

Self-focusing of a LASER beam and nonlinear Schrödinger equations

- An application of the Nelson diffusion -

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Abstract

This note will be focused on some relations between the asymptotic profiles of blowup solutions and blowup rates of those to the pseudo-conformally invariant nonlinear Schrödinger equations. The equation of this type with 2+1 space-time dimension appears as a model of self-focusing of a LASER beam in a Kerr medium. This phenomenon is believed to be well described by blowup solutions of the equation to some extent. We will see that so-called Nelson diffusions bring us some information on the asymptotic behavior and limiting profiles of blowup solutions.

1 Introduction

We are concerned with the following pseudo-conformally^{*1} invariant nonlinear Schrödinger equation:

$$2i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{4/d} \psi = 0, \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+. \quad (1)$$

Here $i = \sqrt{-1}$ and Δ is the Laplace operator on \mathbb{R}^d . We associate this equation with initial datum ψ_0 from $H^1(\mathbb{R}^d)$, which is the set of all square integrable functions on \mathbb{R}^d whose distributional derivatives up to 1st order are also square integrable. We summarize basic, mathematical facts as to this Cauchy problem in Section 2.

The equation of this type with 2+1 space-time dimension appears as a model of a LASER beam propagating along “ t -axis” (the third axis of our space \mathbb{R}^3 , say z) in a nonlinear medium (see, e.g., [1, 2, 15, 45, 40]).

We are assuming that neither charges, currents, nor magnetization exist in a nonlinear material like an optical fiber. Our basic equation describing a LASER light beam in the

^{*1} We shall discuss this property in Section 2.

material is Maxwell's equations: the electric field \mathbb{E} satisfies:^{*2}

$$\epsilon_0 \mu_0 \frac{\partial^2 \mathbb{E}}{\partial t^2} - \Delta \mathbb{E} = -\mu_0 \frac{\partial^2 \mathbb{P}}{\partial t^2}. \quad (2)$$

The electric polarization field \mathbb{P} will depend on the electric field \mathbb{E} nonlinearly (the Kerr effect). We simply assume here that^{*3}

$$\mathbb{P} = \epsilon_0 \left(\chi_e^{(1)} + \chi_e^{(3)} |\mathbb{E}|^2 \right) \mathbb{E}. \quad (3)$$

Now we suppose that monochromatic field having angular frequency ω and wave number $(0, 0, k)$ is applied to the material, so that, introducing a complex amplitude φ , we may make an ansatz as follows:

$$\mathbb{E}(x, y, z, t) = \epsilon \varphi (\epsilon x, \epsilon y, \epsilon^2 z) e^{i(kz - \omega t)} \mathbf{e}_x, \quad (4)$$

where $\mathbf{e}_x = (1, 0, 0)$ and $\epsilon > 0$ is a small constant.^{*4}

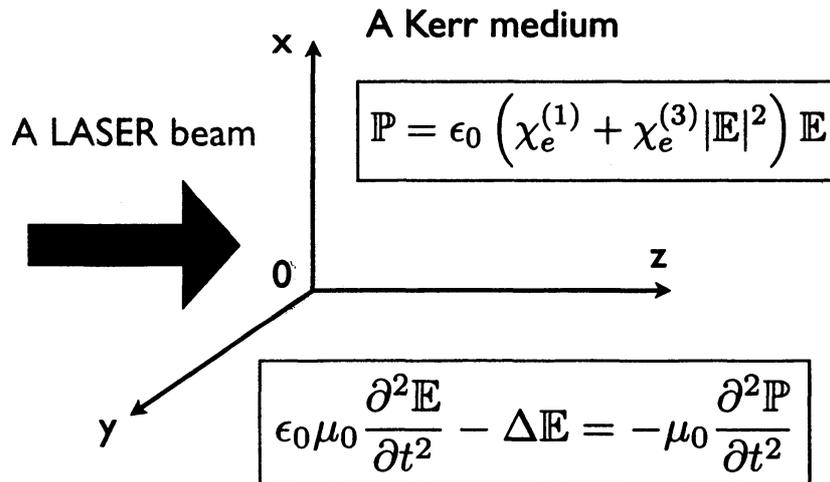


Figure1 A LASER beam propagating in a nonlinear material.

Putting this $\mathbb{E}(x, y, z, t)$ of (4) in the wave equation (2) with (3), making a table of coefficients of powers of ϵ ,^{*5} and equating those coefficients of the same power, we get the dispersion

^{*2} ϵ_0 and μ_0 are the vacuum permittivity and vacuum permeability, respectively. Hence $c_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ is the speed of light in vacuum.

^{*3} $\chi_e^{(n)}$ is the n-th order component of electric susceptibility of the material which is assumed to be isotropic. $\chi_e^{(1)}$ is the linear susceptibility, and $\chi_e^{(2)}$ is dropped out by the inversion symmetry of the material. Hence the $\chi_e^{(3)}$ exhibit the first non-negligible nonlinear effect.

^{*4} $\epsilon > 0$ may be regard as $\epsilon = \frac{k_0}{k}$ ($k \gg 1$) with the "specific wave length" $\frac{1}{k_0}$.

^{*5} Only ϵ , ϵ^3 and ϵ^3 terms will appear.

relation from the ε -term, so that the following nonlinear Schrödinger equation shows up from the ε^3 -term (abandoning the ε^5 -term)^{*6}:

$$2i\frac{1}{k}\frac{\partial\varphi}{\partial Z} + \frac{1}{k^2}\Delta_{XY}\varphi + \frac{n_3}{n_0}|\varphi|^2\varphi = 0. \quad (5)$$

Here,

$$n_0 = 1 + \chi_e^{(1)}, \quad n_3 = \chi_e^{(3)}.$$

These are relevant to the refractive index n of the media as follows:^{*7}

$$n = n_0 + n_3|\mathbb{E}|^2.$$

Analogous arguments of Nelson's stochastic quantization procedure [35] (see also [8]) give us another derivation of (5) from the geometrical optical path obtained through refraction index n [30]. In this note, we shall not discuss this aspect. But the process introduced by Nelson will play a central role in our analysis (see Section 5). This point could be a novelty of this note.

In modern understanding, self-focusing of a LASER beam is well described to some extent by the nonlinear Schrödinger equation (5); blowup solutions^{*8} are considered to describe the phenomena (see, e.g., [25, 9]). Because of mathematical generosity, we consider (1) which, in fact, is a "genuine" generalization of (5) with $k = 1$ to higher space-dimensions, keeping the pseudo-conformal invariance of the equations.^{*9}

We may say that recent one of the trend in the study of this type of nonlinear Schrödinger equation is to determine their blowup rates of the solutions, and to find relevance between their asymptotic behavior and blowup rates (e.g., [10, 23, 9, 29] etc.).

2 The NLS: basic facts

We summarize the basic properties of the Cauchy problem for the nonlinear Schrödinger equation (abbreviated to NLS) of the form:

$$\begin{cases} 2i\frac{\partial\psi}{\partial t} + \Delta\psi + |\psi|^{p-1}\psi = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\ \psi(0) = \psi_0 \in H^1(\mathbb{R}^d). \end{cases}$$

^{*6} We are ignoring the backscattering effect, or assuming the slowly varying approximation.

^{*7} In case of anisotropic or random media, these are not constant but "functions".

^{*8} The solutions explode their L^2 norm of the gradients in finite time. For the precise definition, see Section 2.

^{*9} The invariance property is inherited to the structure of solutions of (1) regardless of the difference of the space-dimension d . This will be discussed in section 2.

Here, the index p in the nonlinear term satisfies: $p \in (1, 2^* - 1)$, where $2^* = \frac{2d}{d-2}$ for $d \geq 3$; $2^* = \infty$ for $d = 1, 2$. The unique local existence theorem is well known (see, e.g., [14, 6, 40]): for any $\psi_0 \in H^1(\mathbb{R}^d)$, there exists a unique solution ψ in $C([0, T_{\max}); H^1(\mathbb{R}^d))$ for some $T_{\max} \in (0, \infty]$ (maximal existence time) such that ψ satisfies the following three conservation laws of L^2 -norm (charge), momentum, energy (Hamiltonian) in this order:

$$\begin{aligned} \|\psi(t)\|^2 &= \|\psi(0)\|^2, \\ \Im \int_{\mathbb{R}^d} \overline{\psi(x,t)} \nabla \psi(x,t) dx &= \Im \int_{\mathbb{R}^d} \overline{\psi_0(x)} \nabla \psi_0(x) dx = \Im \langle \psi_0, \nabla \psi_0 \rangle, \\ \mathcal{H}_{p+1}(\psi(t)) &\equiv \|\nabla \psi(t)\|^2 - \frac{2}{p+1} \|\psi(t)\|_{p+1}^{p+1} = \mathcal{H}_{p+1}(\psi_0). \end{aligned}$$

It is worth while noting that a certain number $p > 1$ (the index appearing in the nonlinear term) divides the world of solutions of NLS into two parts:

- When $1 < p < 1 + \frac{4}{d}$, every solution exists globally in time, i.e., $T_{\max} = \infty$.

For: we have an a priori bound on $\|\nabla \psi(t)\|$ by virtue of the energy conservation law and the Gagliardo-Nirenberg inequality:

$$\|f\|_{p+1}^{p+1} \leq C_{p,d} \|f\|^{p+1 - \frac{4}{d}(p-1)} \|\nabla f\|^{\frac{4}{d}(p-1)}.$$

- When $2^* - 1 > p \geq 1 + \frac{4}{d}$, there exists a class of initial data which give rise to blowup solutions, that is,

$$T_{\max} < \infty \quad \text{and} \quad \lim_{t \uparrow T_{\max}} \|\nabla \psi(t)\| = \infty.$$

Hence, our equation (1) is the borderline case for the existence of blowup solutions. This fact can be easily seen in a weighted energy space $H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; |x|^2 dx)$ ^{*10}: If we assume in addition that $|x|\psi_0 \in L^2(\mathbb{R}^d)$, then the corresponding solution ψ of NLS satisfies

$$|x|\psi(\cdot) \in C([0, T_{\max}); L^2(\mathbb{R}^d))$$

and

$$\begin{aligned} \||x|\psi(t)\|^2 &= \||x|\psi(0)\|^2 + 2t \Im \langle \psi(0), x \cdot \nabla \psi(0) \rangle + t^2 \mathcal{H}_{p+1}(\psi(0)) \\ &\quad - \frac{d}{p+1} \left(p+1 - \left(2 + \frac{4}{d} \right) \right) \int_0^t (t-\tau) \|\psi(\tau)\|_{p+1}^{p+1} d\tau. \end{aligned}$$

This identity (sometimes called the virial identity) shows that every negative energy solution has to blow up in a finite time, provided that $p \geq 1 + \frac{4}{d}$. For $p = 1 + \frac{4}{d}$, the last term in

^{*10} The form domain of harmonic oscillators, $-\Delta + c|x|^2$ ($c > 0$).

the right hand side vanishes; this is one of the appearance of the invariance property of our equation under the pseudo-conformal transformations.

In what follows, we will quote our equation (1) as (NSC). We write it again here:

$$2i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^{4/d} \psi = 0. \quad (\text{NSC})$$

We use the following symbol for the energy of (NSC):

$$\mathcal{H}(\psi(t)) \equiv \|\nabla \psi(t)\|^2 - \frac{2}{2 + \frac{4}{d}} \|\psi(t)\|_{2 + \frac{4}{d}}^{2 + \frac{4}{d}}.$$

We need some knowledge about standing wave solutions of NLS. The standing waves are solutions of variable separation type of the form: $\psi(x, t) = Q(x) \exp(it/2)$.^{*11} We collect necessary ingredients for our equation (NSC) here. Of course, Q solves the following nonlinear scalar field equation:

$$\Delta Q - Q + |Q|^{4/d} Q = 0, \quad Q \in H^1(\mathbb{R}^d) \setminus \{0\}. \quad (6)$$

Especially, the ground state Q_g is significant among other standing waves (usually called bound states). The ground state is characterized as the minimal action solution of (6):^{*12}

$$\mathcal{N}_1 := \inf_{\substack{f \in H^1(\mathbb{R}^d) \\ f \neq 0}} \left\{ \|\nabla f\|^2 + \|f\|^2 - \frac{2}{2 + \frac{4}{d}} \|f\|_{2 + \frac{4}{d}}^{2 + \frac{4}{d}} \mid \mathcal{H}(f) = 0 \right\}.$$

In this case, this variational problem is equivalent to each of the followings:

$$\mathcal{N}_1 = \inf_{\substack{f \in H^1(\mathbb{R}^d) \\ f \neq 0}} \left\{ \|f\|^2 \mid \mathcal{H}(f) \leq 0 \right\}, \quad \mathcal{N}_2 := \inf_{\substack{f \in H^1(\mathbb{R}^d) \\ f \neq 0}} \frac{\|f\|^{\frac{4}{d}} \|\nabla f\|^2}{\|f\|_{2 + \frac{4}{d}}^{2 + \frac{4}{d}}},$$

where these variational values are relevant to each other [43] (see also [27]):

$$\mathcal{N}_2 = \frac{2}{2 + \frac{4}{d}} \mathcal{N}_1^{\frac{2}{d}},$$

and \mathcal{N}_2 gives the best constant for the following Gagliardo-Nirenberg inequality,

$$\|f\|_{2 + \frac{4}{d}}^{2 + \frac{4}{d}} \leq \frac{1}{\mathcal{N}_2} \|f\|^{\frac{4}{d}} \|\nabla f\|^2. \quad (7)$$

Here the important thing is that the ground state Q_g gives these variational values^{*13} such that:

$$\mathcal{N}_2 = \frac{2}{2 + \frac{4}{d}} \|Q_g\|^{\frac{4}{d}}, \quad \mathcal{H}(Q_g) = 0.$$

^{*11} We may consider a frequency $\omega > 0$ of the standing waves as $Q_\omega(x) \exp(i\omega t/2)$. Then, Q_ω solves $\Delta Q_\omega - \omega Q_\omega + |Q_\omega|^{4/d} Q_\omega = 0$. But this doesn't matter for our analyses in the sequel: Consider the dilations, $\mathbb{R}_+ \ni \omega \mapsto \sqrt{\omega}^{d/2} Q(\sqrt{\omega}x)$.

^{*12} We abuse the terminology here. We should say that $Q_g e^{it/2}$ is the ground state of (NSC), and that the other standing waves of the form $Q e^{it/2}$ should be referred as bound states.

^{*13} $\sqrt{\omega}^{d/2} Q_g(\sqrt{\omega}x)$ gives these values as well.

Furthermore, we know that Q_g is positive, so that it is radially symmetric and monotonically decreasing.*¹⁴ Such a shape of the ground state is referred to as a *Townes profile* in the field of nonlinear optics; it is reported that such a profile appears in self-focusing singularities in LASER beams under general circumstances [25]. Some numerical computations also support this fact (see, e.g., [9]). However, we always have exceptions.*¹⁵ Furthermore, another type of singularities are observed in numerically for (NSC) with $d = 2$ [10] and in real experiments in LASER beams [9]. We shall briefly discuss this aspect in the next section.

From the fact that $\frac{1}{\sqrt{2}}$ is the best constants for (7), one can easily verify that if $\|\psi_0\| < \|Q_g\|$, then we always have an H^1 -bounded, global-in-time solution of (NSC), i.e., $T_{\max} = \infty$:*¹⁶ the size of L^2 -norm alone control the H^1 norm. This is one of the peculiarities of our NLS equation with $p = 1 + \frac{4}{d}$, that is our equation (NSC). We will see at the end of this section that this estimate is sharp [44] in the sense that there exists a blowup solutions whose L^2 -norm is just the same as $\|Q_g\|$.

Now we shall discuss the pseudo-conformal invariance of our equation (NSC). Pragmatically we can safely say that pseudo-conformal invariance is the invariant property under the following space-time transformations:*¹⁷ $T > 0$,

$$[\mathcal{G}(T)\psi](x, t) = (T - t)^{-d/2} \exp\left\{-\frac{i|x|^2}{2(T-t)}\right\} \psi\left(\frac{x}{T-t}, \frac{t}{T(T-t)}\right), \quad T > 0.$$

That is, if ψ solves (NSC), then $\mathcal{G}(T)\psi$ also solves (NSC).

Applying this transformation to a standing wave solution $Q(x)e^{i\frac{t}{2}}$, we obtain an explicit blowup solution of (NSC):

$$\tilde{Q}(x, t) = (T - t)^{-d/2} \exp\left\{-\frac{i|x|^2}{2(T-t)}\right\} Q\left(\frac{x}{T-t}\right) \exp\left(\frac{it}{2T(T-t)}\right), \quad (8)$$

which blows up at $T > 0$ such that:

$$\lim_{t \uparrow T} \|\nabla \tilde{Q}(t)\| = \infty \quad \text{with} \quad \|\nabla \tilde{Q}(t)\| \asymp \frac{1}{T-t}, \quad (9)$$

and

$$\lim_{t \uparrow T} \int_{\mathbb{R}^d} |x|^2 |\tilde{Q}(x, t)|^2 dx = 0, \quad \|\tilde{Q}(t)\| = \|Q\|, \quad (10)$$

so that we have: as $t \uparrow T$,

$$|\tilde{Q}(x, t)|^2 dx \rightarrow \|Q\|^2 \delta_0(dx). \quad (11)$$

*¹⁴ This is a classical, beautiful result due to Gidas-Ni-Nirenberg [11], and Kwong [18] proved that the positive solution is unique up to space-translations.

*¹⁵ As we will see just below, there are blowup solutions in which the singularities are described by any bound states other than the ground state.

*¹⁶ "A LASER beam of weak intensity is dispersed in the medium where it propagates."

*¹⁷ It is also referred to as Talanov lens transformations [41].

The whole intensity of a LASER beam concentrates at the origin. However, such a behavior as (11) is not “generic” for blowup solutions. We can say that L^2 -concentration phenomena in blowup solutions are peculiar to (NSC), but every blowup solution does not concentrate its L^2 mass at a single point.*¹⁸ “Single point blowup” as in (11) occurs in a very restrictive situations: these two theorems are a kind of inverse problem:

Theorem 1 ([33]). *We assume that $\psi_0 \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; |x|^2 dx)$. If the corresponding solution ψ blows up at a time $T > 0$ and satisfies*

$$\lim_{t \uparrow T_{\max}} \| |x - a| \psi(t) \| = 0 \quad \text{for some } a \in \mathbb{R}^d,$$

*then ψ should be of the form: up to Gallilei transformations,*¹⁹*

$$\psi(x, t) = (T - t)^{-d/2} \exp \left\{ -\frac{i|x|^2}{2(T - t)} \right\} \Psi \left(\frac{x}{T - t}, \frac{t}{T(T - t)} \right)$$

for some solution $\Psi \in C([0, \infty); H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; |x|^2 dx))$ of (NSC) such that $\mathcal{H}(\Psi) = 0$.

Theorem 2 ([34]). *Suppose one of the following two conditions holds:*

- (i) $d = 1$,
- (ii) $d \geq 2$, and ψ_0 being radially symmetric.

If we have, for some $T > 0$ and $a \in \mathbb{R}^d$,

$$|\psi(x, t)|^2 dx \rightarrow \|\psi_0\|^2 \delta_a(dx) \quad \text{as } t \uparrow T,$$

then

$$|x| \psi_0 \in L^2(\mathbb{R}^d) \quad \text{and} \quad \lim_{t \uparrow T} \| |x - a| \psi(t) \| = 0 \quad \text{as } t \uparrow T.$$

Now we discuss the sharpness of the estimate $\|\psi_0\| \leq \|Q_g\|$: Choosing $Q = Q_g$ in (8), we see that this threshold value $\|Q_g\|$ is sharp for the existence of blowup solutions as we mentioned before. Merle [21] proved that the explicit blowup solution of (8) with $Q = Q_g$ is the only blowup solution*²⁰ in $\{ \psi \in H^1(\mathbb{R}^d) \mid \|\psi\| = \|Q_g\| \}$.*²¹

3 The loglog law

Before going to discuss the generic behavior of blowup solutions of (NSC), we recall some known facts and results as to the blowup rates.

*¹⁸ We shall discuss the generic behavior of blowup solutions in Section 4.

*¹⁹ $\psi(x, t) \mapsto e^{i(vx - \frac{1}{2}|v|^2 t)} \psi(x - vt, t)$ for $v \in \mathbb{R}^d$.

*²⁰ up to space translations, Galilei transformations, dilations and multiplication of $e^{i\theta}$ for $\theta \in [0, 2\pi)$

*²¹ We do not need the weight condition.

It had been long conjectured that the rate of blowup (speed of blowup) is:

$$\|\nabla\psi(t)\| \asymp \sqrt{\frac{\ln \ln(T_{\max} - t)^{-1}}{T_{\max} - t}},$$

and the singularities are believed to be described by a Townes profile. This behavior is called “the loglog law”. But, explicit blowup solutions constructed in the previous section behave as:

$$\|\nabla\psi(t)\| \asymp \frac{1}{T_{\max} - t}.$$

Hence, we are in an odd and messy situation. For a short history of the quest for the loglog law, see, e.g., [40]. It was Perelman [39] who first succeeded in constructing a blowup solution of (NSC) with $d = 1$ near *the ground state level* which obey the loglog law in a rigorous mathematical way. Subsequently, Merle and Raphael had been studying with vigor [22, 23, 24] that, for $d = 1, 2, 3, 4$, every blowup solution slightly above *the ground state level* obeys the loglog law. For general class of (large) blowup solutions, the validity of the loglog law is still an open question, however. One of the key fact of their analyses is that Townes profile describe the singularity.*²²

Now we have, at least, two types of blowup rates, which makes the situation complicated. More worse, Fibich-Gavsh-X.P.Wang [10] suggests the existence of blowup solutions that show “self-similar” rate:

$$\|\nabla\psi(t)\| \asymp \sqrt{\frac{1}{T_{\max} - t}}.$$

They [10] find that the “self-similar solution” of (NSC) showed up instead of Townes profile, when we rescaled the singularities.

Their numerical observation in [10] together with the results of Perelman [39] and Merle-Raphael [22, 23, 24] also suggests that the asymptotic profile of blowup solutions and their blowup rates are closely relevant. It seems that these aspects cannot be considered separately at all.

Thus, it seems natural to ask that: under the following two conditions of blowup rates:*²³

$$\int_0^{T_{\max}} \|\nabla\psi(t)\| dt < \infty \quad \text{and} \quad \lim_{t \uparrow T_{\max}} \sqrt{T_{\max} - t} \|\nabla\psi(t)\| = \infty,$$

do we always have

$$\|\nabla\psi(t)\| \asymp \sqrt{\frac{\ln \ln(T_{\max} - t)^{-1}}{T_{\max} - t}}$$

*²² “Near” the ground state level, we have only one L^2 -concentration point (see Theorem 3 in Section 4).

*²³ The lower bound is known ([7, 42, 6]): $\|\nabla\psi(t)\| \gtrsim \frac{1}{\sqrt{T_{\max} - t}}$.

with a certain universal structure of singularities?^{*24}

In the last section, we shall consider this problem by means of Nelson diffusions.

4 Asymptotic Profiles of Blowup Solutions

In order to investigate the generic behavior of blowup solutions, we employ a kind of renormalization technique. Let ψ be a blowup solution of (NSC). We choose a time sequence as:

$$t_n \uparrow T_{\max}, \quad \sup_{t \in [0, t_n]} \|\psi(t)\|_{2+\frac{d}{2}} = \|\psi(t_n)\|_{2+\frac{d}{2}},$$

and define the scaling parameter

$$\lambda_n = \frac{1}{\|\psi(t_n)\|_{2+\frac{d}{2}}^{1+\frac{d}{2}}}.$$

Using this λ_n , we investigate the asymptotic behavior of

$$\psi_n(x, t) = \lambda_n^{\frac{d}{2}} \overline{\psi(\lambda_n x, t_n - \lambda_n^2 t)}$$

in some functional spaces.^{*25} We have:

Theorem 3 ([27, 28]). *The renormalized solution ψ_n behaves like a finite superposition of 0-energy, 0-momentum, global-in-positive-time solution of (NSC) accompanied by a “tail” (or “shoulder”). Precisely, we have:*

$$\psi_n(x, t) - \left(\sum_{j=1}^L \psi^j(x - \gamma_n^j, t) + \varphi_n(x, t) \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

in the strong topology of $C([0, T]; L^2(\mathbb{R}^d))$ (for any $T > 0$). Here,

(i) “Singularities” are carried by $\psi^j(x, t)$ ’s, which are solutions of (NSC) in $C_b(\mathbb{R}_+; H^1(\mathbb{R}^d))$ with $\mathcal{H}(\psi^j) = 0$ and $\mathfrak{S}(\psi^j, \nabla \psi^j) = 0$;

(ii) The “tail” $\varphi_n(x, t)$ solves:

$$\begin{cases} 2i \frac{\partial \varphi_n}{\partial t} + \Delta \varphi_n = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\ \varphi_n(x, 0) = \psi_n(x, 0) - \sum_{j=1}^L \psi^j(x - \gamma_n^j, 0), & x \in \mathbb{R}^d, \end{cases}$$

that is, $\varphi_n(x, t)$ ’s are solutions of the free Schrödinger equation; and

(iii) the sequences $\{\gamma_n^1\}, \{\gamma_n^2\}, \dots, \{\gamma_n^L\}$ are in \mathbb{R}^d such that $\lim_{n \rightarrow \infty} |\gamma_n^j - \gamma_n^k| = \infty$ ($j \neq k$).

In the original world of ψ , we have

$$\lim_{n \rightarrow \infty} \sup_{t \in [t_n - \lambda_n^2 T, t_n]} \left\| \overline{\psi(\cdot, t)} - \sum_{j=1}^L \psi_n^j(\cdot, t) - \tilde{\varphi}_n(\cdot, t) \right\| = 0$$

^{*24} A Towns profile is expected to appear under an appropriate scaling at each singularity.

^{*25} Information of asymptotic behavior of blowup solutions is encoded in that of the sequence $\{\psi_n\}$.

with

$$\lim_{n \rightarrow \infty} \lambda_n^2 \sup_{t \in [t_n - \lambda_n^2 T, t_n]} \|\tilde{\varphi}_n(t)\|_{2+\frac{4}{d}}^{2+\frac{4}{d}} = 0,$$

where

$$\begin{aligned} \psi_n^j(x, t) &= \frac{1}{\lambda_n^{d/2}} \psi^j \left(\frac{x - \gamma_n^j \lambda_n}{\lambda_n}, \frac{t_n - t}{\lambda_n^2} \right), \\ \tilde{\varphi}_n(x, t) &= \frac{1}{\lambda_n^{d/2}} \varphi_n \left(\frac{x}{\lambda_n}, \frac{t_n - t}{\lambda_n^2} \right). \end{aligned}$$

If the family of Radon measures defined by $\{|\psi(x, t)|^2 dx\}_{0 \leq t < T_{\max}}$ is tight, then we can show that: along $s_n := t_n - \lambda_n^2 T$, we have

$$|\psi(x, s_n)|^2 dx \rightarrow \sum_{j=1}^L \|\psi^j(0)\|^2 \delta_{a^j}(dx) + \mu(dx) \quad \text{as } n \rightarrow \infty$$

in the sense of measures, where μ comes from $|\tilde{\varphi}_n|^2 dx$ which has a different nature from the other part which produces the Dirac measures. However, there remains possibilities that we have $a^i = a^j$ for $i \neq j$ ("resonance") and that μ itself involves Dirac masses as well.

There arises a simple question here:

Do we always have the tightness of $\{|\psi(x, t)|^2 dx\}_{0 \leq t < T_{\max}}$?

Of course, in the weighted space $H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d; |x|^2 dx)$, we always have the tightness, provided that $T_{\max} < \infty$. Without such a weight-condition, we have:

Theorem 4 ([28]). *Suppose one of the following two conditions holds:*

- (i) $d = 1$ and $\mathcal{H}(\psi_0) < 0$,
- (ii) $d \geq 2$, $\mathcal{H}(\psi_0) < 0$ and ψ_0 being radially symmetric.

Then we have $T_{\max} < \infty$, that is, the corresponding solution ψ of (NSC) blows up in finite time T_{\max} , and the family of Radon measures $\{|\psi(x, t)|^2 dx\}_{0 \leq t < T_{\max}}$ is tight.

Remark 1. *Only the nonexistence part of global-in-time solution is proved by Ogawa-Y. Tsutsumi [37, 38].*

We should note here that this type of primary problem of proving the nonexistence of global solutions seems in fact closely relevant to the asymptotic profile of blowup solutions. Indeed, the following Theorem 5 (weak form of Theorem 4) plays a crucial role in proving the finiteness of ψ^j 's.

Theorem 5 ([26, 28]). *If ψ_0 has negative energy:*

$$\mathcal{H}(\psi_0) < 0,$$

then the corresponding solutions of (NSC) satisfies

$$\sup_{t \in [0, T_{\max})} \|\nabla \psi(t)\| = \infty.$$

Suppose that $T_{\max} = \infty$. Then we have that, for any $R > 0$,

$$\limsup_{t \uparrow \infty} \int_{|x| > R} |\nabla \psi(x, t)|^2 dx = \infty$$

This theorem is the main ingredient to prove the finiteness of ψ^j : If $L = \infty$ in the course of tracing the compactness loss of ψ_n , we have:

$$\limsup_{L \rightarrow \infty} \sum_{j=1}^L \mathcal{H}(\psi^j) \leq 0.$$

Thus, Theorem 5 implies^{*26} $\mathcal{H}(\psi^j) = 0$ for any j , so that we have

$$\|\psi^j\| \geq \|Q_g\| \quad \text{for each } j$$

by the variational characterization of the ground state Q_g . This fact implies the finiteness, because we have

$$\limsup_{L \rightarrow \infty} \sum_{j=1}^L \|\psi^j\|^2 \leq \|\psi_0\|^2.$$

Now we are back to the tightness problem for $\{|\psi(x, t)|^2 dx\}_{0 \leq t < T_{\max}}$. As we saw, the problem does not seem to be easy. However, once we know the blowup rate, we immediately obtain:

Theorem 6 ([29]). *Suppose that*

$$\int_0^{T_{\max}} \|\nabla \psi(t)\| dt < \infty. \quad (12)$$

Then the family of Radon measures $\{|\psi(x, t)|^2 dx\}_{0 < t < T_{\max}}$ is tight, and we have:

$$|\psi(x, t)|^2 dx \rightarrow \sum_{j=1}^L A_j \delta_{a^j}(dx) + \mu(dx) \quad \text{as } t \uparrow T_{\max}. \quad (13)$$

The number of singularities L , and their locations $\{a^j\}_{j=1}^L$ and amplitudes $\{A^j\}_{j=1}^L$ are uniquely determined.

^{*26} Each ψ^j belongs to $L^\infty([0, \infty); H^1(\mathbb{R}^d))$.

We shall give a “simple” proof of Theorem 6 by using the Nelson diffusion (constructed in Section 5) corresponding to the solution ψ , while we have another proof without using the probabilistic stuff [32].

The blowup rates are also relevant to the asymptotic profiles of the blowup solutions.

Theorem 7. *Suppose that ψ_0 gives rise to the blowup solution ψ of (NSC) such that $\lim_{t \uparrow T_{\max}} \|\nabla\psi(t)\| = \infty$. We put:*

$$\|\nabla\psi(t)\| \asymp \frac{1}{\sqrt{T_{\max} - t}} \quad (\text{SS})$$

This condition (SS) is incompatible with the following condition (B):

$$\text{We have } L = 1, \quad \varphi_n \equiv 0 \quad \text{and} \quad |x|\psi^1 \in L^2(\mathbb{R}^d) \quad \text{in Theorem 3.} \quad (\text{B})$$

The proof roughly goes as follows [32]: We assume both of the conditions (SS) and (B). It follows from Theorem 6 with the aid of an argument used in proving Theorem 2 in [34] that

$$\exists a \in \mathbb{R}^d; \quad \lim_{t \uparrow T_{\max}} |\psi(x, t)|^2 dx = \|\psi_0\|^2 \delta_a(dx)$$

with

$$|x|\psi_0 \in L^2(\mathbb{R}^d).$$

Hence, by Theorem 1, we have another expression of ψ :

$$\psi(x, t) = (T_{\max} - t)^{-d/2} \exp \left\{ -\frac{i|x|^2}{2(T_{\max} - t)} \right\} \Psi \left(\frac{x}{T_{\max} - t}, \frac{t}{T_{\max}(T_{\max} - t)} \right), \quad (14)$$

for some zero-energy, zero-momentum, global-in-time solution Ψ up to Galilei transformations and space translations. Applying renormalization procedure as in Theorem 3 to RHS, we have another sequence which should have the same asymptotic behavior as ψ_n , and we have by the explicit form of blowup solution above and (SS) that

$$\psi^1 \in C(\mathbb{R}; H^1(\mathbb{R}^d)) \quad \text{and} \quad |t|\psi^1 \in L^1((-\infty, 1); H^1(\mathbb{R}^d)).$$

On the other hand, ψ must obey

$$\Im \langle \psi^1, x \cdot \nabla \psi^1 \rangle = 0.$$

These contradicts each other. ^{*27}

From Theorem 7, it holds that:

$$\|\nabla\psi(t)\| \asymp \frac{1}{\sqrt{T_{\max} - t}} \implies L \geq 2 \quad \text{or} \quad \varphi_n \neq 0 \quad \text{or} \quad |x|\psi^1 \in L^2(\mathbb{R}^d).$$

^{*27} It is easy from the virial identity with $p = 1 + \frac{4}{d}$.

This suggests that the blowup profile could be different from a Townes profile as is suggested in [10]

We could expect the following stronger “theorem” :

“Theorem” 8. Suppose that ψ_0 gives rise to the blowup solution ψ of (NSC) such that $\lim_{t \uparrow T_{\max}} \|\nabla\psi(t)\| = \infty$. We put:

$$\begin{cases} \int_0^{T_{\max}} \|\nabla\psi(t)\| dt < \infty, \\ \lim_{t \uparrow T_{\max}} \sqrt{T_{\max} - t} \|\nabla\psi(t)\| = \infty, \\ \lim_{t \uparrow T_{\max}} (T_{\max} - t) \|\nabla\psi(t)\| = 0. \end{cases} \quad (\text{A})$$

This condition (A) is incompatible with the following:

$$\text{We have } L = 1, \quad \varphi_n \equiv 0 \quad \text{and} \quad |x|\psi^1 \in L^2(\mathbb{R}^d) \quad \text{in Theorem 3.} \quad (\text{B})$$

This conjecture suggests that:

$$\|\nabla\psi(t)\| \asymp \sqrt{\frac{\ln \ln(T_{\max} - t)^{-1}}{T_{\max} - t}} \implies L \geq 2 \quad \text{or} \quad \varphi_n \neq 0;$$

and roughly speaking

$$L = 1 \quad \text{and} \quad \varphi_n \equiv 0 \implies \|\nabla\psi(t)\| \gtrsim \frac{1}{T_{\max} - t}.$$

These properties might be what we expect according to known results and some numerical simulations.

Simple but Important Obsevation for “Theorem” 8

The 2nd condition of (A) in “Theorem” 8 gives us some information of singularities. In Theorem 3, suppose that

$$\lim_{t \uparrow T_{\max}} \sqrt{T_{\max} - t} \|\nabla\psi(t)\| = \infty \quad (15)$$

and that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{T_{\max} - a_n} \|\nabla\psi(a_n)\|}{\sqrt{T_{\max} - b_n} \|\nabla\psi(b_n)\|} = 1$$

for any sequence $\{a_n\}$ and $\{b_n\}$ both converging to 0 as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1.$$

Then we have that ψ^j ($j = 1, 2, \dots, L$) in Theorem 3 are defined on the whole real line \mathbb{R} , and they are bounded in $H^1(\mathbb{R}^d)$ for $t \in \mathbb{R}$, that is:

$$\psi^j \in C \cap L^\infty(-\infty, +\infty; H^1(\mathbb{R}^d)) \quad (j = 1, 2, \dots, L)$$

with

$$\mathcal{H}(\psi^j) = 0, \quad \Im\langle \psi^j, \nabla \psi^j \rangle = 0.$$

If $|x|\psi^j \in L^2(\mathbb{R}^d)$ further, then we obtain

$$\Im\langle \psi^j, x \cdot \nabla \psi^j \rangle = 0,$$

so that the Virial identity yields

$$\| |x| \psi^j(t) \| = \| |x| \psi^j(0) \| \quad \text{for any } t \in \mathbb{R}.$$

These facts above seem to be strongly suggesting that ψ^j 's are bound states, which is plausible.

In the next section, we shall see that a weak version of this conjecture holds valid (see Theorems 10 and 11).

5 Nelson Diffusions and its applications

Let ψ be a solution of (NSC) in $C([0, T_{\max}); H^1(\mathbb{R}^d))$. We can construct a measure on the path space $\Gamma \equiv C([0, T_{\max}); \mathbb{R}^d)$ which gives us the same prediction as standard Quantum Mechanics does. In order to state it precisely, put:

$$u(x, t) \equiv \begin{cases} \Re \frac{\nabla \psi(x, t)}{\psi(x, t)}, & \text{if } \psi(x, t) \neq 0 \\ 0, & \text{if } \psi(x, t) = 0, \end{cases}$$

$$v(x, t) \equiv \begin{cases} \Im \frac{\nabla \psi(x, t)}{\psi(x, t)}, & \text{if } \psi(x, t) \neq 0 \\ 0, & \text{if } \psi(x, t) = 0, \end{cases}$$

and define

$$b(x, t) \equiv u(x, t) + v(x, t).$$

Under this notation, we have:

Theorem 9. *Let u , v , and b be defined through the solution ψ of (NSC) on $[0, T_{\max})$. We associate $\Gamma \equiv C([0, T_{\max}); \mathbb{R}^d)$ with its Borel σ -algebra \mathcal{F} with respect to the Fréchet topology. Let $(\Gamma, \mathcal{F}, \mathcal{F}_t, X_t)$ be evaluation stochastic process $X_t(\gamma) \equiv \gamma(t)$ for $\gamma \in \Gamma$ with natural filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Then there exists a Borel "probability" measure P on Γ such that:*

- (i) $(\Gamma, \mathcal{F}, \mathcal{F}_t, X_t, P)$ is a Markov process,
- (ii) the image of P under X_t has density, that is,

$$P[X_t \in dx] = \frac{|\psi(x, t)|^2 dx}{\|\psi(0)\|^2}, \quad (16)$$

- (iii) The following process B_t is a $(\Gamma, \mathcal{F}_t, P)$ -Brownian motion:

$$B_t \stackrel{\text{def}}{=} X_t - X_0 - \int_0^t b(X_\tau, \tau) d\tau. \quad (17)$$

Carlen ([3, 4, 5]) proved this theorem for linear Schrödinger equations with appropriate potentials which give rise to the finite energy solutions^{*28} satisfying L^2 -norm conservation law.^{*29} His proof works well for our finite energy solutions of NLS, a fortiori (NSC) (see [29, 32]).

The process $(\Gamma, \mathcal{F}, \mathcal{F}_t, X_t, P)$ constructed in Theorem 9 is a so-called weak solution of Ito-type stochastic differential equation:

$$dx_t = b(x_t, t)dt + dB_t,$$

that is, a kind of martingale problem (see, e.g. [13]): “Find a measure P on Γ which make the functional B_t in (17) a Brownian motion!”^{*30} Nelson [35] (see also [36]) proposed such a process in his theory of stochastic quantization.^{*31} So, the process is referred to as a “Nelson diffusion”, which is pragmatically a measure defined on the path space Γ associated to each solution of the Schrödinger equation in consideration.

We shall not discuss the problem of the stochastic quantization. The important thing here is the measure P on Γ does exist for each solution, although the notorious “Feynman measure”, which is in nature universal, does not exist mathematically as a canonical measure on Γ (see, e.g., [16]).

The first benefit of considering the process is that we have a “simple” proof of Theorem 6. Under the assumption of (12), one can show with the aid of Borel-Canteli argument that the process has the limit: $\lim_{t \uparrow T_{\max}} X_t$ a.s. The key fact used here is the following estimate:

$$E \left[\int_0^t |b(X_\tau, \tau)| d\tau \right] \leq 2\|\psi_0\| \int_0^t \|\nabla\psi(t)\| dt \quad (18)$$

It is well known that the convergence of processes implies that of the distributions:^{*32}

$$\exists \lim_{t \uparrow T_{\max}} P[X_t \in dx] \equiv \lim_{t \uparrow T_{\max}} |\psi(x, t)|^2 dx,$$

so that $\{|\psi(x, t)|^2 dx\}_{0 \leq t < T_{\max}}$ is tight and we have (13), since we have the limiting profile under some sequences.^{*33}

The second benefit is the following theorem [32], which is a weak version of “Theorem” 8.

^{*28} the solution belongs to $C(\mathbb{R}; H^1(\mathbb{R}^d))$

^{*29} Carlen proved this type of theorem for finite energy solutions with weight-condition (that is: solutions in the form domain of harmonic oscillators) in [3], and subsequently in [4] presented the considerably detailed outline of the proof for purely finite energy solutions. We can complete his proof in [4] with some modifications (see [17]),

^{*30} Note that $\{X_t\}$ is a family of given evaluation maps.

^{*31} Fényes [8] also proposed such a concept of Quantization.

^{*32} For simplicity, we “normalize” the total probability to be $\|\psi_0\|^2$.

^{*33} See the arguments below Theorem 3.

Theorem 10. Assume that $\sqrt{|x|}\psi_0 \notin L^2(\mathbb{R}^d)$. Then (12) implies “nontrivial” φ_n in Theorem 3: Precisely,

$$\int_0^{T_{\max}} \|\nabla\psi(t)\| dt < \infty \implies \int_{\mathbb{R}^d} |x| \mu(dx) = \infty$$

where μ is the measure found in (13).

The proof is rather simple by using the stochastic differential equation (17). Without the aid of stochastic stuff, we have the following analogous theorem [32]:

Theorem 11. Assume that $|x|\psi_0 \notin L^2(\mathbb{R}^d)$. Then we have:

$$\begin{cases} \int_0^{T_{\max}} \|\nabla\psi(t)\| dt < \infty \\ \sup_{0 < t < T_{\max}} (T_{\max} - t) \|\nabla\psi(t)\| < \infty \end{cases} \implies \int_{\mathbb{R}^d} |x|^2 \mu(dx) = \infty$$

where μ is the measure found in (13).

The idea of proving Theorems 2 and 4 works well under the assumptions made on the blowup rates of Theorem 11. Both in Theorem 10 and Theorem 11, we may be able to remove the weight-conditions made on initial data. In [20], Merle constructed blowup solutions such that we have (13) with $L \geq 2$ and $\mu \equiv 0$. But all those blowup rates are the same as that of explicit blowup solutions. These solution also suggest that the existence of nontrivial μ is closely related to the blowup rate.

The third benefit will be that we could reveal the hidden mechanism of the loglog law of the blowup rate for (NSC). Even though the weak solution of (17), once we have a Brownian motion, the Brownian motion B_t satisfies the law of iterated logarithm (LIL) (see, e.g., [12, 13, 19]):

$$\limsup_{s \downarrow 0} \frac{1}{\sqrt{s \ln \ln \frac{1}{s}}} |B_{T_{\max}} - B_{T_{\max}-s}| < \infty.$$

From this, we have:

$$\begin{aligned} & \limsup_{t \uparrow T_{\max}} \sqrt{\frac{T_{\max} - t}{\ln \ln (T_{\max} - t)^{-1}}} \left| \frac{B_{T_{\max}} - B_t}{T_{\max} - t} \right| \\ &= \limsup_{t \uparrow T_{\max}} \frac{1}{\sqrt{(T_{\max} - t) \ln \ln (T_{\max} - t)^{-1}}} |B_{T_{\max}} - B_t| < \infty, \quad \text{a.s.,} \end{aligned}$$

This property could be a hidden mechanism of the loglog law. We might expect the following “Theorem” 12, considering a disguise of (17).^{*34}

$$\frac{B_{T_{\max}} - B_t}{T_{\max} - t} = \frac{X_{T_{\max}} - X_t}{T_{\max} - t} - \frac{1}{T_{\max} - t} \int_t^{T_{\max}} b(X_\tau, \tau) d\tau.$$

^{*34} If a blowup solution ψ belongs to $C([0, T_{\max}); H^1(\mathbb{R}^d) \cap L^2(|x|^2 dx))$, then it can be expressed by the formula (14) with some $\Psi \in C([0, \infty); H^1(\mathbb{R}^d) \cap L^2(|x|^2 dx))$ such that $\mathcal{H}(\Psi) \geq 0$. Computing the

“Theorem” 12. Let ψ be a blowup solution of (NSC) such that $\lim_{t \uparrow T_{\max}} \|\nabla\psi(t)\| = \infty$. Suppose that

$$\begin{cases} \int_0^{T_{\max}} \|\nabla\psi(t)\| dt < \infty, \\ \lim_{t \uparrow T_{\max}} \sqrt{T_{\max} - t} \|\nabla\psi(t)\| = \infty, \\ \lim_{t \uparrow T_{\max}} (T_{\max} - t) \|\nabla\psi(t)\| = 0. \end{cases} \quad (\text{A})$$

Then we have:

$$\limsup_{t \uparrow T_{\max}} \sqrt{\frac{T_{\max} - t}{\ln \ln(T_{\max} - t)^{-1}}} \left(\frac{1}{T_{\max} - t} \int_t^{T_{\max}} \|\nabla\psi(\tau)\| d\tau \right) \asymp 1. \quad (19)$$

We may call the assertion (19) the weak-loglog law. In order to prove this conjecture, we need to know the sample path trajectories of the Nelson diffusions. For simplicity, we assume that the origin is one of the L^2 concentration point, that is, $a^1 \equiv 0$ in (13). Hence, considering a subset Γ_0 of Γ defined by

$$\Gamma_0 := \{ \gamma \in \Gamma \mid \gamma(t) \rightarrow 0 \text{ as } t \uparrow T_{\max} \},$$

we have $P(\Gamma_0) = A_1 > 0$.^{*35}

Now we introduce:

$$\Gamma_1(R) := \bigcup_{\eta > 0} \bigcap_{\eta < t < T_{\max}} \left\{ \gamma \in \Gamma \mid |\gamma(t)| \leq R \frac{1}{\|\nabla\psi(t)\|} \right\}$$

and

$$\Gamma_2(R) := \bigcup_{\eta > 0} \bigcap_{\eta < t < T_{\max}} \left\{ \gamma \in \Gamma \mid |\gamma(t)| \leq R \int_t^{T_{\max}} \|\nabla\psi(\tau)\| d\tau \right\}.$$

If one can prove, under the condition (A) in “Theorem” 12,

$$P(\Gamma_1) > 0 \text{ or } P(\Gamma_2) > 0,$$

then the lower estimate, i.e., “ \gtrsim ”-part in (19) could be proved. The upper estimate of “ \lesssim ”-part could be proved, if one can show that $P(\Gamma_3) > 0$ under the condition (A), where

$$\Gamma_3 := \Gamma_0 \cap \mathcal{C} \left\{ \bigcup_{R > 0} \bigcup_{\eta > 0} \bigcap_{\eta < t < T_{\max}} \left\{ \gamma \in \Gamma \mid |\gamma(t)| \leq R \int_t^{T_{\max}} \|\nabla\psi(\tau)\| d\tau \right\} \right\}.$$

drift b from this expression, we have an unpleasant term: $\frac{-x}{T_{\max}-t}$ coming from accompanying function $e^{-\frac{|x|^2}{2(T_{\max}-t)}}$. If $\mu \neq 0$, this term causes a contradiction under the assumption of (12) (see Theorem 6 and (18)), so that such a term should be canceled out with another term coming from Ψ . Thus the leading term might be

$$\int_t^{T_{\max}} \|\nabla\psi(t)\| \left(\frac{\nabla\psi^1}{\psi^1} \right) \left(\frac{X_\tau}{\|\nabla\psi(\tau)\|}, \tau \right) d\tau.$$

^{*35} We are using the “normalization” of $P(\Gamma) = \|\psi_0\|^2$.

Here, \complement denotes the operation of taking the complement of the set appearing to just right of the symbol. The paths in Γ_3 curve “wildly”, reaching $0 \in \mathbb{R}^d$ finally at T_{\max} .

However, we have not succeeded in proving these subsets $\Gamma_1, \Gamma_2, \Gamma_3$ of Γ_0 having positive probabilities.*³⁶ At the present, we have [32]:

Theorem 13. *We assume (12) and (15) for a blowup solution ψ of (NSC). Then, we have:*

$$\liminf_{t \uparrow T_{\max}} \frac{\sup_{T < t < T_{\max}} |X_t - X_{T_{\max}}|}{\int_T^{T_{\max}} \|\nabla\psi(\tau)\| d\tau} < \infty, \quad a.s.,$$

and, for any $\varepsilon > 0$,

$$\lim_{t \uparrow T_{\max}} \frac{\sup_{T < t < T_{\max}} |X_t - X_{T_{\max}}|}{\left(\int_T^{T_{\max}} \|\nabla\psi(\tau)\| d\tau\right)^{1-\varepsilon}} = 0, \quad a.s.$$

The first assertion is considerably easy by Fatou’s lemma, while the second needs the Borel-Canteli lemma.

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*³⁶ For the lower estimate, we need to know the sample path property in shrinking balls converging to the singularity. In contrast, the upper estimate requires subtler information on sample path behavior “outside” the shrinking balls.

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