Title
Periodic solutions of the model equation describing electrodynamics of the Josephson junction (Mathematical Physics and Application of Nonlinear Wave Phenomena)

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Abstract

A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are obtained in the form of parametric representation. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions. Last, the exact method is applied to the sine-Hilbert equation to obtain periodic solutions. The detail of this report has been published in J. Phys. A: Math. Theor. 42 (2009) 025401.

1. Model equation
1.1 Nonlocal model equation

Consider a Josephson junction with a thin layer between two superconductors. The phase difference $\phi(x, t)$ across the Josephson junction is described by the following model equation:

$$
\omega_j^{-2} \phi_{tt} + \omega_j^{-2} \eta \phi_t = -\sin \phi + \frac{\lambda_j^2}{2 \lambda_L} \int_{-\infty}^{\infty} K_0 \left( \frac{|x - x'|}{\lambda_L} \right) \phi_{x'}(x', t) dx' + \gamma. \quad (1)
$$

$K_0$: modified Bessel function of order zero, $\omega_j$: Josephson plasma frequency, $\lambda_L$: London penetration depth, $\lambda_J$: Josephson penetration depth, $\gamma$: bias current density across the junction, $\eta$: positive parameter characterizing the resistance of a unit area of the tunneling junction.

Let $l$ be the characteristic space scale of $\phi$. When $\lambda_L << l$, then $K_0(x) \sim \pi \delta(x)$ and Eq. (1) reduces to the perturbed sine-Gordon equation

$$
\omega_j^{-2} \phi_{tt} + \omega_j^{-2} \eta \phi_t = -\sin \phi + \frac{\lambda_j^2}{\lambda_L} \phi_{xx} + \gamma. \quad (2)
$$
If \( l \ll \lambda_L \), then \( K(|x|) \sim -\ln |x| \) and Eq. (1) becomes
\[
\omega^2_j \phi_{tt} + \omega^2 \eta \phi_t = -\sin \phi + \frac{\lambda_L^2}{\pi \lambda_L} \int_{-\infty}^{\infty} \frac{\phi(x',t)}{x' - x} \, dx' + \gamma. \tag{3}
\]
In the following, we consider the overdamped case \( \eta \gg 1 \) and the zero bias current \( \gamma = 0 \). Eq. (3) can then be written in an appropriate dimensionless form as
\[
\phi_t = -\sin \phi + H \phi_x, \quad H \phi_x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi'(x',t)}{x' - x} \, dx'. \tag{4}
\]

1.2 Remarks
- Equation (1) is derived from Maxwell’s equations combined with the London equation and the Josephson equation:
- Equation (4) has been proposed for the first time in a purely mathematical context:
- As for a review on nonlocal Josephson electrodynamics:
  - A.A. Abdumalikov et al, Superconductor Science and Technology, \textbf{22} (2009) 023001

2. Exact method of solution
2.1 A nonlinear dynamical system
- Dependent variable transformation
  We seek periodic solution of (4) of the form
\[
\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin \beta(x - x_j), \tag{5}
\]
where \( x_j = x_j(t) \) are complex functions of \( t \) with \( \text{Im} \, x_j(t) > 0 \), \( \beta \) is a positive parameter, \( N \) is an arbitrary positive integer and \( f^* \) denotes the complex conjugate expression of \( f \). Using a formula for the Hilbert transform, one has \( H \phi_x = -(\ln f^* f)_x \). Substitution of this expression and (5) into (4) gives the following bilinear equation for \( f \) and \( f^* \)
\[
i(f^*_t f - f^* f_t) = \frac{i}{2} (f^2 - f^{*2}) - f^*_x f - f^* f_x. \tag{6}
\]
A system of nonlinear ODEs for \( x_j \)

We divide (6) by \( f^*f \), substitute \( f \) from (5) and then evaluate the residue at \( x = x_j \) on both sides. This gives a system of nonlinear ODEs for \( x_j \)

\[
\dot{x}_j = -\frac{1}{2\beta} \prod_{l=1}^{N} \sin \left( \frac{\beta(x_j - x_l^*)}{2} \right) + i, \quad j = 1, 2, ..., N, 
\]

(7)

where an overdot denotes differentiation with respect to \( t \).

We introduce the following notations:

\[
z = e^{2i\beta x}, \quad \xi_j = e^{2i\beta x_j}, \quad \eta_j = e^{2i\beta x_j^*}, \quad j = 1, 2, ..., N, 
\]

(8a)

\[
s_1 = \sum_{j=1}^{N} x_j, \quad s_2 = \sum_{j<l} x_j x_l, \quad ..., \quad s_N = \prod_{j=1}^{N} x_j, 
\]

(8b)

\[
u_1 = \sum_{j=1}^{N} \xi_j, \quad u_2 = \sum_{j<l} \xi_j \xi_l, \quad ..., \quad u_N = \prod_{j=1}^{N} \xi_j, 
\]

(8c)

\[
v_1 = \sum_{j=1}^{N} \eta_j, \quad v_2 = \sum_{j<l} \eta_j \eta_l, \quad ..., \quad v_N = \prod_{j=1}^{N} \eta_j, 
\]

(8d)

\[
ts_j = \sum_{i=1}^{N} \xi_i^j, \quad j = 1, 2, ..., N. 
\]

(8e)

In terms of \( u_j(j=1,2,...,N) \) and \( s_1 \), \( f \) can be written as

\[
f = e^{-i\beta(Nx-s_1)} \left( z^N - u_1 z^{N-1} + u_2 z^{N-2} + ... + (-1)^N u_N \right). 
\]

(9)

Thus, \( u_j(j = 1, 2, ..., N) \) and \( s_1 \) determine the function \( f \) completely.

Let us derive a system of equations for \( u_j \). To this end, We rewrite (7) in terms of \( \xi_j \) and \( \eta_j \) as

\[
\dot{\xi}_j = -\frac{1}{2\alpha} \prod_{l=1}^{N} \frac{\xi_l - \eta_l}{\prod_{l=1}^{N} (\xi_j - \xi_l)} - 2\beta \xi_j, \quad j = 1, 2, ..., N, 
\]

(10a)

where

\[
\alpha = \prod_{j=1}^{N} (\xi_j \eta_j)^{-1/2} = e^{-i\beta(s_1+s_1^*)}, \quad u_N = \prod_{j=1}^{N} \xi_j = e^{2i\beta s_1}. 
\]

(10b)

Later, we show that \( \alpha \) is a constant independent of \( t \) and \( u_N \) obeys a single nonlinear ODE.
2.2 Linearization

The system of nonlinear ODEs (10) can be linearized in terms of the variables \( u_j \) defined by (8c). We multiply \( \xi_j^{n-1} \) on both sides of (10a) and sum up with respect to \( j \) from 1 to \( N \) to obtain

\[
\frac{1}{n} \dot{t}_n = -\frac{\alpha}{2} u_N \sum_{s=0}^{n} (-1)^s v_s I_{n-s} - 2\beta t_n, \quad n = 1, 2, ..., N, \tag{11a}
\]

where \( I_{n-s} \) is defined by

\[
I_{n-s} = \sum_{j=1}^{N} \frac{\xi_j^{N+n-s-1}}{\prod_{i=1}^{N} (\xi_j - \xi_i)}. \tag{11b}
\]

In deriving (11), we have used the identity

\[
I_n = 0, \quad -N + 1 \leq n \leq -1. \tag{11c}
\]

- Time evolution of \( u_j \)

The time evolution of \( u_n \) follows from (11a) with the help of the formulas

\[
u_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j u_j t_{n-j}, \quad 1 \leq n \leq N, \quad \sum_{j=0}^{n} (-1)^j u_j I_{n-j} = 0, \quad n \geq 1, \tag{12}\]

where \( u_0 = 1 \) and \( I_0 = 1 \). In fact, differentiating the first formula in (12) by \( t \) and substituting (11a) for \( \dot{t}_{n-j} \), we can show that the quantity \( h_n \) defined by

\[
h_n = \dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{n-n} + 2\beta n u_n, \quad n = 1, 2, ..., N, \tag{13}\]

satisfies the relation

\[
h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j} + \frac{(-1)^{n+1} r_n}{2n\alpha}, \tag{14a}\]

where

\[
r_n = \sum_{j=1}^{n} u_{N-j+n} \left[ -\sum_{s=1}^{j} (-1)^{n-s} s I_{j-s} + (-1)^{n-j} t_j \right]. \tag{14b}\]

The quantity in the brackets on the right-hand side of (14b) can be shown to vanish identically so that \( r_n \equiv 0 \). It follows from this and (14a) that

\[
h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j}, \quad n = 1, 2, ..., N. \tag{15}\]
Solving (15) with the initial condition \( h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0 \), we obtain the relations \( h_n \equiv 0 \) \( (n = 1, 2, ..., N) \). Thus, we see that \( u_n \) evolves according to the following system of ODEs

\[
\dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{N-n}^* + 2\beta n u_n = 0, \quad n = 1, 2, ..., N.
\] (16)

It is remarkable that \( u_N \) obeys a single nonlinear ODE of the form

\[
\dot{u}_N + \frac{\alpha}{2} u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \quad u_N = e^{2i\beta s_1}, \quad \alpha = \sqrt{\frac{u_N^*}{u_N}},
\] (17)

and other \( N - 1 \) variables \( u_1, u_2, ..., u_{N-1} \) constitute a system of linear ODEs. Rewriting (17) in terms of \( s_1 \), we can put it into a nonlinear ODE for \( s_1 \)

\[
\dot{s}_1 = \frac{1}{2i\beta} \sinh(2\beta \text{Im} s_1) + iN,
\] (18)

where \( \text{Im} s_1 \) implies the imaginary part of \( s_1 \).

3. Periodic solutions

3.1 Construction of periodic solutions

The first step for constructing periodic solutions is to integrate (18). It follows from the real and imaginary parts of (18) that

\[
\text{Re} \dot{s}_1 = 0, \quad \text{Im} \dot{s}_1 = -\frac{1}{2\beta} \sinh(2\beta \text{Im} s_1) + N.
\] (19)

Thus, the real part of \( s_1 \) becomes a constant \( \text{Re} s_1(t) = \text{Re} s_1(0) \equiv b \) whereas integration of the equation for \( \text{Im} s_1 \) yields an explicit expression. In terms of a new variable \( y = 2\beta \text{Im} s_1 \), it is given by

\[
e^{-y} = \frac{2\nu_N (-\tanh \frac{y_0}{2} + 1) \cosh \nu_N t + \{(2\beta N + 1) \tanh \frac{y_0}{2} - 2\beta N + 1\} \sinh \nu_N t}{2\nu_N (\tanh \frac{y_0}{2} + 1) \cosh \nu_N t + \{(2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1\} \sinh \nu_N t},
\] (20)

where \( \nu_N = \sqrt{(\beta N)^2 + (1/4)} \) and \( y_0 = y(0) = 2\beta \text{Im} s_1(0) \). For \( n = 1, 2, ..., N - 1 \), on the other hand, (16) can be written in the form

\[
\dot{u}_n = -\left(\frac{1}{2} e^{-2\beta \text{Im} s_1} + 2\beta n\right) u_n + \frac{\alpha^{-1}}{2} u_{N-n}^*.
\] (21)

Note from (10b) and \( \text{Re} s_1 = b \) that \( \alpha = e^{-2i\beta b} \) becomes a constant. The solution of the initial value problem for (21) can be put into the form of a rational function

\[
u_n(t) = \frac{G_n}{F}, \quad n = 1, 2, ..., N - 1,
\] (22a)
with

\[ F = 2\nu_N \left( \tanh \frac{y_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1 \right\} \sinh \nu_N t, \tag{22b} \]

\[ G_n = 2\nu_N \left( \tanh \frac{y_0}{2} + 1 \right) \left[ u_n(0) \cosh \nu_n t \right. \\
+ \left. \frac{1}{\nu_n} \left\{ \beta(N - 2n)u_n(0) + \frac{\alpha^{-1}}{2} u_{n-n}^*(0) \right\} \sinh \nu_n t \right], \tag{22c} \]

where \( \nu_n = \sqrt{\beta^2(N - 2n)^2 + (1/4)} \). We see that the expression (22) with \( n = N \) produces (20) and hence it can be used for all \( u_n \).

3.2 Properties of solutions

- Asymptotic form of the solution as \( t \to \infty \)

\[ u_n \to 0, \quad n = 1, 2, ..., N - 1, \quad u_N \to e^{2i\beta b} (\sqrt{4(\beta N)^2 + 1} - 2\beta N), \tag{23} \]

\[ \phi \sim 2 \tan^{-1} \left[ \frac{\sqrt{4(\beta N)^2 + 1} - 1}{2\beta N} \tan \beta \left( N x - b - \frac{N\pi}{2\beta} \right) \right], \tag{24} \]

\[ u \equiv \phi_x \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2 + 1} + (-1)^N \cos 2\beta(N x - b)}. \tag{25} \]

- Novel features of solutions

1) The asymptotic form of \( u \) does not depend on initial conditions except for a phase constant \( b \). It represents a train of nonlinear periodic standing waves.

2) The initial profile of \( u \) with a spatial period \( \pi/\beta \) evolves into a periodic wave with a period \( \pi/N\beta \).

3) The amplitude of the wave \( A(= u_{\text{max}} - u_{\text{min}}) \) is a constant independent of the wavenumber. Indeed, \( u_{\text{max}} = \sqrt{4(\beta N)^2 + 1} + 1, u_{\text{min}} = \sqrt{4(\beta N)^2 + 1} - 1 \) and hence \( A = 2 \).

4) The steady profile (25) satisfies the Peierls equation \( H\phi_x = \sin \phi \) in the theory of dislocation

Example 1: $N = 1$, $x_1(0) = 3i$, $\beta = 0.2$

Figure 1. Time evolution of $u$ for $N = 1$ (periodic case).

Example 2: $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$, $\beta = 0.4$

Figure 2. Time evolution of $u$ for $N = 2$ (periodic case).

3.3 Long-wave limit $\beta \to 0$

The long-wave limit $\beta \to 0$ of the periodic solutions can be derived easily. We quote the results:

$$\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} (x - x_j) = \sum_{j=0}^{N} s_j(t) x^{N-j}, \quad (s_0 = 1),$$

$$\dot{s}_j = -i \text{Im } s_j + i(N - j + 1)s_{j-1}, \quad j = 1, 2, \ldots, N.$$

(26)

(27)
For $N = 2$, the solution reads as follows:

\[ f = x^2 - s_1 x + s_2, \]  

\[ s_1 = b_1 + i[-(a_1 - 2)(1 - e^{-t}) + a_1], \]
\[ s_2 = -2t - (a_1 - 2)(1 - e^{-t}) + b_2 + i[-(a_2 - b_1)(1 - e^{-t}) + a_2]. \]

The large time asymptotic of the solution $u \equiv \phi_x$ is given by a superposition of $N$ Lorentzian pulses

\[ u \sim \sum_{j=1}^{N} \frac{2}{(x - \sqrt{2t} x_{j,N})^2 + 1}, \]

where $x_{j,N}$ is the $n$th root of the Hermite polynomial of order $N$. These results have been detailed in Matsuno (1992).

**Example 1:** Nonperiodic case $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$

![Figure 3](image)

**Figure 3.** Time evolution of $u$ for $N = 2$ (nonperiodic case).

4. **Application**

The exact method of solution developed so far can be applied to obtain periodic solutions of the sine-Hilbert ($sH$) equation

\[ H \theta_t = - \sin \theta, \quad \theta = \theta(x, t). \]

4.1 **Remark**

- The $sH$ equation was introduced by Degasperis and Santini in a purely mathematical context:
• The reduction to a Riemann-Hilbert scattering problem was given by Degasperis et al:
• An exact method of solution by means of bilinear transformation method was
developed by Matsuno:

4.2 Periodic solutions

Here, we summarize the procedure for constructing periodic solutions of the sH
  equation. We seek periodic solutions of the form (5)

\[ \theta = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^{N} \frac{1}{\beta} \sin(\beta(x - x_j)), \]

(31)

The corresponding bilinear equation for \( f \) is given by

\[ (f^*f)_t = \frac{1}{2}(f^2 - f^{*2}). \]

(32)

The system of equations for \( x_j \) becomes

\[ \dot{x}_j = \frac{1}{2i\beta} \sum_{l=1}^{N} \sin(\beta(x_j - x_l)), \quad j = 1, 2, \ldots, N, \]

(33)

and \( u_j \) satisfies the system of equations

\[ \dot{u}_j = i \left( \frac{-c}{2} u_N u_j + \frac{1}{2c} u_{N-j}^* \right), \quad c = \sqrt{\frac{u_N^*}{u_N}}, \quad j = 1, 2, \ldots, N. \]

(34)

The above system can be solved analytically and solutions are given explicitly.

**Example:** \( N = 1 \)

Substituting \( u_1 = e^{2i\beta s_1} \) into (34)

\[ \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1), \]

(35a)

\[ \text{Re} \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1), \quad \text{Im} \dot{s}_1 = 0, \]

(35b)

\[ x_1 = s_1 = at + b + \frac{1}{2\beta} \sinh^{-1}(2\beta a), \quad a = \frac{1}{2\beta} \sinh(2\beta \text{Im} s_1), \quad b = \text{Res}_1(0), \]

(35c)

\[ u \equiv \theta_x = \frac{4\beta^2 a}{\sqrt{1 + 4\beta^2 a^2} - \cos 2\beta(x - at - b)}. \]

(36)

Note that the solution is not a standing wave but a traveling wave.
5. Summary

- We have constructed periodic solutions of a resistive model describing Josephson electrodynamics by means of a novel linearization procedure.

- The large time asymptotic of the periodic solution has a steady profile which is formed by a balance between nonlinearity and dissipation. This feature is in striking contrast to periodic solutions of nonlinear dispersive wave equations.

- The exact method of solution developed here was applied to the sine-Hilbert equation to obtain periodic solutions.