数理解析研究所講究録
第 1705 巻 2010 年 88-94

Quantum Mutual Entropy Defined by Liftings and Violation of the Shannon Inequality

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Abstract

A lifting is a continious map from a system to a compound system introduced by Ohya and Accardi [1], and we can represent several dynamical processes by using it. Liftings show the relation between two systems in the compound system clearly, and it is useful to discuss not only the communication process, but also an entanglement of it. In this study we define a quantum mutual entropy using liftings and investigate the property. We show that there exist some cases where the quantum mutual entropy violates the Shannon inequality.

1 Introduction

In order to discuss the relation between two systems, we construct a map from the state space of a system to the state space of another system. The map is called a channel. Channels from the state space of a system to the state space of a compound system are very important class, such channels are called liftings. An example of liftings are the duals of transition expectation.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$. For the set $\mathfrak{S}(\mathcal{H})$ of all density operators on $\mathcal{H}$; $\mathfrak{S}(\mathcal{H}) = \{\rho; \rho \geq 0, \mathrm{tr}\rho = 1\}$, we call a map from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$ a channel. If Λ^* is affine, we call it a linear channel. We denote $\Lambda : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ by a dual map of Λ^*; i.e., $\mathrm{tr}\Lambda^*\rho A = \mathrm{tr}\rho\Lambda A$ for all $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and $A \in \mathcal{B}(\mathcal{H}_2)$. If Λ is a complete positive map (i.e., for all $n \in \mathbb{N}, A_j \in \mathcal{B}(\mathcal{H}_2), B_k \in \mathcal{B}(\mathcal{H}_1)$ holding $\sum_{j,k=1}^{n} B_j^*\Lambda\left(A_j^*A_k\right)B_k \geq 0$), Λ^* is called a complete positive channel. Channel is a mathematical tool to describe various physical processes[10].

Lifting was introduced by Accardi and Ohya in C^*-dynamical systems[1] to integrate various channels and open system dynamics. Here let our C^*- algebras are realized on some separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. A continious map $\mathcal{E}^*$ from the state space $\mathfrak{S}(\mathcal{H}_1)$ to the compound state space $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called a lifting:

$$\mathcal{E}^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

The concept of lifting can be used to understand noncommutative probability.

If $\mathcal{E}^*$ is affine and its dual is a completely positive map, we call it a CP linear lifting. If it maps pure states into pure states, we call it pure. Remark that a pure lifting sends a mixed state to either a pure or a mixed state. A lifting from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called *non-demolition* for a state $\rho_1 \in \mathfrak{S}(\mathcal{H}_1)$ if $\mathcal{E}^*$ holds the following condition

$$\mathrm{tr}_2 \mathcal{E}^* \rho_1 = \rho_1$$

Given a state $\rho_1 \in \mathfrak{S}(\mathcal{H}_1)$ and a channel $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_2)$, the following problem is important, that is, to find a standard lifting $\mathcal{E}^* : \mathfrak{S}(\mathcal{H}_1) \to \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that it describe the correlation between of ρ_1 and $\Lambda^* \rho_1 = \mathrm{tr}_1 \mathcal{E}^* \rho_1$. There are several solutions of this problem in the papers [1, 8, 10].

2 Quantum Mutual Entropy

The classical mutual entropy was introduced by Shannon to discuss the transmission of information from an input system to an output system[5], then Kolmogorov[6], Gelfand and Yaglom[2] gave a measure theoretic expression for the mutual entropy by means of the relative entropy defined by Kullback and Leibler. Shannon's expression for mutual entropy was generalized for the finite-dimensional quantum (matrix) case by Holevo[3, 4] and Lebtin[7]. Ohya took the measure theoretic expression of KGY and defined quantum mutual entropy by means of quantum relative entropy[8, 10].

Let $\mathfrak{S}$ be the set of all states in a certain C^*-algebra (or von Neumann algebra) describing a quantum system, and μ a measure decomposing the state φ into extremal orthogonal states in $\mathfrak{S}$. Ohya's definition of quantum mutual entropy(QME in short) entropy is

Definition 1 *QME w.r.t. φ and Λ^* is defined as[8, 10]*

$$I(\varphi; \Lambda^*) \equiv \sup \left\{ \int_{\mathfrak{S}} S^{Araki}(\Lambda^* \omega, \Lambda^* \varphi)\, d\mu; \varphi = \int_{ex\mathfrak{S}} \omega d\mu \right\}$$

where S^{Araki} is Araki's relative entropy.

Definition 2 *In the case that the C^*-algebra is $B(\mathcal{H})$ and $\mathfrak{S}$ is the set of all density operators, the above definition goes to*

$$I(\rho; \Lambda^*) \equiv \sup \left\{ \sum_n \lambda_n S^{Umegaki}(\Lambda^* E_n, \Lambda^* \rho); \rho = \sum_n \lambda_n E_n \right\}$$

where ρ is a density operator, $S^{Umegaki}$ is Umegaki's mutual entropy and $\rho = \sum_n \lambda_n E_n$ is the Schatten decomposition. The Schatten decomposition is no always unique, so we take the supremum over all possible decompositions.

Both are quantum input and quantum output case. When the input is classical, i.e., the state is a probability distribution, the von Neumann-Schatten decomposition is unique

$$\rho = \sum_n \lambda_n \delta_n$$

and if the channel is written as $\Lambda^* = \Gamma_2^* \Gamma_1^*$ where Γ_1^* is one for quantum coding, i.e., $\Gamma_1^* \delta_n = \rho_n$, then the above mutual entropy generalizes Holevo's one

$$I(\rho; \Lambda^*) = S(\Lambda^* \rho) - \sum_n \lambda_n S(\Lambda^* \rho)$$

Moreover, let $\rho = \sum_k \lambda_k E_k$ be a Schatten decomposition of $\rho \in \mathfrak{S}(\mathcal{H})$ and let σ_E be a compound state of ρ and $\Lambda^* \rho$

$$\sigma_E = \sum_k \lambda_k E_k \otimes \Lambda^* E_k$$

Theorem 3 *[8]The QME $I(\rho; \Lambda^*)$ is*

$$I(\rho; \Lambda^*) = \sup \left\{ \sum_n \lambda_n S(\Lambda^* E_n, \Lambda^* \rho); E = \{E_n\} \right\}$$

where $\sigma_0 = \rho \otimes \Lambda^* \rho$.

Theorem 4 *[8]$I(\rho; \Lambda^*)$ satisfies the following property:*

1. *If a channel Λ^* is an i.d., $I(\rho; \Lambda^*)$ is equal to $S(\rho)$*
2. *If the system is classical, $I(\rho; \Lambda^*)$ is equal to classical mutual entropy*
3. *(The Shannon inequality) $0 \le I(\rho; \Lambda^*) \le \min\{S(\rho), S(\Lambda^* \rho)\}$*

These are discussed precisely in [11, 12].

3 Quantum Mutual Entropy defined by Lifting

In this section, we define the QME by using a lifting with the marginal condition. Then we study under which conditions this QME satisfies the Shannon inequality.

Let Λ^* be a complete positive channel from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$ and $\mathcal{E}^*$ a lifting from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Here, we take the following two marginal conditions:

(M1) For an input state $\rho \in \mathfrak{S}(\mathcal{H}_1)$, it holds $\mathrm{tr}_2 \mathcal{E}^* \rho = \rho$ (non-demolition property).

(M2) For a given channel Λ^*, $\mathrm{tr}_1 \mathcal{E}^* \rho = \Lambda^* \rho$.

We define the QME w.r.t. $\mathcal{E}^*$ as

$$I_L(\rho; \mathcal{E}^*) \equiv S(\mathcal{E}^* \rho, \rho \otimes \Lambda^* \rho)$$

Taking a supremum of $I_L(\rho;\mathcal{E}^*)$ on the liftings $\mathcal{E}^*$, the QME for a channel Λ^* is defined as

$$I_L(\rho;\Lambda^*) \equiv \sup_{\mathcal{E}^*}\{I_L(\rho;\mathcal{E}^*);\mathrm{tr}_2\mathcal{E}^*\rho=\rho,\ \mathrm{tr}_1\mathcal{E}^*\rho=\Lambda^*\rho\}$$

Let us check whether $I_L(\rho;\Lambda^*)$ satisfies the Shannon inequality

$$0\le I_L(\rho;\Lambda^*)\le S(\rho).$$

For a channel Λ^* we can consider the following three liftings $\mathcal{E}_i^*$ $(i=1,2,3)$ with M1 and M2:

Case1: $\mathcal{E}_1^*\rho=\sum_k\lambda_kE_k\otimes\Lambda^*E_k$, where $\rho=\sum_k\lambda_kE_k$ is a Schatten decomposition.

Case2: $\mathcal{E}_2^*\rho=\sum_k p_k\rho_k\otimes\Lambda^*\rho_k$ for $\rho=\sum_k p_k\rho_k$. $\sum_k p_k=1, p_k\ge 0$

Case3: $\mathcal{E}_3^*$ is a pure lifting.

Concerning the Shannon inequality, we obtain the results below[13].

Theorem 5 *$\mathcal{E}_1^*$ satisfies the marginal condition M1 and M2 and the Shannon inequality.*

Theorem 6 *$\mathcal{E}_2^*$ satisfies M1, M2, and the Shannon inequality.*

From the above two theorems, we may conclude that if the lifting is a separable type, that is, $\mathcal{E}^*\rho$ is a separable state, then the Shannon inequality is satisfied. On the contrary, there exists several entangled type pure liftings, that is, $\mathcal{E}^*\rho$ is a pure entangled state, that does not satisfy the Shannon inequality. In the rest of our paper, we give three examples of pure lifting $\mathcal{E}_3^*$; one is for satisfying the Shannon inequality and two others are for not.

Example 7 *In the case that a channel Λ^* is written as*

$$\Lambda^*\rho=V\rho V^*$$

where V is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, the lifting $\mathcal{E}^*\rho=\rho\otimes V\rho V^*$ is pure. Let $\rho=\sum_k\lambda_kE_k$ be a Schatten decomposition of ρ, the lifting

$$\mathcal{E}_3^*\rho=\sum_k\lambda_kE_k\otimes VE_kV^*$$

is also pure. This is same as $\mathcal{E}_1^*$ so that it holds the Shannon' inequality.

Example 8 *Let $\{e_k^1\}$ and $\{e_k^2\}$ be two CONSs in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, such that $\{e_k^1\}$ gives the Schatten decomposition of ρ:*

$$\rho=\sum_k\lambda_kE_k$$

$$E_k = \left|e_k^1\right\rangle \left\langle e_k^1\right|$$

We can give a pure lifting $\mathcal{E}_3^*$ as

$$\mathcal{E}_3^* \rho = \left(\sum_k \sqrt{\lambda_k} \left| e_k^1 \otimes e_k^2 \right\rangle \right) \left(\sum_l \sqrt{\lambda_l} \left\langle e_l^1 \otimes e_l^2 \right| \right)$$

This pure lifting $\mathcal{E}_3^*$ does not satisfy the Shannon inequality.

Proof. In this case, $\mathcal{E}_3^* \rho$ can be written as

$$\mathcal{E}_3^* \rho = |\xi\rangle \langle \xi|$$

$$|\xi\rangle = \sum_k \sqrt{\lambda_k} \left| e_k^1 \otimes e_k^2 \right\rangle .$$

Since $\mathcal{E}_3^* \rho$ is a pure state, and $S(\mathcal{E}_3^* \rho) = 0$. For a general (i.e., pure or mixed) state ρ, one has

$$S(\rho) = S(\Lambda^* \rho)$$

where

$$\rho = \mathrm{tr}_2 \mathcal{E}_3^* \rho$$

$$\Lambda^* \rho = \mathrm{tr}_1 \mathcal{E}_3^* \rho$$

Then,

$$\begin{aligned} I_L(\rho; \mathcal{E}_3^*) &= -S(\mathcal{E}_3^* \rho) + S(\rho) + S(\Lambda^* \rho) \\ &= 2S(\rho) \end{aligned}$$

which does not satisfy the Shannon inequality. ■

Example 9 *Let a linear map $V : \mathcal{H}_1 \to \mathcal{H}_2$ which defines a channel*

$$\Lambda \rho^* = V \rho V^*,$$

We can define a pure lifting $\mathcal{E}_3^*$ as

$$\mathcal{E}_3^* \rho = \sum_{k,l} \sqrt{\lambda_k} \sqrt{\lambda_l} \left| e_k^1 \right\rangle \left\langle e_l^1 \right| \otimes V \left| e_k^1 \right\rangle \left\langle e_l^1 \right| V^*$$

Then $\mathcal{E}_3^*$ does not satisfy the Shannon inequality

Proof. $\mathcal{E}_3^* \rho$ holds marginal condition in fact:

$$\begin{aligned} \mathrm{tr}_2 \mathcal{E}_3^* \rho &= \sum_{m,k,l} \sqrt{\lambda_k} \sqrt{\lambda_l} \left\langle e_m^2, \left| e_k^1 \otimes e_k^2 \right\rangle \left\langle e_l^1 \otimes e_l^2 \right| e_m^2 \right\rangle \\ &= \sum_k \lambda_k \left| e_k^1 \right\rangle \left\langle e_k^1 \right| = \rho \end{aligned}$$

$$\begin{aligned}\mathrm{tr}_1\mathcal{E}_3^*\rho &= \sum_{n,k,l}\sqrt{\lambda_k}\sqrt{\lambda_l}\left\langle e_n^1, \left| e_k^1 \otimes e_k^2 \right\rangle \left\langle e_l^1 \otimes e_l^2 \right| e_n^1 \right\rangle \\ &= \sum_l \lambda_l \left| e_l^2 \right\rangle \left\langle e_l^2 \right| = \Lambda^*\rho\end{aligned}$$

Since $\mathcal{E}_3^*\rho$ is a pure state for a general state, one has $S\left(\mathcal{E}_3^*\rho\right) = 0$. As the same discussion as 8, we obtain

$$I_L\left(\rho; \mathcal{E}_3^*\right) = 2S\left(\rho\right).$$

Therefore it does not satisfy the Shannon inequality. ■

4 Conclusion

We generalized a quantum mutual entropy by using liftings, so that we can represent the relation between input and output precisely. In some cases, there exists pure liftings which do not satisfy the Shannon inequality make an entangled state.

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