Title: Asymptotic property of solutions of nonautonomous Lotka-Volterra model for $N$-competing species (Theory of Biomathematics and its Applications VI)

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1 Introduction and Statements of the main results

In this paper we consider the system of differential equations

$$u'_i = u_i \left[ a_i(t) - \sum_{j=1}^{N} b_{ij}(t)f_{ij}(u_i, u_j) \right], \quad i = 1, \ldots, N, \ N \geq 2,$$

(GLV)

where the functions $a_i(t), 1 \leq i \leq N,$ and $b_{ij}(t), 1 \leq i, j \leq N,$ are assumed to be continuous and nonnegative on $\mathbb{R}$. Furthermore, let the functions $f_{ij}(x, y), 1 \leq i, j \leq N,$ be continuously differentiable on $\mathbb{R}^2_+ = (0, \infty)^2$, and we impose the following conditions on $f_{ij}$s:

$$\begin{cases}
    f_{ii}(x, y), \ 1 \leq i \leq N, \text{ is continuously differentiable on } [0, \infty) \times [0, \infty);
    f_{ij}(x, y) > 0, \ (x, y) \in \mathbb{R}^2_+, \ 1 \leq i, j \leq N;
    (D_1 f_{ii} + D_2 f_{ii})(x, x) > 0, \ x \in \mathbb{R}_+, \ 1 \leq i \leq N;
    D_1 f_{ij}(x, y) \geq 0, \ (x, y) \in \mathbb{R}^2_+, \ 1 \leq i, j \leq N;
    D_2 f_{ij}(x, y) \geq 0, \ (x, y) \in \mathbb{R}^2_+, \ 1 \leq i, j \leq N;
    f_{ii}(0, 0) = 0, \ 1 \leq i \leq N;
    \lim_{x \to \infty} f_{ii}(x, x) = \infty, \ 1 \leq i \leq N,
\end{cases}$$

(1.1)

where $D_i, i = 1, 2$, denotes the differentiation with respect to the $i$-th variable.

System (GLV) is a generalization of the following nonautonomous $N$-dimensional Lotka-Volterra competition system which S. Ahmad and A. C. Lazer [2] considered:

$$u'_i = u_i \left[ a_i(t) - \sum_{j=1}^{N} b_{ij}(t)u_j \right], \quad i = 1, \ldots, N, \ N \geq 2.$$  

(LV)

An prototype of system (LV), as well as (GLV), is the classical Lotka-Volterra competition model for two species:

$$\begin{cases}
    u'_1 = u_1(a_1 - b_{11}u_1 - b_{12}u_2),
    u'_2 = u_2(a_2 - b_{21}u_1 - b_{22}u_2),
\end{cases}$$

(1.2)

where $a_i, i = 1, 2,$ and $b_{ij}, i, j = 1, 2,$ are positive constants. When the growth rates $a_i, i = 1, 2,$ and the interaction coefficients $b_{ij}, i, j = 1, 2,$ satisfy

$$a_1 - b_{12} \left( \frac{a_2}{b_{22}} \right) > 0, \quad a_2 - b_{21} \left( \frac{a_1}{b_{11}} \right) > 0,$$

(1.3)
there exists a unique equilibrium point \((u_1^*, u_2^*) \in \mathbb{R}_+^2\). It is known that, if (1.3) hold, then any solution \((u_1(t), u_2(t))\) of system (1.2) with \((u_1(t_0), u_2(t_0)) \in \mathbb{R}_+^2\) satisfies

\[ u_1(t) \to u_1^* \text{ and } u_2(t) \to u_2^* \text{ as } t \to \infty. \]

In [2]–[4] it is shown that analogous results still hold for the nonautonomous equation (LV), as seen below. In this paper we intend to generalize such results further.

We introduce notation. Put \(c_M := \sup_{t \in \mathbb{R}} c(t)\) for bounded functions \(c(t)\) on \(\mathbb{R}\). For \(i = 1, \ldots, N\), we put

\[ \tilde{f}_{ii}(x) = f_{ii}(x, x), \quad x \in \mathbb{R}_+. \]

By assumption (1.1) \(\tilde{f}_{ii}, i = 1, \ldots, N\), have the inverse function \(\tilde{f}_{ii}^{-1} : \mathbb{R}_+ \to \mathbb{R}_+.\) The assumptions employed in the paper will be selected from the following list:

\begin{enumerate}[\textit{(A1)}]
    \item \(b_{ii}(t) > 0, \quad t \in \mathbb{R}, \ 1 \leq i \leq N;\)
    \item \(\int_0^\infty b_{ii}(s)ds = \infty, \quad 1 \leq i \leq N;\)
    \item \(\left( \frac{a_{ii}}{b_{ii}} \right)_M < \infty, \quad 1 \leq i \leq N;\)
    \item \(\inf_{t \in \mathbb{R}} \frac{a_{ii} - \sum_{j \neq i} b_{ij}(t)(a_{j}/b_{jj})_M}{b_{ii}(t)} > 0, \quad 1 \leq i \leq N;\)
    \item \(\inf_{t \in \mathbb{R}} \frac{a_{ii} - \sum_{j \neq i} b_{ij}(t)f_{ij}((\tilde{f}_{ii}^{-1}(a_{ij}/b_{jj})_M), \tilde{f}_{jj}^{-1}(a_{jj}/b_{jj})_M))}{b_{ii}(t)} > 0, \ 1 \leq i \leq N;\)
    \item \(f_{ij}(x, y) \leq \tilde{f}_{jj}(y), \quad (x, y) \in \mathbb{R}_+^2, \ 1 \leq i, j \leq N;\)
    \item \(\text{for any } s > 1 \text{ sufficiently close to } 1;\)
    \[ f_{ij}(\tilde{f}_{ii}^{-1}(sz), \tilde{f}_{jj}^{-1}(sy)) \leq sf_{ij}(\tilde{f}_{ii}^{-1}(x), \tilde{f}_{jj}^{-1}(y)), \quad (x, y) \in \mathbb{R}_+^2, \ 1 \leq i, j \leq N.\]
\end{enumerate}

\textbf{REMARK 1.1.} As in the case of (LV) and (1.2), if \(f_{ij}(x, y), 1 \leq i, j \leq N\), are independent of \(x\), (A6) is satisfied. For (LV) we can take \(f_{ij}(x, y) = y, 1 \leq i, j \leq N\), which satisfy (A6) and (A7).

\textbf{REMARK 1.2.} Let

\[ f_{ij}(x, y) = \begin{cases} 
    x_1^{a_{ij}}y^{\beta_{ij}}, & i \neq j, \\
    x_1^{a_{ij}}y^{\beta_{ij}}, & i = j,
\end{cases} \]

where \(\alpha_{ij}, \beta_{ij} \in \mathbb{R}_+.\) If for \(i \neq j, \beta_{ij} = \alpha_{jj} + \beta_{jj}\), then the functions \(f_{ij}, 1 \leq i, j \leq N\), satisfy (A6).

\textbf{REMARK 1.3.} Let

\[ f_{ij}(x, y) = x_1^{a_{ij}}y^{\beta_{ij}}, \quad (x, y) \in \mathbb{R}_+^2, \ 1 \leq i, j \leq N, \]

where \(\alpha_{ij}, \beta_{ij} \in \mathbb{R}_+.\) If \(\alpha_{ij} + \beta_{ij} \leq \min\{\alpha_{ii} + \beta_{ii}, \alpha_{jj} + \beta_{jj}\}\), then the functions \(f_{ij}, 1 \leq i, j \leq N\), satisfy (A7).

S. Ahmad and A. C. Lazer [2] supposed that the functions \(a_i(t), 1 \leq i \leq N\) and \(b_{ij}(t), 1 \leq i, j \leq N\), satisfy conditions (A1)–(A3) and (A4). Under these conditions they have shown the following [2]:

\begin{enumerate}[\textbf{(I)}]
    \item If \(u = (u_i, \ldots, u_N)\) is a solution of (LV) with \(u_i(t_0) > 0, 1 \leq i \leq N, t_0 \in \mathbb{R}\), then
    \[ 0 < \inf_{t \geq t_0} u_i(t) \leq \sup_{t \geq t_0} u_i(t) < \infty, \quad \text{for } 1 \leq i \leq N. \]
If $A$ is a compact subset of $\mathbb{R}_+^N$, then the Lebesgue measure of the set \{u(t) | u is a solution of (LV) satisfying $u(t_0) \in A$\} tends to 0 as $t \to \infty$.

Our main aim is to show that (I) and (II) are still valid for (GLV). To state the results we introduce the symbol: For compact subset $A$ of $\mathbb{R}_+^N$ and $t_0 \in \mathbb{R}$ we set

$$u(t, t_0, A) = \{u(t) | u is a solution of (GLV) satisfying $u(t_0) \in A$\}.$$ 

By $\mu(\cdot)$ we denote the Lebesgue measure of measurable sets in $\mathbb{R}_+^N$. We can show the following:

**Theorem 1.4.** Let conditions (A1)–(A3), (A4), and (A6) hold. Let $A$ be a compact subset of $\mathbb{R}_+^N$ and let $t_0 \in \mathbb{R}$. Then,

$$\mu(u(t, t_0, A)) \to 0 \quad \text{as} \quad t \to \infty.$$ 

**Theorem 1.5.** Let conditions (A1)–(A3), (A5), and (A7) hold. Let $A$ be a compact subset of $\mathbb{R}_+^N$ and let $t_0 \in \mathbb{R}$. Then,

$$\mu(u(t, t_0, A)) \to 0 \quad \text{as} \quad t \to \infty.$$ 

We give examples of systems (GLV) for which above conditions hold.

**Example 1.6.** We consider system (GLV) for two species

\[ \begin{align*}
    u_1' &= u_1 \left[ (\cos t + 7) - (\sin t + 7) \cdot u_1^2 - (\sin t + 1) \cdot \left( \frac{u_1^3}{1 + u_1^3} \cdot u_2^2 \right) \right], \\
    u_2' &= u_2 \left[ (\cos t + 9) - (\sin t + 2) \cdot \left( \frac{u_2^4}{1 + u_2^4} \cdot u_1^2 \right) - (\sin t + 9) \cdot u_2^3 \right].
\end{align*} \]

Obviously (A6) holds. We have

\[ a_1(t) - b_{12}(t) \left( \frac{a_2}{b_{22}} \right)_M > \cos t + 7 - (\sin t + 1) \cdot \frac{10}{8} > 2, \]

\[ a_2(t) - b_{21}(t) \left( \frac{a_1}{b_{11}} \right)_M > \cos t + 9 - (\sin t + 2) \cdot \frac{8}{6} > 2. \]

So conditions (A1)–(A3) and (A4) hold. Of course condition (1.1) hold.

**Example 1.7.** We consider system (GLV) for two-species

\[ \begin{align*}
    u_1' &= u_1 \left[ (\cos t + 7) - (\sin t + 7) \cdot u_1^4 - (\sin t + 1) \cdot u_1 u_2^2 \right], \\
    u_2' &= u_2 \left[ (\cos t + 9) - (\sin t + 2) \cdot u_2^2 u_1^2 - (\sin t + 9) \cdot u_2^6 \right].
\end{align*} \]

Obviously (A7) holds. We have

\[ a_1(t) - b_{12}(t)f_{12} \left( \tilde{f}_{11}^{-1}((\frac{a_1}{b_{11}})_M), \tilde{f}_{22}^{-1}((\frac{a_2}{b_{22}})_M) \right) > \cos t + 7 - (\sin t + 1) \cdot \left( \frac{8}{6} \right)^{1/4} \cdot \left( \frac{10}{8} \right)^{2/6} > 2, \]

\[ a_2(t) - b_{21}(t)f_{21} \left( \tilde{f}_{22}^{-1}((\frac{a_2}{b_{22}})_M), \tilde{f}_{11}^{-1}((\frac{a_1}{b_{11}})_M) \right) > \cos t + 9 - (\sin t + 2) \cdot \left( \frac{10}{8} \right)^{2/6} \cdot \left( \frac{4}{3} \right)^{2/4} > 2. \]

So conditions (A1)–(A3), (A5) hold. Of course condition (1.1) hold.
2 The sketch of the proof of the main results

In this section we give the sketch of the proof of the main results. As a first step, we note that every solutions \( u \) of (GLV) with \( u(t_0) \in \mathbb{R}^N_+ \) remains here as long as it exists. To see this we rewrite system (GLV) in the form

\[ u'_i(t) = p_i(t)u_i(t), \quad i = 1, 2, \ldots, N, \]

where the functions \( p_i(t) \), \( 1 \leq i \leq N \), are given by

\[ p_i(t) = a_i(t) - \sum_{j=1}^{N} b_{ij}(t)f_{1j}(u_i(t), u_j(t)). \]

Since \( p_i, 1 \leq i \leq N \), is continuous on the domain of \( u \), for \( t \) in the domain of \( u \) we obtain

\[ u_i(t) = u_i(t_0) \exp \int_{t_0}^{t} p_i(s)ds > 0. \]

Hence \( u(t) \in \mathbb{R}_+^N \). Next we rewrite system (GLV) in the form

\[ u' = g(u, t), \]

where \( u(t) = (u_1(t), \ldots, u_N(t)) \in \mathbb{R}^N \), and \( g(u, t) = (g_1(u, t), \ldots, g_N(u, t)) \) is given by

\[ g_i(x, t) = x_i \left[ a_i(t) - \sum_{j=1}^{N} b_{ij}(t)f_{1j}(x_i, x_j) \right], \quad 1 \leq i \leq N, \]

for \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \). Since the functions \( a_i, 1 \leq i \leq N \), and \( b_{ij}, 1 \leq i, j \leq N \), are continuous on \( \mathbb{R} \) and the functions \( f_{ij}, 1 \leq i, j \leq N \), are continuously differentiable on \( \mathbb{R}_+^2 \), for every \( \xi = (\xi_i) \in \mathbb{R}_+^N \) and \( \tau \in \mathbb{R} \), there exists a unique solution \( u(t) \) of (GLV) with \( u(\tau) = \xi \). We denote it by \( u(t, \tau, \xi) = (u_i(t, \tau, \xi)) \).

Recall that we have introduced the notation:

\[ u(t, t_0, A) = \{u(t, t_0, \xi) | \xi \in A\} \]

for \( A \subset \mathbb{R}_+^N \). Furthermore, since the functions \( g_i(x, t), 1 \leq i \leq N \), are continuously differentiable with respect to the components of \( x \in \mathbb{R}^N \), \( u(t, \tau, \xi) \) are continuously differentiable with respect to the components of \( \xi \in \mathbb{R}^N \). Therefore we can introduce the following notations. We denote by \( D_\xi u(t, \tau, \xi) \) the \( N \times N \) matrix with \( (i, j) \)th entry equal to \( \partial u_i(t, \tau, \xi)/\partial \xi_j \):

\[ D_\xi u(t, \tau, \xi) = \left[ \frac{\partial u_i(t, \tau, \xi)}{\partial \xi_j} \right], \]

where \( \xi \in \mathbb{R}_+^N \). Similarly we define \( N \times N \) matrix \( D_x g(x, t) \) by

\[ D_x g(x, t) = \left[ \frac{\partial g_i(x, t)}{\partial x_j} \right], \]

where \( x \in \mathbb{R}_+^N \).

Now for \( t \geq t_0 \) and \( \xi_0 \in \mathbb{R}_+^N \), we set \( u_0(t) = u(t, t_0, \xi_0) \). Then it is well known [6] that

\[ X'(t) = A(t)X(t), \quad X(t_0) = I, \]

where

\[ X(t) = D_\xi u(t, t_0, \xi_0), \quad A(t) = D_x g(u_0(t), t), \]

and \( I \) is the \( N \times N \) identity matrix. Furthermore we know that

\[ \det X(t) = \exp \int_{t_0}^{t} \text{tr} A(s)ds. \]
Therefore, we have
\[ \det D_{\xi}u(t, t_0, \xi_0) = \exp \int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds. \]
Hence it follows from the change of variables formula that
\[ \mu(u(t, t_0, A)) = \int_{u(t, t_0, A)} \exp \int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds d\xi_0. \]
Therefore, by (A2) and (1.1), in order to prove Theorems 1.4 and 1.5, it is sufficient to prove the following claim:
\[ \text{Claim (see Taniguchi [1, Lemmas 3.1 and 4.1]). If either conditions (A1)-(A3), (A4), and (A6) or conditions (A1)-(A3), (A5) and (A7) hold, there exists some numbers } \delta_{A} > 0 \text{ and } t_{A} \geq t_{0} \text{ such that for } t \geq t_{A}, i = 1, \ldots, N, \text{ and } \xi_{0} \in A, \]
\[ \delta_{A} \leq u_{i}(t, t_0, \xi_0) \leq M_{A}. \]  
(2.1)
In fact, by (1.1), (2.1), we have
\[ \int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds \leq \int_{t_0}^{t} \sum_{i=1}^{N} b_{ii}(s) ds = \int_{t_0}^{t} \sum_{i=1}^{N} b_{ii}(s) \frac{\delta_{A}}{\delta_{A}} ds = N \log \frac{M_{A}}{\delta_{A}} - \delta_{A} \frac{\delta_{A}}{\delta_{A}} \int_{t_0}^{t} \sum_{i=1}^{N} b_{ii}(s) ds, \]
where \( \delta_{A} := \min \{ \tilde{f}_{ii}(\delta_{A}) | 1 \leq i \leq N \} \). Therefore, by (A2), we have
\[ \int_{t_0}^{t} \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds \to -\infty \text{ as } t \to \infty \]
uniformly with respect to \( \xi_0 \in A \); that is
\[ \mu(u(t, t_0, A)) \to 0 \text{ as } t \to \infty. \]
This completes the proof.

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