数理解析研究所講究録
第 1705 巻 2010 年 212-216

Exact RG flow of point interactions in one dimension

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Abstract

We find the exact renormalization group flow of $U(2)$ family of point interactions in one-dimensional quantum mechanics. We show that the scale-independent subfamily of point interactions first discovered by Fülöp and Tsutsui are realized as nontrivial fixed points.

1 Introduction

Point interactions, or interactions of zero range, may be the simplest yet ubiquitous interactions in low energy physics. We (naively) expect that any short-ranged interaction could be approximated by a point interaction in the long-wavelength limit. In order to get detailed information about what the localized potential is, we need to use a probe particle whose de Broglie wavelength is shorter than the size of the localized potential. In other words, the longer the probe particle's wavelength is, the less information we can get about the short-ranged interaction. This naive consideration leads to the following question: *Does there exist any universality classes of short-ranged interactions whose long-wavelength limits appear to be the same?* In this contribution we review our recent work [1] and try to give an answer by investigating the renormalization group (RG) flow of point interactions.

To this end, let us imagine one-dimensional quantum mechanics for a particle on $\mathbb{R}$ in the presence of a single localized potential centered at the origin, whose spatial extent is characterized by a length scale a. In the long-wavelength limit $\lambda \gg a$ with λ being the de Broglie wavelength of a probe particle, any localized potential could be approximated by a point interaction at the origin. In this limit a particle would freely propagate in the bulk yet interact at the origin. The Schrödinger equation describing this situation is given by

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = E\psi(x), \quad x \neq 0. \tag{1.1}$$

(In this paper we will work in the units where $\hbar = 2m = 1$.) It is known that allowed point interaction at the origin is described by the following boundary condition [2]

$$\vec{\Psi}(0) - iL_0\vec{\Psi}'(0) = U\left[\vec{\Psi}(0) + iL_0\vec{\Psi}'(0)\right], \quad U \in U(2), \tag{1.2}$$

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where L_0 is an *arbitrary* real length scale that implements the length dimension of the equation (1.2). $\vec{\Psi}$ and $\vec{\Psi}'$ are the 2-component column vectors defined by

$$\vec{\Psi}(x) := \big(\psi(x), \psi(-x)\big)^T, \tag{1.3a}$$

$$\vec{\Psi}'(x) := \big(\psi'(x), -\psi'(-x)\big)^T. \tag{1.3b}$$

Eq.(1.2) is the $U(2)$ family of boundary conditions that describe all possible point interactions in one-dimensional quantum mechanics. Any point interaction will be specified by a certain unitary matrix $U \in U(2)$. In this sense we can say that in an appropriate long-wavelength limit, the theory space of one-dimensional quantum mechanics for a particle with a single localized potential is equivalent to the parameter space of two-dimensional unitary group $U(2)$.

For the following discussions it is suitable to parameterize the matrix U into the following spectral decomposition form:

$$U = \mathrm{e}^{i\alpha_+}P_+ + \mathrm{e}^{i\alpha_-}P_-, \quad P_\pm := \frac{\mathbb{1} \pm \vec{e}\cdot\vec{\sigma}}{2}, \tag{1.4}$$

where $0 \le \alpha_\pm < 2\pi$ and $\vec{e} = (e_x, e_y, e_z)^T$ is a real unit vector satisfying $e_x^2 + e_y^2 + e_z^2 = 1$. $P_\pm$ are the hermitian projection operators fulfilling $P_+ + P_- = \mathbb{1}$, $(P_\pm)^2 = P_\pm$, $P_\pm P_\mp = 0$ and $P_\pm^\dagger = P_\pm$.

The rest of this paper is organized as follows. In section 2 we will derive the one-particle scattering matrix (S-matrix) exactly. In section 3 we will derive the exact RG flow of point interactions by using the exact S-matrix. There we will see that L_0 turns out to play a role of renormalization scale if we require physical quantities should not depend of the choice of L_0. Section 4 is devoted to our conclusions.

2 Exact S-matrix

In this section we solve the Schrödinger equation (1.1) with the boundary conditions (1.2) and then derive the exact S-matrix.

The general solution to the Schrödinger equation (1.1) for positive energy $E > 0$ is the linear combination of the plane waves

$$\psi(x;k) = \begin{cases} A_+^{\mathrm{in}}(k)\mathrm{e}^{-ikx} + A_+^{\mathrm{out}}(k)\mathrm{e}^{ikx}, & \text{for } x > 0, \\ A_-^{\mathrm{in}}(k)\mathrm{e}^{ikx} + A_-^{\mathrm{out}}(k)\mathrm{e}^{-ikx}, & \text{for } x < 0, \end{cases} \tag{2.1}$$

where $k := \sqrt{E} > 0$. Note that the coefficients $A_\pm^{\mathrm{in}}(\mathrm{k})$ and $A_\pm^{\mathrm{out}}(k)$ may depend on k. The superscripts 'in' and 'out' mean the incoming waves towards the origin and the outgoing waves against the origin, respectively (see Figure 1). Substituting these into (1.2) with the parameterization (1.4) we get

$$\vec{A}^{\mathrm{out}}(k) = S(k)\vec{A}^{\mathrm{in}}(k), \tag{2.2}$$

where

$$\vec{A}^{\mathrm{in}}(k) := \big(A_+^{\mathrm{in}}(k), A_-^{\mathrm{in}}(k)\big)^T, \tag{2.3a}$$

$$\vec{A}^{\mathrm{out}}(k) := \big(A_+^{\mathrm{out}}(k), A_-^{\mathrm{out}}(k)\big)^T, \tag{2.3b}$$

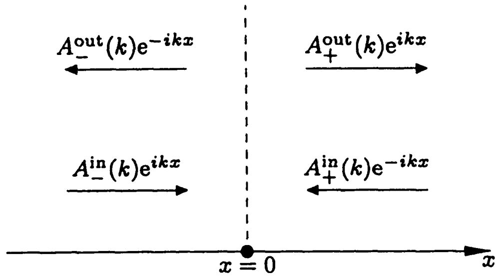

Figure 1: One-particle scattering from a point interaction.

and $S(k)$ is a 2×2 unitary matrix defined as

$$S(k) := \sum_{j=\pm} \frac{ikL_j - 1}{ikL_j + 1} P_j, \quad L_\pm := L_0 \cot \frac{\alpha_\pm}{2}. \tag{2.4}$$

Eq.(2.2) shows that the matrix $S(k)$ plays a role of an evolution map between the "in-state" $\vec{A}^{\rm in}(k)$ and the "out-state" $\vec{A}^{\rm out}(k)$. The ij-component of the matrix $S(k)$ is nothing but the transition amplitude for a particle traveling from $\mathbb{R}_j$ to $\mathbb{R}_i$, where $i,j = +$ or $-$. Thus, the diagonal element $S_{\pm\pm}$ should be interpreted as the reflection coefficient $R_\pm(k)$ for a particle of momentum k on the positive (negative) half-line $\mathbb{R}_\pm$. Similarly, the off-diagonal element $S_{\pm\mp}$ should be interpreted as the transmission coefficient $T_\pm(k)$ for a particle incoming from $\mathbb{R}_\mp$ and scattered to $\mathbb{R}_\pm$. Hence we can interpret $S(k)$ as the one-particle S-matrix. It should be emphasized that this S-matrix is *exact*.

3 Exact RG flow of point interactions

As noted before, L_0 is an arbitrary reference scale so that the physical quantities, that is, the S-matrix elements must be independent of the choice of L_0. The lack of dependence of L_0 can be expressed as an invariance of the theory under the RG transformation

$$R_t : L_0 \mapsto \bar{L}(t) := L_0 e^{-t}, \quad -\infty < t < \infty. \tag{3.1}$$

Any change of L_0 must be equivalent to changes in the $U(2)$ parameters $g_i = \{\alpha_\pm, e_i\}$. This requirement is expressed as

$$S(k; g_i, L_0) = S(k; \bar{g}_i(t), \bar{L}(t)), \tag{3.2}$$

where $\bar{g}_i(t) = \{\bar{\alpha}_\pm(t), \bar{e}_i(t)\}$ are the running $U(2)$ parameters, which are determined by the following homogeneous RG equation

$$\left(-\bar{L}\frac{\partial}{\partial \bar{L}} + \sum_{\bar{g}_i = \bar{\alpha}_\pm, \bar{e}_i} \beta_{g_i}(\bar{g}_i(t)) \frac{\partial}{\partial \bar{g}_i} \right) S(k; \bar{g}_i(t), \bar{L}(t)) = 0, \tag{3.3}$$

where the β-functions are given by

$$\beta_{\alpha_\pm}(\bar{\alpha}_\pm(t)) = -\sin\bar{\alpha}_\pm(t), \tag{3.4a}$$
$$\beta_{e_i}(\bar{e}_i(t)) = 0. \tag{3.4b}$$

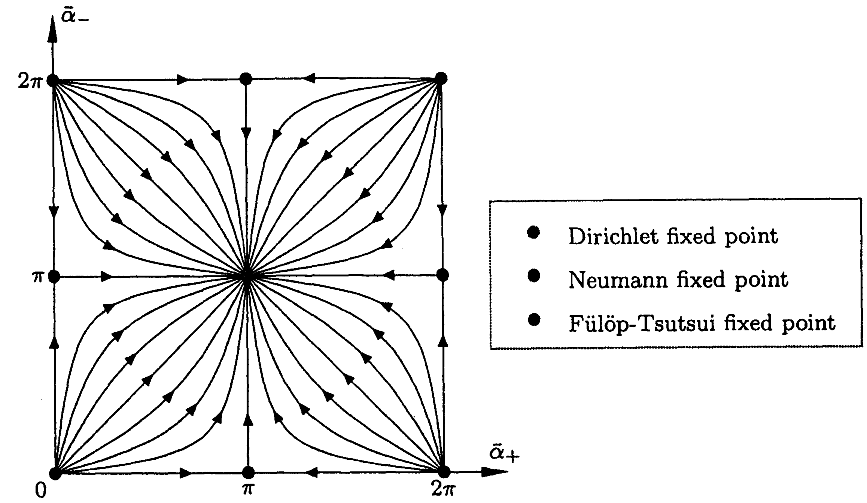

Figure 2: Exact RG flow of $\alpha_\pm$. Arrows indicate the directions toward the infrared.

We emphasize that these β-functions are *exact*. Eq.(3.4b) implies that the unit vector $\vec{e}$ is the exactly marginal parameter. From Eq.(3.4a), on the other hand, we see that the eigenphases of U are renormalized and have fixed points. As depicted in Figure 2, in the parameter space of $(\bar{\alpha}_+, \bar{\alpha}_-)$ there exist the following three distinct types of fixed points:

1. *Neumann fixed point (UV stable fixed point).*
   There is a ultraviolet stable fixed point at $(\alpha_+, \alpha_-) = (0, 0)$. At this point the unitary matrix becomes the identity matrix, $U = P_+ + P_- = \mathbb{1}$. Thus we see that this fixed point corresponds to the Neumann-Neumann boundary conditions at the origin:
$$\psi'(0_-) = 0 = \psi'(0_+). \tag{3.5}$$

2. *Dirichlet fixed point (IR stable fixed point).*
   There is an infrared stable fixed point at $(\alpha_+, \alpha_-) = (\pi, \pi)$. At this point the unitary matrix is $U = -P_+ - P_- = -\mathbb{1}$. Thus we see that this fixed point corresponds to the Dirichlet-Dirichlet boundary conditions:
$$\psi(0_-) = 0 = \psi(0_+). \tag{3.6}$$

3. *Fülöp-Tsutsui fixed point.*
   There are nontrivial fixed points at $(\alpha_+, \alpha_-) = (0, \pi)$ and $(\pi, 0)$. Since these two fixed points are related by the exchange '+' $\leftrightarrow$ '$-$', in the following we will concentrate on the case $(\alpha_+, \alpha_-) = (0, \pi)$. This fixed point is IR stable in the α_--direction and unstable only in the α_+-direction; see Figure 2. At this point the unitary matrix becomes $U = P_+ - P_- = \vec{e}\cdot\vec{\sigma}$. It follows immediately that this fixed point corresponds to the boundary conditions $P_-\vec{\Psi}(0) = \vec{0}$ and $P_+\vec{\Psi}'(0) = \vec{0}$. With the parameterization
$$\vec{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi, \tag{3.7}$$
   the projection operators become
$$P_+ = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}, \tag{3.8a}$$
$$P_- = \frac{1}{2}\begin{pmatrix} 1-\cos\theta & -e^{-i\varphi}\sin\theta \\ -e^{i\varphi}\sin\theta & 1+\cos\theta \end{pmatrix} = \begin{pmatrix} \sin^2\frac{\theta}{2} & -e^{-i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix}. \tag{3.8b}$$

Thus, with this parameterization, the boundary conditions $P_-\vec{\Psi}(0) = \vec{0}$ and $P_+\vec{\Psi}'(0) = \vec{0}$ are cast into the following forms:

$$\psi(0_-) = e^{i\varphi} \tan\tfrac{\theta}{2}\psi(0_+), \tag{3.9a}$$

$$\psi'(0_-) = e^{i\varphi} \cot\tfrac{\theta}{2}\psi'(0_+), \tag{3.9b}$$

which are the scale-independent boundary conditions first discovered by Fülöp and Tsutsui [3].

4 Conclusions

In this paper we study the *exact* RG flow of $U(2)$ family of point interactions in the framework of one-particle quantum mechanics. We find that there are three distinct fixed points, where the so-called scale-independent point interactions [2] are realized. We believe that this result strongly suggests that in one spatial dimension there exist three types of universality classes of localized potentials: If UV theory is tuned to lie on the critical point $(\alpha_+, \alpha_-) = (0, 0)$ it remains on the Neumann fixed point. If UV theory is happened to lie on the critical lines $\alpha_+ = 0$ or $\alpha_- = 0$, it flows into the Fülöp-Tsutsui fixed point. All other short-raged interactions flow into the Dirichlet fixed point in the long-wavelength limit, which implies that without fine-tuning most of localized potentials will be effectively described by the infinite Dirichlet wall in the low energy regime.

Acknowledgment

The work of SO is supported in part by JSPS Research Fellowships for Young Scientists and by Excellent Young Researchers Overseas Visit Program conducted by JSPS.

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