_j \) satisfies the system of equations

\[ \dot{u}_j = i \left( -\frac{c}{2} u_N u_j + \frac{1}{2c} u_{N-j}^* \right), \quad c = \sqrt{\frac{u_N^*}{u_N}}, \quad j = 1, 2, \ldots, N. \tag{34} \]

The above system can be solved analytically and solutions are given explicitly.

Example: \( N = 1 \)

Substituting \( u_1 = e^{2i\beta s_1} \) into (34)

\[ \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im } s_1), \tag{35a} \]

\[ \text{Re } \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im } s_1), \quad \text{Im } \dot{s}_1 = 0, \tag{35b} \]

\[ x_1 = s_1 = at + b + i \frac{1}{2\beta} \sinh^{-1}(2\beta a), \quad a = \frac{1}{2\beta} \sinh(2\beta \text{Im } s_1) \quad b = \text{Res}_1(0), \tag{35c} \]

\[ u \equiv \theta_x = \frac{4\beta^2 a}{\sqrt{1 + 4\beta^2 a^2 - \cos 2\beta(x - at - b)}}. \tag{36} \]

Note that the solution is not a standing wave but a traveling wave.
5. Summary

- We have constructed periodic solutions of a resistive model describing Josephson electrodynamics by means of a novel linearization procedure.

- The large time asymptotic of the periodic solution has a steady profile which is formed by a balance between nonlinearity and dissipation. This feature is in striking contrast to periodic solutions of nonlinear dispersive wave equations.

- The exact method of solution developed here was applied to the sine-Hilbert equation to obtain periodic solutions.