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Construction of non-associative algebras in algebras generated by Chomsky sentences

By

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Abstract. *A strucutre of binary product is introduced to the context free language(Chomsky algebra)and the binary strucutre of an arbitrary algebra is .realized in the algebra. Non-associative algebras are discussed by introducing the contexts to Chomsky sentences and flexible algebra and Jordan algebra are constructed*

Introduction

In this paper we introduce an algebraic structure in the context free language which is called Chomsky algebra and discuss non-associative algebras in this algebra. Here we call the context free language Chomsky language when a dictionary is given([1]). For an algebra A, we make the Chomsky algebra $C(A)$ of the dictionary A. Then we see that there exists a homomorphism :

$$\Phi : C(A) \to A$$

which is called the versatility homomorphism. This implies that the Chomsky algebra is a kind of a free algebra of A. Then we can realize the binary structure of an arbitrary algebra on the Chomsky algebra and discuss associativity or non-asscoiativity structure by use of the binary strucutre of Chomsky algebra. In order to discuss the given algebra, we have to analyze the kernel of the homorphism. Here we notice that the analysis of the kernel of the morophism is nothing but the introduction of the „context" to the context free sentences. Hence we see that we can discuss the non-associativity of algebras in terms of the introduction of the contexts to the context free sentences. Our discussion is divided into three parts:

(1) Generation of non-associative algebras in Chomsky sentences.
(2) Generation of associative sentences by shift operation
(3) Realization of non associative algebra by sentence algebra and Mendel algebra.

(1)In the first part we will introduce a non-associaitve strucutre in Chomsky sentences as much as possible. Here we choose two algebras, „sentence algebra" and

„Mendel algebra" which may describe the full non-associative algebraic structure. The sentence algebra is motivated by the structure of Chomsky sentences([1]). We can introduce the sentence algebra for a given Chomsky sentence and we can describe the non-associativity of the sentences explicitly. The second algebra is motivated by the Mendelian genetics([3]). We call the linear space M with generators $S_1, S_2, \ldots, S_n$ Mendel algebra, when generators satisfying the following commutation relations and the distributive law([4]):

$$S_i * S_j = \frac{1}{2}\{S_i + S_j\}$$

We can introduce the non-associativity strucutre in Chomsky sentences by use of the non-associativity of Mendel algebras.

(2) In the second part we will treat well known non-associative algebras. Here we want to treat the following two algebras as examples([2]):

flexible algebra: $(XY)X = X(YX)$
Jordan algebra: $(((XX)Y)X) = (XX)(YX)$

for any pair of elements $^\forall X, ^\forall Y \in A$. For this we restrict ourselves to a special classes of non-assosiative algebras including flexible algebra and Jordan algebra. We can introduce a concept of shift operations in Chomsky sentences and describe commutation relations in terms of shift operations. Then we can describe „associative structure" of non-associative algebras in terms of the „shift invarant sentences". At first we discuss algebras with following commutation relations:

$$(XY)Z = X(YZ) \text{ or } (((XY)Z)W) = (XY)(ZW).$$

We can obtain shift invariant sentences making symmetric elements:

$$\frac{1}{2}\{(XY)Z + X(YZ)\} \text{ or } \frac{1}{2}\{(X(Y(ZW))) + (XY)(ZW)\}$$

In order to get the commutation relations of flexible algebras or Jordan algebras

$$(XY)X = X(YX) \text{ or } (((XX)Y)X) = (XX)(YX),$$

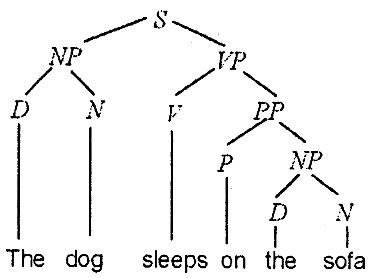
we have to choose shift operations of restricted type and make symmetrizations.

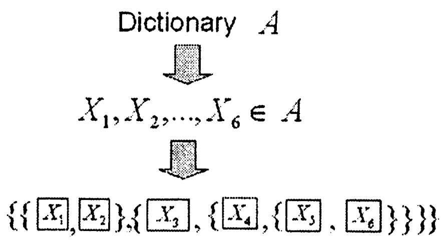
(3) Next we proceed to the realization of non-associativity of Chomsky sentences in an explicit manner. By use of sentence algebras we can realize flexible algebras or Jordan algebras by special choices of sentence algberas, but their choices are strongly restricted. Hence we will consider another realization by use of Mendel algebras. This is one of the important contribution to the theory of non-associative algebra, if there exists. Then we can show that the shift invariant algebras on Mendel algebra are automatically derive algebras including flexible algebra, Jordan algebra and others. As for a genetic generation of Jordan algebras will be given in the forthcoming paper.

By these discussions we may conclude that a method of formal language and genetics is quite effective for the theory of non-associative algebrs.

1. An algebra of Chomsky sentences(Chomsky algebra)

In this section we recall the context free language and introduce several algebraic strucutres on it. At first we recall basic facts on Chomsky sentences. Chomsky has discovered that a tree strucutre is essential in the theory of languages([1]). We take a sentence: „The cat sleeps on the sofa" and make the decomposition:





By this observation we can introduce the following concept:

Definition

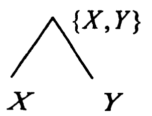
We take a set of words A which is called dictionary. Choosing words $A_1, A_2, ..., A_n$, we make a sentence: $\{\{A_1, A_2\}, \{A_3, ..., A_n\}\}$ which is called Chomsky sentence of A.

We notice that we do not care about the context of the sentence. As for the acceptability condition and generation of Chomsky sentences, see Appendix.

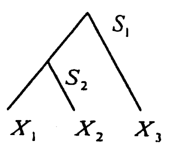
Next we proceed to the algebraic structure of sentences. We choose an algebra A which is generated by $e_1, e_2, ..., e_n$ over R which is denoted by $A = R[e_1, e_2, ..., e_n]$. Choosing generators, we make sentences, which we call Chomsky sentences of algebra A. Making operations of sum and constant multiplication, we can define an algebra which is called Chomsky algebra $C(A)$ .i.e.,

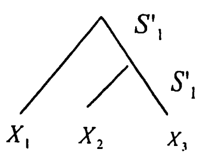
$$\begin{cases} X, Y \in C(A) \Rightarrow \{X, Y\} \in C(A) \\ X, Y \in C(A) \Rightarrow X + Y \in C(A) \\ \alpha \in R, X \in C(A) \Rightarrow \alpha X \in C(A) \end{cases}$$

The product of X and Y can be described in terms of the tree structure as follows:



We notice that this algebra is non-associative. We give a simple example:





We see that $\{\{X_1, X_2\}, X_3\} \neq \{\{X_1, X_2\}, X_3\}$. As for an explicit introduction of non-associativity structure to Chomsky sentences, we will discuss in Section 3.

2. The versatility of the Chomsky algebra for an arbitray algebra

In this section we show that the Chomsky algebra $C(A)$ generated by an arbitrary algebra A has the versatility property, even if it is a non-associative algebra. Namely we have a homomorphism $\Phi : C(A) \to A$. We begin with some basic notations. We assume that the algebra is finitely generated: $A = R[e_1, e_2, ..., e_n]$. Then we can represent an arbitray element X as follows:

$$X=\sum_{k=0}^{n} X_k,\ X_k=\sum \alpha_{(i_1(i_2\ldots i_k)}(e_{i_1}(e_{i_2}(\ .\ .e_{i_k}),$$

where the sum is taken through all possible sentences. In the following we put the following notations: $\Omega_{(i_1(i_2\ldots i_k)}=(e_{i_1}(e_{i_2}(\ldots.))e_{i_k})$.

We proceed to the construction of desired homomorphism. We define

$$(*)\begin{cases}\Phi[\{X_{i_1}\{X_{i_2}\{\ldots.\}\}X_{i_k}\}]=(X_{i_1}(X_{i_2}(\ldots.))X_{i_k}))\\ \Phi[X+Y]=\Phi[X]+\Phi[Y],\ \Phi[\alpha X]=\alpha\Phi[X]\,(\alpha\in R)\end{cases}$$

Then we have an algebraic homomorphism: $\Phi: C(A)\to A$. By this correspondence we can prove the following theorem:

Theorem I

Let A be a finitely generated algebra over R, i.e., $A=R[e_1,e_2,\ldots,e_n]$ and let C(A) be the Chomsky algebra of A. Then we can prove the following assertions:

(1) We have the algebraic homomorphism $\Phi: C(A)\to A$ defined by (*). (2) We have the following homomorphism theorem: Namely there exists an ideal I such that the following commutative diagram holds

$$\begin{array}{ccc}\Phi: C(A)\to & A & \\ \downarrow & \| & (\hat{\Phi}\text{ is isomorphism})\\ \hat{\Phi}: C(A)/I\to A & & \end{array}$$

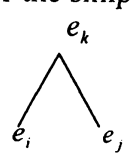
The main task of this paper is to describe the ideal I in connection to the Chomsky sentences.

3. Generation of non-associative algebras by sentence algebra and Mendel algebra

In this section we introduce two concept of algebras. The first one is sentence algebra which is motivated by the graph strucutre of Chomsky sentences, and the second one is Mendel algebras which is motivated by the separation law of Mendel's law respectively. Then we can describe the non-associativity condition of Chomsky sentences by these algebras.

(Sentence algebra)

At first, we consider the simplest sentence and associate a product structure on it:



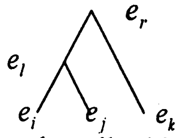
$$\Longrightarrow\quad e_ie_j=e_k\ (e_k=\{e_i,e_j\})$$

Next we introduce generators of an algebra with the product table:

	e_i	e_j
e_i	δe_i,	e_k
e_j	εe_k	$\delta' e_j$

$$\Longleftrightarrow\quad e_ie_i=\delta e_i\ ,e_ie_j=\varepsilon e_k\qquad e_je_j=\delta' e_j\ (\delta,\delta',\varepsilon=\pm 1\ or\ 0)$$

When $\delta\delta'\varepsilon=0$ holds, then the sentence is called degenerate. Then we can generate an algebra by use of compositions of generators of basic sentence algebras. We notice that we can associate associative algebras when $\varepsilon=1$ and non-associative algebra when $\varepsilon=-1$ respectively. We call the algebra the basic sentence algebra which is denoted by $S(e_i,e_j:\varepsilon e_k)$ when we can endow an algebraic structure, for example, $k=i\ or\ j$. We proceed to more complicated sentences:

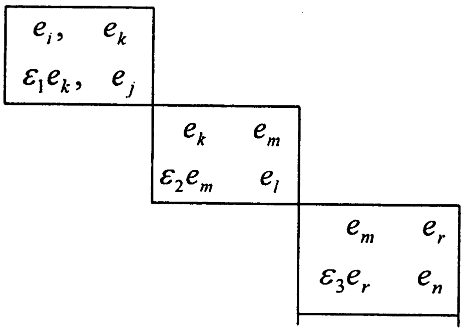


$\Leftrightarrow \quad ((e_i,e_j),e_k)$

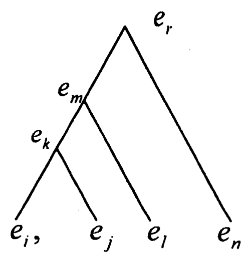
This sentence can be described by use of two basic sentence algebra: Preparing $S(e_i,e_j:\varepsilon_1 e_l)$ and $S(e_l,e_k:\varepsilon_2 e_r)$ and introducing a product strucutre $((e_i,e_j),e_k)$, we can introduce an algebra which is denoted by

$$S(e_i,e_j:\varepsilon_1 e_l)\times S(e_l,e_k:\varepsilon_2 e_r)$$

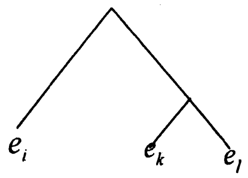
The graphic description of the algebra can be given in the following manner:



$\Leftrightarrow$

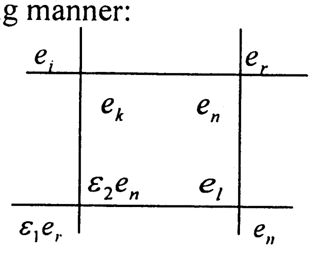


Next we proceed to an association of an algebra to sentences of more general type



$\Leftrightarrow \quad ((e_l,(e_k,e_l))$

To this sentence we associate the following algebra which is denoted by $S(e_i,e_n:\varepsilon_1 e_r)\circ S(e_k,e_l:\varepsilon_2 e_n)$. The product table of the algebra is given in the following manner:



$\Longrightarrow \quad S(e_i,e_n:\varepsilon_1 e_r)\circ S(e_k,e_l:$

Then we can prove the following theorem:

Theorem II

(1) We can associate an algebra for an arbitray given Chomsky sentence which is generated by the two kinds of products:

$$((S(e_i,e_j:\varepsilon_1 e_l)\times\ldots\times S(e_l,e_k:\varepsilon_2 e_r))\circ\ldots\circ S(e_l,e_k:\varepsilon_2 e_r))\times\ldots$$

(2) Choosing $\varepsilon_l\,(l=1,2,\ldots)$, we can realize the associativity or non-associativity of the sentence.

(3) For a sentence X, we make an element of the sentence algebra $X(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$ and we have

$$X(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)=\pm X(1,1,\dots,1).$$

(Mendel algebra)
In order to treat flexible algebras and Jordan algebras, it is convienient to introduce a concept of Mendel algebra([4]):

Definition
Let $A(=R[S_1,S_2,\dots,S_n])$ be a linear space. Introducing the product by

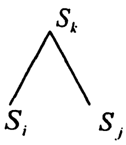
$$\begin{cases} S_i * S_j = \frac{1}{2}\{S_i + S_j\} \\ X * Y = \sum_{i,j=1}^{n} \alpha_i \beta_j S_i * S_j (X = \sum_{i=1}^{n} \alpha_i S_i, Y = \sum_{i=1}^{n} \beta_i S_i) \end{cases}$$

Then we have an algebra $M^{(n)}$ which is called Mendel algebra .

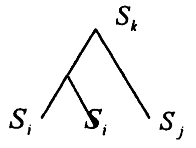
We notice that the Mendel algebra is non-associative and commutative algebra. In fact we can give a simple example:

$$((S_i * S_j) * S_k) = \frac{1}{4}(S_i + S_i + 2S_k), \quad ((S_i * (S_j * S_k)) = \frac{1}{4}(2S_i + S_i + S_k).$$

Following the scheme in the sentence algebra construction, we can make sentences in Mendel algebras. We give an example.



$\Rightarrow \quad S_i * S_j = S_k$



$\Rightarrow \quad (S_i * S_j) * S_k$

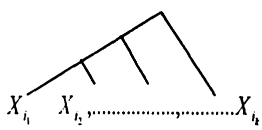
With this Mendel algebra, we willl treat flexible algebra and Jordan algebra in S. 5,6.

4. Shift operations and description of associaitive elements

In this section we give an description of elements of the Chomsky algebra in terms of shift operations.

(Shift operation)
We consider an element of the Chomsky algebra which is called element of L-type: $\tilde{\Omega}_{(i_1(i_2\dots i_k)} = (X_{i_1}(X_{i_2}(\dots))X_{i_k})$. We denote the linear space over R by $L(A)$: $L(A)=\{\sum \alpha_{(i_1(i_2\dots i})} \tilde{\Omega}_{(i_1(i_2\dots i_k)}\}$. The tree representation is given by

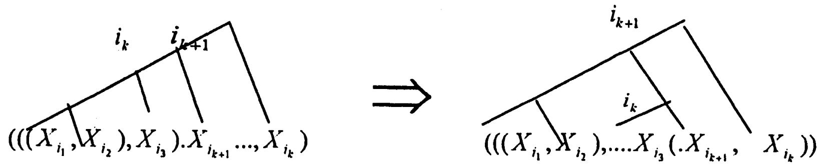
$$\tilde{\Omega}_{(i_1(i_2\dots i_k)} =$$



Next we proceed to shift operations in Chomsky sentences. At first we define a shift operation from $L(A)$ to $C(A)$. We define $\sigma_{i_k i_{k+1}} : L(A) \to C(A)$ by

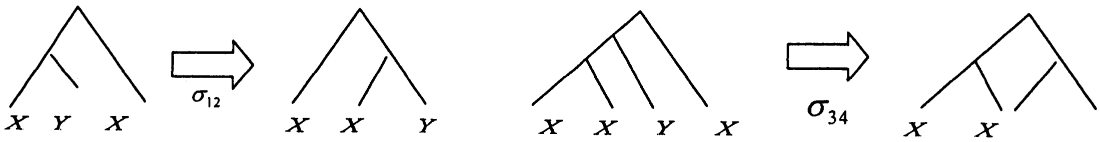
$$\sigma_{i_k i_{k+1}}[(\dots((X_{i_1}, X_{i_2})\dots)X_{i_k})X_{i_{k+1}}\dots X_{i_t})] = \breve{(}\dots((X_{i_1}, X_{i_2})\dots)X_{i_k}(X_{i_{k+1}}\dots X_{i_t})),$$

where $\breve{(}$ implies the taking off the bracket. The graphical description is given as follows:



Examples

We give two examples of shift operations which are connected to non-associative algebras([2])



flexible algbera Jordan algbera

Then we see the following proposition:

Proposition

Any element of $C(A)$ can be obtained from that of $L(A)$ by the successive operations of shift operators.

(Shift invariant non-associative algebra)

Next we introduce a class of non-associative algebra which can be described in terms of shift operations. Hence we have the following definition:

Definition

A non-associative algebra of shift type, or homogenous non-associative algebra, if associative elements are given by shift invariant elements: $\sigma(X) = X$, where σ is the shift operation. It is extended to the identity for remained elements. The element $\hat{X}(or\ X')$ which is defined from $X(\in A)$ in the following manner is called symmetric (or anti-symmetric) element:

$$\hat{X} = \frac{1}{2}(\sigma(X) + X)\ \left(resp.\ X' = \frac{1}{2}(\sigma(X) - X)\right)$$

In fact, we see that the typical non-associativity condition connected to the flexible and Jordan algebras can be described in terms of the shift invariant condition;

$$\sigma(((X,Y),Z)) = (X,(Y,Z)), \quad \sigma(((X,Y),Z),W)) = (X,Y),(Z,W))$$

(Non associative algebra generated by shift operations)

We construct an algebra which is generated by the shift invariance condition. We consider an algebra which is generated by $e_1, e_2,...,e_m$. We take a shift operation and make a system of shift invariant sentences: $S_1, S_2,...,S_m$. Making the symmetrization, we can obtain non-associative algebras. We give an explicit generation of the algebra. We consider the following set:

$$C_k = \{ \sum \alpha_{(i_1(i_2...)i_k)}((e_{i_1}(.....)e_{i_k}) \}$$

We decompose elements by use of $e_1, e_2,...,e_m$ and $\hat{S}_i, S'_i$ $(i = 1,2,,,,m)$:Putting one of them as θ_i

$$C_k = \{ \sum \alpha_{(i_1(i_2...)i_k)}((\theta_{i_1}(.\theta_{i_2}...\theta_{k;l}..)\theta_{i_k}) \}$$

Then we can prove the following theorem:

Theorem III
(1)Every element of Chomsky algebras can be obtained from that of $L(A)$ by shift operations and symmetrization.
(2) Making the symmertization of elements of an algebra A

$$\hat{C}_k = \{ \sum \alpha_{(i_1(i_2 \dots)i_k)}((\hat{\theta}_{i_1}(\hat{\theta}_{i_2} \dots \hat{\theta}_{k,l} \dots)\hat{\theta}_{i_k})\}$$

and we can obtain a new algebra $\hat{A}$ introducing product strucutre: $\circ : \hat{A} \circ \hat{A} \rightarrow \hat{A}$ by

$$\forall x \in \hat{C}_k, \forall y \in \hat{C}_l \quad \Rightarrow \quad x \circ y = \hat{(xy)} \in \hat{C}_{l+m}$$

(3) We have $\hat{C}_k \subset \sum_{\alpha+\beta=k} \hat{C}_\alpha \circ \hat{C}_\beta$. Hence we see that the algebra is determined by finite $\hat{C}_k \ (k = 1,2,..,M)$

(Shift operations of restricted type)
Next we proceed to the shift operation of restricted type. In order to treat flexible algebra and Jordan algebras, we have to treat shift operation of restricted type

$$\sigma(((X,Y),X)) = (X,(Y,X)), \quad \sigma(((X,X),Y),X)) = (X,X),(Y,X))$$

In order to treat this type of shift invariant sentences we have to introduce shift operations with bigger symmetries. We just indicate its idea by considering an example $((XY)Z)X = (XY)(ZX)$:Putting $X = \sum \alpha_i e_i, Y = \sum \beta_i e_i, Z = \sum \gamma_i e_i$, we can write the invarinat condition. Then we have

$$\sum \alpha_i \beta_j \gamma_k \alpha_l ((e_i e_j)e_k)e_l = \sum \alpha_i \beta_j \gamma_k \alpha_l ((e_i e_j)(e_k e_l)) \text{ for } \forall \alpha_i, \forall \beta, \forall \gamma \in R.$$

Rewriting this condition in the following form

$$\sum \alpha_i \beta_j \gamma_k \alpha_l \{((e_i e_j)e_k)e_l + ((e_l e_j)e_k)e_i\} = \sum \alpha_i \beta_j \gamma_k \alpha_l \{((e_i e_j)(e_k e_l) + ((e_l e_j)(e_k e_i)\}$$

Hence we obtain the shift invariance condition:

$$\{((e_i e_j)e_k)e_l + ((e_l e_j)e_k)e_i\} = \{((e_i e_j)(e_k e_l) + ((e_l e_j)(e_k e_i\}\}.$$

Making the symmetrization for these elements, we can describe elements of the algebras.

5. Flexible algebra

In this section we treat flexible algebra from our point of view([2]). We begin with the definition:

Definition
An algebra is called flexible algebra, if the following commutation relation holds:

$$\forall X, \forall Y \in A \Rightarrow (XY)X = X(YX)$$

At first we choose a basic sentence algebra and make a flexible algebra. We choose a degenerate sentence algebra: $S(e_1, e_2 : e_2 : 2,0,1)$.

	e_1	e_2
e_1	$2e_1$	e_2
e_2	e_2	0

$\Longrightarrow \quad e_2(e_1 e_1) \neq (e_2 e_1)e_1$

Proposition
Algebra $S(e_1, e_2 : e_2 : 2,1,0))$ is a flexible algbera.

Proof
Putting $X=\sum\alpha_i e_i$, $Y=\sum\beta_i e_i$, we check the condition: $(XY)X=X(YX)$. Since $XY=YX=2x_1y_1e_1+(x_1y_2+x_2y_1)e_2$. Hence we have $(XY)X=2x_1^2y_1e_1+x_1x_2y_1e_2$ and $(X(YX)=2x_1^2y_1e_1+x_1x_2y_1e_2$ which proves the assertion.

Next we introduce the Mendelian algebra $M^{(n)}$ and prove the following Theorem:

Theorem IV
(1)Shift invariant algebra of associative type is a flexible algebra. Namely, if we assume that $X*(Y*Z)=(X*Y)*Z$ for $\forall X,\forall Y,\forall Z\in M(A)$, then we have $X*=Z*$. Hence we obtain a flexible algebra from the shift invariance condition.
(2) Mendel algebra $M^{(n)}$ $(n\geq 3)$ is a flexible algebra,but not associative algebra

Proof of(1) Putting $X=\sum\alpha_i S_i, Y=\sum\beta_i S_i, Z=\sum\gamma_i S_i$ we consider the shift invariant condition: $X*(Y*Z)=(X*Y)*Z$. Restricting special elements, $X=S_i, Y=S_j, Z=S_k$, we consider $((S_i*S_j)*S_k)=((S_i*(S_j*S_k))$. Then we have $S_i=S_k$. Hence putting $((XY)X)=\sum\alpha_i\beta_j\alpha_k\delta_{ik}(S_i*S_j)*S_k$, and $(X(YX))=\sum\alpha_i\beta_j\alpha_k\delta_{ik}S_i*(S_j*S_k)$, we obtain $X*(Y*X)=(X*Y)*X$.

Proof of(2) Putting $X=\sum\alpha_i S_i, Y=\sum\beta_i S_i$, we see

$$((XY)X)=\sum\alpha_i\beta_j\alpha_k(S_i*S_j)*S_k, \text{ and } (X(YX))=\sum\alpha_i\beta_j\alpha_k S_i*(S_j*S_k),$$

Hence to prove the assertion, it is enough to prove the following equality:

$$\sum\alpha_i\beta_j\alpha_k(S_i*S_j)*S_k=\sum\alpha_i\beta_j\alpha_k S_i*(S_j*S_k).$$

For this we decompose the both sides in the following manner:

$$\sum\alpha_i\beta_j\alpha_k(S_i*S_j)*S_k=\sum_{i=k}\alpha_i\beta_j\alpha_k(S_i*S_j)*S_k+\sum_{i\neq k}\alpha_i\beta_j\alpha_k(S_i*S_j)*S_k$$
$$\sum\alpha_i\beta_j\alpha_k S_i*(S_j*S_k)=\sum_{i=k}\alpha_i\beta_j\alpha_k S_i*(S_j*S_k)+\sum_{i\neq k}\alpha_i\beta_j\alpha_k S_i*(S_j*S_k)$$

Since $((S_i*S_j)*S_i)=((S_i*(S_j*S_i))$, the first term of the both sides are identical. The second terms of the both sides can be written as follows:

$$\sum_{i\neq k}\alpha_i\beta_j\alpha_k(S_i*S_j)*S_k=\sum_{i\neq k}\alpha_i\beta_j\alpha_k\{(S_i*S_j)*S_k+(S_k*S_j)*S_i\}$$
$$\sum_{i\neq k}\alpha_i\beta_j\alpha_k S_i*(S_j*S_k)=\sum_{i\neq k}\alpha_i\beta_j\alpha_k\{S_i*(S_j*S_k)+S_k*(S_j*S_i)\}$$

By use of the commutativity of Mendel algebra, we see the both sides are identical. Hence we have proved the assertion.

6. Jordan algebra

In this section we give an understanding commutation relations of Jordan algebra from the point of Chomsky sentence and make a Jordan algebra by use of basic sentence algebra and Mendel algebra. We recall the definition of Jordan algebra([2]):

Definition
An algebra J is called Jordan algebra if the following commutation relation holds:

$$\forall X,\forall Y\in A\Rightarrow(((XX)Y)X)=((XX)(YX))$$

When it is commutative, it is Jordan algebra simply, otherwise it is called non-commutative Jordan algebra.

Next we proceed to make a Jordan algebra by use of basic sentence algebra. At first we notice the following proposition:

Proposition
The following sentence algebra $S(e_1, e_2 : -1e_3)$ is a non-commutative Jordan algebra

	e_1	e_2
e_1	e_1	e_2
e_2	$-e_2$	e_1

$\Longrightarrow \quad (e_2 e_1)e_2 \neq e_2(e_1 e_2)$

Proof
The proof is a direct calculation . Putting $X = \sum x_i e_i, Y = \sum y_j e_j$, we have

$$(XX) = (x_1^2 + x_2^2)e_1, \text{and} (YX) = (x_1 y_1 + x_2 y_2)e_1 + (x_2 y_1 + x_1 y_2)e_2$$

From $(XX)Y = (x_1^2 + x_2^2)(y_1 e_1 + y_2 e_2)$, we have

$$((XX)Y)X = (x_1^2 + x_2^2)\{(x_1 y_1 + x_2 y_2)e_1 + (x_2 y_1 - x_1 y_2)e_2)\}.$$

On the other side we have

$$((XX)(YX) = (x_1^2 + x_2^2)\{(x_1 y_1 + x_2 y_2)e_1 + (x_2 y_1 - x_1 y_2)e_2)\}.$$

Hence we have the assertion.

Next we proceed to the realization of Jordan algebra by use of the Mendelian algebra $M^{(n)}$. We can prove the following:

Theroem V
(1)We assume that $(((X*Y)*Z)*W) = ((X*Y)*(Z*W))$ then we have $X = Z = W$. Hence we obtain a non-commutative Jordan algebra from the shift invariance condition
(2) Mendel algebra $M^{(n)}$ is a Jordan algebra

Proof of (1) :From

$$(****)(((S_i * S_j) * S_k) * S_l) = \frac{1}{8}(S_i + S_j + 2S_k + 4S_l), ((S_i * S_j) * (S_k * S_l)) = \frac{1}{4}(S_i + S_j + S_k + S_l)$$

and $(((S_i * S_j) * S_k) * S_l) = ((S_i * S_j) * (S_k * S_l))$, we have $S_i = S_l = S_l$. Hence putting $X = \sum \alpha_i S_i, Y = \sum \beta_i S_i$, we have the commutation relation of a Jordan algebra as in the proof.of Theorem IV.

Proof of (2) : Putting $X = \sum \alpha_i S_i, Y = \sum \beta_i S_i$, we see

$$(((XX)Y)X) = \sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l,$$
$$((XX)(YX)) = \sum \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l),$$

Hence to prove the assertion, it is enough to prove the following equality:

$$\sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l).$$

For this we decompose the both sides in the following manner:

$$\sum \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum_{i=j=l} \alpha_i \beta_j \alpha_k \alpha_l ((S_i * S_j) * (S_k * S_l) + \sum{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l)$$

$$\sum \alpha_i \alpha_j \beta_k \alpha_k (S_i * (S_j * S_k)) * S_l$$
$$= \sum_{i=k=l} \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l) + \sum{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l)$$

where, the second sum is remained sum. Since $((S_i * S_i) * S_k) * S_i = ((S_i * S_i) * (S_k * S_i))$, the first term of the both sides are identical. Next we decompose the remained sum into two parts: $\Sigma' = \Sigma'_1 + \Sigma'_2$:The first sum is taken for the case of two of the indices(i,j,l) are identical and the remained sum is taken for the three indices are different.The second terms of the both sides can be written as follows:

$$\sum_2{}' \alpha_i \alpha_j \beta_k \alpha_l ((S_i * S_j) * S_k) * S_l = \sum_\sigma \alpha_{\sigma(i)} \alpha_{\sigma(j)} \beta_k \alpha_{\sigma(l)} \{(S_{\sigma(i)} * S_{\sigma(j)}) * S_k * S_{\sigma(l)}\}$$

$$\sum_2{}' \alpha_i \alpha_j \beta_k \alpha_l (S_i * S_j) * (S_k * S_l) = \sum_\sigma \alpha_{\sigma(i)} \alpha_{\sigma(j)} \beta_k \alpha_{\sigma(l)} \{(S_{\sigma(i)} * S_{\sigma(j)}) * (S_k * S_{\sigma(l)})\}$$

where the sum is taken through the permutations of three words. By use of (****) we see the both sides are identical. In a completely analogous manner, we have the first equality for Σ'_1 . Hence we have proved the assertion.

Appendix

In this appendix we give the acceptability condition and the generation of Chomsky sentences. We begin with the tree strucutre of Chomsky sentences:

(Acceptability condition)
We can give the acceptability condition of Chomsky sentences in the following manner:We choose a sequence, for example, {{,},{,}}or {{,},,} . We make a numbering:

$$\underset{1\,2\,3\,4\,5\,6\,7\,8\,9}{\{\{,\},\{,\}\}}, \quad \underset{1\,2\,3\,4\,5\,6\,7}{\{\{,\},,\}}$$

Then we can give the following criterion of the acceptability condition:

For the sequence $\underset{1\,2\,3\,4}{\{\,\{\,,\,\}}$......
(1) $\#(\{\,)_k \ge \#(,)_k \ge \#(\})_k$
(2) $\#(\{\,)_n = \#(,)_n = \#(\})_n$
for only n(=length)
(3) $\#(,)_{k+2} - \#(,)_k \le 1$

$\#(\{\,)_k$ implies the sum of open brackets from the first to the k-step. The other notations are similarly used

We give several examples:

$\{\{X_1, X_2\}.X_3\}$

$\#(\{\,)_k$	1	2	2	2	2	2
$\#(,)_k$	0	0	1	1	2	2
$\#(\})_k$	0	0	0	1	1	2

Acceptable

$\{\{X_1..X_2\}\}$

$\#(\{\,)_k$	1	2	2	2	2	2
$\#(,)_k$	0	0	1	2	2	2
$\#(\})_k$	0	0	0	0	1	2

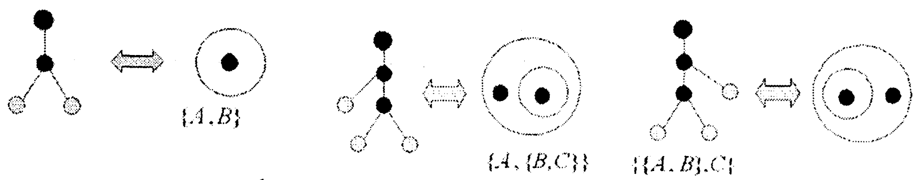
Non acceptable

$\{X_1.X_2\}\{X_3.X_4\}$

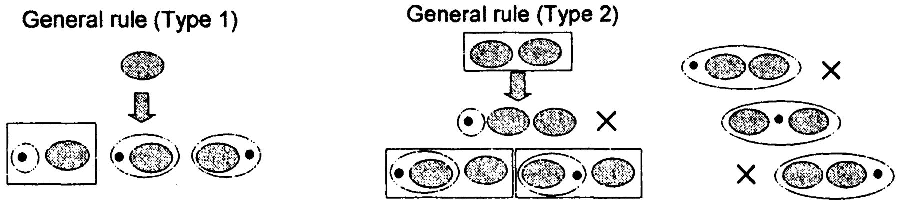
$\#(\{\,)_k$	1	1	1	2	2	2
$\#(,)_k$	0	1	1	1	2	2
$\#(\})_k$	0	0	1	1	1	2

Non acceptable

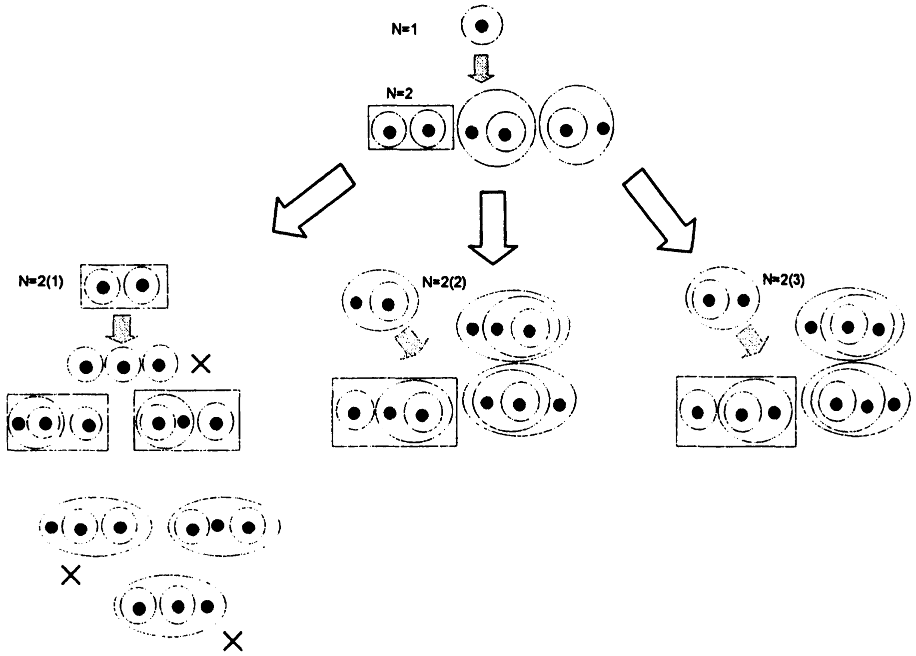
(Generationof Chomskysentences)
We proceed to a generation of Chomsky sentences.We introduce the circle representation of centences.



Then we can state the generation rule in the following manner. Choosing an acceptable sentence and putting points between each connected compornents and making circles surrounding each connected components succesively. We can give its generation rule in the following two types:



where the square implies the candidates whose acceptability will be determined in the further steps and the circle implies the acceptable sentences. The sentence with × implies that the sentences can not be acceptable. Hence we see that we have three possible sentences by the first type and three possible sentences by the second type. We give the generations in the first three steps:



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