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Hilbert Space Representations of Quantum Phase Spaces with General Degrees of Freedom

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Abstract

For each integer \( n \geq 2 \) and a parameter \( \Lambda = (\theta, \eta) \) with \( \theta \) and \( \eta \) being \( n \times n \) real anti-symmetric matrices, a quantum phase space (QPS) (or a non-commutative phase space) with \( n \) degrees of freedom, denoted \( \text{QPS}_n(\Lambda) \), is defined, where \( \theta \) and \( \eta \) are parameters measuring non-commutativity of the QPS. Some results on Hilbert space representations of \( \text{QPS}_n(\Lambda) \) are reported.

Keywords: Quantum phase space; non-commutative phase space; canonical commutation relations; quantum deformation.

Mathematics Subject Classification 2000: 81D05, 81R60, 47L60, 47N50

1 Introduction

As is well-known, one of the fundamental principles in von Neumann's axiomatic quantum mechanics is that a subset of physical quantities of a quantum system with \( n \) external degrees of freedom (\( n \in \mathbb{N} \)) are constructed from a self-adjoint representation of the canonical commutation relations (CCR) with \( n \) degrees of freedom, which is given by a triple \((\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^{n})\) consisting of a complex Hilbert space \( \mathcal{H} \), a dense subspace \( \mathcal{D} \) of \( \mathcal{H} \) and a set \( \{Q_j, P_j\}_{j=1}^{n} \) of self-adjoint operators on \( \mathcal{H} \) satisfying (i) \( \mathcal{D} \subset \cap_{j,k=1}^{n} D(Q_jQ_k) \cap D(P_jP_k) \cap D(Q_jP_k) \cap D(P_kQ_j) \), where, for a linear operator \( A \) on a Hilbert space, \( D(A) \) denotes the domain of \( A \); (ii) (CCR)

\[
[Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad [Q_j, P_k] = i\delta_{jk}, \quad j, k = 1, \cdots, n \tag{1.1}
\]

on \( \mathcal{D} \), where \( [X,Y] := XY - YX \) is the imaginary unit and \( \delta_{jk} \) is the Kronecker delta. If \( Q_j \) and \( P_j \ (j = 1, \cdots, n) \) are not necessarily self-adjoint, but symmetric, then the triple \((\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^{n})\) is called a symmetric representation of the CCR with \( n \) degrees of freedom. This class of representations of CCR also plays important roles, e.g., in the theory of time operators ([1, 2, 3], [5, 6], [12]).
In commutation relations (1.1) and (1.2), non-commutativity is imposed only between $Q_j$ and $P_j$ ($j = 1, \cdots, n$). But, from a general mathematical point of view, it may be natural to extend non-commutativity to $Q_j$’s and $P_j$’s too. This idea leads us to a general concept of a quantum phase space (QPS) or a non-commutative phase space\(^1\). In this paper we propose one of possible
QPS’s and report some results on Hilbert space representations of it (for more details, see [4]).

In addition, we remark that non-commutative extensions of CCR have already been discussed in connection with quantum theory on non-commutative space-times (e.g., [7, 8, 9, 15]), non-commutative spaces (e.g., [10, 11]) and non-commutative phase spaces (e.g., [13, 14, 16, 17]). But it seems that representation theoretic investigations on non-commutative extensions of CCR have not yet been fully developed.

2 Hilbert Space Representations of a QPS

Let $n \in \mathbb{N}$ with $n \geq 2$. To define a QPS with $n$ degrees of freedom, we take two $n \times n$ real anti-symmetric matrices \(\theta = (\theta_{jk})_{j,k=1,\cdots,n}\) and \(\eta = (\eta_{jk})_{j,k=1,\cdots,n}\). Then we introduce an algebra generated by $2n$ elements \(\hat{Q}_j, \hat{P}_j (j = 1, \cdots, n)\) and a unit element $I$ obeying deformed CCR with $n$ degrees of freedom

\[
\begin{align*}
[\hat{Q}_j, \hat{Q}_k] &= i\theta_{jk}I, \quad (2.1) \\
[\hat{P}_j, \hat{P}_k] &= i\eta_{jk}I, \quad (2.2) \\
[\hat{Q}_j, \hat{P}_k] &= i\delta_{jk}I, \quad j, k = 1, \cdots, n. \quad (2.3)
\end{align*}
\]

We call this algebra the QPS or the non-commutative phase space with $n$ degrees of freedom and parameter

\[
\Lambda := (\eta, \theta). \quad (2.4)
\]

We denote it by $\text{QPS}_n(\Lambda)$.

It is obvious that $\hat{Q}_j$ and $\hat{Q}_k$ (resp. $\hat{P}_j$ and $\hat{P}_k$) with $j \neq k$ do not commute if and only if $\theta_{jk} \neq 0$ (resp. $\eta_{jk} \neq 0$). Hence the parameter $\Lambda$ “measures” the non-commutativity of $\hat{Q}_j$’s and $\hat{P}_j$’s respectively. Moreover $\text{QPS}_n(\Lambda)$ in the case $\theta = \eta = 0$ reduces to the algebra of the CCR with $n$ degrees of freedom. Hence $\text{QPS}_n(\Lambda)$ can be regarded as a deformation of the algebra of the CCR with $n$ degrees of freedom.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ (linear in the second variable) and norm $|| \cdot ||$. Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ and $\hat{Q}_j, \hat{P}_j$ be symmetric operators on $\mathcal{H}$.

**Definition 2.1** We say that the triple $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ is a representation (on $\mathcal{H}$) of the algebra $\text{QPS}_n(\Lambda)$ if $\mathcal{D} \subset \cap_{j,k=1}^n D(\hat{Q}_j \hat{Q}_k) \cap D(\hat{P}_j \hat{P}_k) \cap D(\hat{Q}_j \hat{P}_k) \cap D(\hat{P}_j \hat{Q}_k)$ and it satisfy (2.1)–(2.3) on $\mathcal{D}$ with $I$ being the identity on $\mathcal{H}$ (we sometimes omit the identity $I$ below).

\(^1\)Note that the components $x_j$ and $p_j$ ($j = 1, \cdots, n$) of each element $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ in the classical phase space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ can be regarded as multiplication operators acting in $L^2(\mathbb{R}^{2n})$. They form a commutative algebra.
If all $\hat{Q}_j$ and $\hat{P}_j$ ($j = 1, \cdots, n$) are self-adjoint, we say that the representation $\left( \mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \right)$ is self-adjoint.

In every representation $\left( \mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \right)$ of QPS$_n(\Lambda)$, we have commutation relations (2.1)–(2.3) on $\mathcal{D}$. Hence the following Heisenberg uncertainty relations follow: for all $\psi \in \mathcal{D}$ with $\|\psi\| = 1$ and $j, k = 1, \cdots, n$,

\[
(\Delta \hat{Q}_j)_{\psi} (\Delta \hat{Q}_k)_\psi \geq \frac{1}{2}|\theta_{jk}|,
\]
(2.5)

\[
(\Delta \hat{P}_j)_{\psi} (\Delta \hat{P}_k)_\psi \geq \frac{1}{2}|\eta_{jk}|,
\]
(2.6)

\[
(\Delta \hat{Q}_j)_{\psi} (\Delta \hat{P}_k)_\psi \geq \frac{1}{2}|\delta_{jk}|,
\]
(2.7)

where, for a symmetric operator $A$ and a vector $\psi \in D(A)$ with $\|\psi\| = 1$,

\[
(\Delta A)_{\psi} := \| (A - \langle \psi, A\psi \rangle) \psi \|,
\]

the uncertainty of $A$ in the vector state $\psi$.

3 A Class of Self-Adjoint Representations of QPS$_n(\Lambda)$ on $L^2(\mathbb{R}^n)$

In this section, we show that there exist self-adjoint representations of QPS$_n(\Lambda)$ on $L^2(\mathbb{R}^n)$. This is done by using the Schrödinger representation of the CCR with $n$ degrees of freedom.

We denote by $C_0^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

Let $\left( L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{q_j, p_j\}_{j=1}^n \right)$ be the Schrödinger representation of the CCR with $n$ degrees of freedom, namely, $q_j$ is the multiplication operator by the $j$th variable $x_j$ on $L^2(\mathbb{R}^n)$ and $p_j := -iD_j$ with $D_j$ being the generalized partial differential operator in $x_j$ on $L^2(\mathbb{R}^n)$, so that

\[
[q_j, p_k] = i\delta_{jk},
\]
(3.1)

\[
[q_j, q_k] = 0, \quad [p_j, p_k] = 0, \quad j, k = 1, \cdots, n,
\]
(3.2)

on the subspace $C_0^\infty(\mathbb{R}^n)$.

Lemma 3.1 For all $a_j, b_j \in \mathbb{R}, j = 1, \cdots, n$, $\sum_{j=1}^n (a_j p_j + b_j q_j)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

For an $n$-tuple $L = (L_1, \cdots, L_n)$ of linear operators $L_j, j = 1, \cdots, n,$ on a Hilbert space and an $n \times n$ matrix $A = (A_{jk})_{j,k=1,\cdots,n}$, we define the $n$-tuple $AL = ((AL)_1, \cdots, (AL)_n)$ of linear operators by

\[
(AL)_j := \sum_{k=1}^n A_{jk} L_k.
\]
(3.3)
We say that the parameter $\Lambda = (\theta, \eta)$ is normal if there exist $n \times n$ real matrices $A, B, C$ and $D$ satisfying

\begin{align}
A^t D - B^t C &= I_n, \\
A^t B - B^t A &= \theta, \\
C^t D - D^t C &= \eta,
\end{align}

where $I_n$ is the $n \times n$ unit matrix and $^t A$ denotes the transposed matrix of $A$.

For a normal parameter $\Lambda$ with (3.4)-(3.6), we can define a $(2n) \times (2n)$ matrix:

$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Let

$K(\Lambda) := \begin{pmatrix} \theta I_n & I_n \\ -I_n & \eta \end{pmatrix}$, \hspace{1em} $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Then we have

$G J_n^t G = K(\Lambda)$.

Conversely, if a $(2n) \times (2n)$ real matrix $G$ of the form (3.7) satisfies (3.9), then $A, B, C$ and $D$ obey relations (3.4)-(3.6).

Thus $\Lambda$ is normal if and only if there exists a $(2n) \times (2n)$ real matrix $G$ satisfying (3.9). In that case, we call $G$ a generating matrix of $\Lambda$.

We remark that, for a normal parameter $\Lambda$, its generating matrices are not unique. For example, if $G$ is a generating matrix of $\Lambda$, then, for all orthogonal matrix $M$ commuting with $K(\Lambda)$, $MG$ is a generating matrix of $\Lambda$ too.

Suppose that $\Lambda$ is normal with (3.4)-(3.6). We set

$q = (q_1, \cdots, q_n), \hspace{1em} p = (p_1, \cdots, p_n)$

and define

$\tilde{q} := A q + B p, \hspace{1em} \tilde{p} := C q + D p$.

Then, by Lemma 3.1, the operators $\tilde{q}_j$ and $\tilde{p}_j$ ($j = 1, \cdots, n$) are essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Hence their closures $\tilde{q}_j$ and $\tilde{p}_j$ are self-adjoint\(^2\). Moreover, we have the following result:

**Theorem 3.2** The set $(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\tilde{q}_j, \tilde{p}_j\}_{j=1,\cdots,n})$ is a self-adjoint representation of $QPS_n(\Lambda)$.

We call the representation $(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\tilde{q}_j, \tilde{p}_j\}_{j=1,\cdots,n})$ the quasi-Schrödinger representation of $QPS_n(\Lambda)$ with generating matrix $G$ of the form (3.7).

\(^2\)For a closable linear operator $T$, we denote its closure by $\overline{T}$. 
Remark 3.3  One can write
\[
\begin{pmatrix}
\hat{q}_1 \\
n \\
\hat{p}_1 \\
\vdots \\
\hat{q}_n \\
\hat{p}_n
\end{pmatrix} = G
\begin{pmatrix}
q_1 \\
n \\
p_1 \\
\vdots \\
q_n \\
p_n
\end{pmatrix}
\] (3.12)
on \cap_{j=1}^n D(q_j) \cap D(p_j).  Equation (3.9) is rewritten as follows:
\[
GJ_n G = J_n + \delta(\Lambda)
\] (3.13)
with
\[
\delta(\Lambda) := \begin{pmatrix}
\theta & 0 \\
0 & \eta
\end{pmatrix}.
\] (3.14)

Hence \'G is symplectic if and only if \(\delta(\Lambda) = 0\) (i.e., \(\theta = \eta = 0\)). Therefore the matrix \(\delta(\Lambda)\) represents a difference from the symplectic relation. Note that the diagonal element \(\theta\) (resp. \(\eta\)) of \(\delta(\Lambda)\) gives the non-commutativity of \(\hat{q}_j\)'s (resp. \(\hat{p}_k\)'s) \((j, k = 1, \ldots, n)\).

3.1 The Schrödinger representation of QPS

It may be interesting to consider a special case of \(\Lambda\). Let \(a \geq 0, b \geq 0\) be constants and
\[
\xi := \frac{1}{\sqrt{1 + \frac{ab}{4}}}.
\] (3.15)

Let \(\gamma\) be an \(n \times n\) real anti-symmetric matrix satisfying
\[
\gamma^2 = -I_n.
\] (3.16)

Then the parameter
\[
\Lambda_S := (\xi^2 a\gamma, \xi^2 b\gamma) \quad \text{(the case } \theta = \xi^2 a\gamma, \eta = \xi^2 b\gamma)\] (3.17)
is normal, since the matrix
\[
G_S := \begin{pmatrix}
\xi I_n & -\frac{1}{2} \xi a\gamma \\
\frac{1}{2} \xi b\gamma & \xi I_n,
\end{pmatrix}
\] (3.18)
is a generating matrix of \(\Lambda_S\), as is easily checked. We denote \(\tilde{q}_j\) and \(\tilde{p}_j\) in the present case by \(\hat{q}_j^{(S)}\) and \(\hat{p}_j^{(S)}\) respectively:
\[
\hat{q}_j^{(S)} := \xi \left( q_j - \frac{1}{2} a(\gamma p)_j \right), \quad \hat{p}_j^{(S)} := \xi \left( p_j + \frac{1}{2} b(\gamma q)_j \right), \quad j = 1, \ldots, n.
\] (3.19)

We call this self-adjoint representation \(\left( L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\hat{q}_j^{(S)}, \hat{p}_j^{(S)}\}_{j=1,\cdots,n} \right) \) of \(\text{QPS}_n(\Lambda_S)\) the \textit{Schrödinger representation of QPS}_n(\Lambda_S).

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3.2 Reconstruction of the Schrödinger representation of the CCR with $n$ degrees of freedom

In this subsection, we consider reconstruction of $q_j$ and $p_j$ in terms of $\hat{q}_j$ and $\hat{p}_j$. By (3.12), this problem may be reduced by the invertibility of the matrix $G$. From this point of view, we introduce a class of parameters $\Lambda$.

We say that $\Lambda$ is regular if it is normal and has an invertible generating matrix. It follows from (3.9) that, if $\Lambda$ is regular, then every generating matrix of $\Lambda$ is invertible.

The next lemma characterizes the regularity of $\Lambda$:

**Lemma 3.4** Let $\Lambda$ be normal with a generating matrix $G$ given by (3.7). Then $\Lambda$ is regular if and only if $I_n + \theta \eta$ and $I_n + \eta \theta$ are invertible. In that case, $G$ is invertible and

\[
(G^{-1})_n G^{-1} = - \begin{pmatrix} (I_n + \eta \theta)^{-1} \eta & -(I_n + \eta \theta)^{-1} \\ (I_n + \theta \eta)^{-1} \theta & (I_n + \theta \eta)^{-1} \end{pmatrix}.
\]

Let $\Lambda$ be regular with a generating matrix $G$. Then we can write

\[
G^{-1} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},
\]

where $F_1, F_2, F_3$ and $F_4$ are $n \times n$ real matrices.

Let

\[
\hat{q} := (\hat{q}_1, \ldots, \hat{q}_n), \quad \hat{p} := (\hat{p}_1, \ldots, \hat{p}_n).
\]

**Theorem 3.5** The following equations hold:

\[
q = F_1 \hat{q} + F_2 \hat{p}, \quad p = F_3 \hat{q} + F_4 \hat{p}.
\]

on $\bigcap_{j=1}^n D(q_j) \cap D(p_j)$.

Theorem 3.5 also implies relations of matrix elements of $G^{-1}$:

**Corollary 3.6**

\[
F_1 \theta F_1 + F_2 \eta F_2 + F_1 \eta F_2 - F_2 \theta F_1 = 0,
\]

\[
F_3 \theta F_3 + F_4 \eta F_4 + F_3 \eta F_4 - F_4 \theta F_3 = 0,
\]

\[
F_1 \theta F_3 + F_2 \eta F_4 + F_1 \eta F_4 - F_2 \theta F_3 = I_n.
\]

We now apply Theorem 3.5 to the Schrödinger representation $\{\hat{q}_j^{(S)}, \hat{p}_j^{(S)}\}_{j=1}^n$ of $\text{QPS}_n(\Lambda_S)$:

**Corollary 3.7** Let $a, b, \xi$ and $\gamma$ be as in Subsection 3.1. Suppose that

\[
\chi := 1 - \frac{1}{4} ab \neq 0.
\]
Then
\[ q_j = \frac{1}{\xi \chi} \left( \hat{q}_j^{(S)} + \frac{1}{2} a(\gamma \hat{p}^{(S)})_j \right), \quad (3.28) \]
\[ p_j = \frac{1}{\xi \chi} \left( \hat{p}_j^{(S)} - \frac{1}{2} b(\gamma \hat{q}^{(S)})_j \right), \quad j = 1, \ldots, n, \quad (3.29) \]
on $C_0^\infty(\mathbb{R}^n)$.

4 General Correspondence Between a Representation of $QPS_n(\Lambda)$ and a Representation of the CCR with $n$ Degrees of Freedom

4.1 Construction of a representation of $QPS_n(\Lambda)$ from a representation of the CCR with $n$ degrees of freedom

The contents in Section 2 suggest a general method to construct a representation of $QPS_n(\Lambda)$ from a representation of the CCR with $n$ degrees of freedom.

Let $\left( \mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n \right)$ be a representation of the CCR with $n$ degrees of freedom, namely, $\mathcal{H}$ is a Hilbert space, $\mathcal{D}$ is a dense subspace of $\mathcal{H}$ and $Q_j$ and $P_j$ ($j = 1, \ldots, n$) are symmetric operators on $\mathcal{H}$ such that $\mathcal{D} \subset \cap_{j,k=1}^n D(Q_j Q_k) \cap D(P_j P_k) \cap D(Q_j P_k) \cap D(P_k Q_j)$ and $\{Q_j, P_j\}_{j=1}^n$ obeys the CCR with $n$ degrees of freedom on $\mathcal{D}$: for $j, k = 1, \ldots, n$,
\[ [Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad [Q_j, P_k] = i\delta_{jk} \quad (4.1) \]
on $\mathcal{D}$. Let
\[ Q = (Q_1, \ldots, Q_n), \quad P = (P_1, \ldots, P_n). \]

Let $\Lambda$ be normal and $A, B, C, D$ be $n \times n$ real matrices obeying (3.4)-(3.6). By an analogy with (3.11), we define the $n$-tuples
\[ \hat{Q} := (\hat{Q}_1, \ldots, \hat{Q}_n), \quad (4.2) \]
and
\[ \hat{P} := (\hat{P}_1, \ldots, \hat{P}_n), \quad (4.3) \]
by
\[ \hat{Q} := AQ + BP, \quad \hat{P} := CQ + DP. \quad (4.4) \]

Theorem 4.1 The set $\left( \mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \right)$ defined by (4.4) is a representation of $QPS_n(\Lambda)$.

We remark that the representation $\left( \mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \right)$ of $QPS_n(\Lambda)$ is not necessarily self-adjoint even in the case where all $Q_j$ and $P_j$ ($j = 1, \ldots, n$) are self-adjoint.

As in the case of quasi-Schrödinger representations of $QPS_n(\Lambda)$ discussed in Section 2, we have the following fact:
Theorem 4.2 Let $\Lambda$ be regular with generating matrix $G$ given by (3.7) and $F_1, F_2, F_3$ and $F_4$ be as in (3.21). Then
\[ Q = F_1\hat{Q} + F_2\hat{P}, \quad (4.5) \]
\[ P = F_3\hat{Q} + F_4\hat{P}. \quad (4.6) \]
on $\mathcal{D}$.

4.2 Construction of a representation of the CCR with $n$ degrees of freedom from a representation of $\text{QPS}_n(\Lambda)$

We next consider constructing a representation of the CCR with $n$ degrees of freedom from a representation of $\text{QPS}_n(\Lambda)$. A method for that is suggested by Theorem 4.2.

Let $(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^{n})$ be a representation of $\text{QPS}_n(\Lambda)$ on a Hilbert space $\mathcal{H}$ with $\mathcal{D}$ dense in $\mathcal{H}$. Throughout this subsection, we assume the following:

(A) The parameter $\Lambda$ is regular with generating matrix $G$ given by (3.7).

Let $F_1, F_2, F_3$ and $F_4$ be as in (3.21). Then we can define $Q(\Lambda) = (Q_1(\Lambda), \cdots, Q_n(\Lambda))$ and $P(\Lambda) = (P_1(\Lambda), \cdots, P_n(\Lambda))$ by
\[ Q(\Lambda) := F_1\hat{Q} + F_2\hat{P}, \quad (4.7) \]
\[ P(\Lambda) := F_3\hat{Q} + F_4\hat{P}. \quad (4.8) \]

Theorem 4.3 Assume (A). Then $(\mathcal{H}, \mathcal{D}, \{Q_j(\Lambda), P_j(\Lambda)\}_{j=1}^{n})$ is a representation of the CCR with $n$ degrees of freedom.

The next theorem shows that every representation of $\text{QPS}_n(\Lambda)$ with condition (A) comes from a representation of the CCR with $n$ degrees of freedom:

Theorem 4.4 Assume (A). Let $Q(\Lambda)$ and $P(\Lambda)$ be defined by (4.7) and (4.8) respectively. Then
\[ \hat{Q} = AQ(\Lambda) + BP(\Lambda), \quad \hat{P} = CQ(\Lambda) + DP(\Lambda) \quad (4.9) \]
on $\mathcal{D}$.

5 Irreducibility

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators $B$ on $\mathcal{H}$ with $D(B) = \mathcal{H}$. Let $A$ be a linear operator on $\mathcal{H}$. We say that $A$ strongly commutes with $B \in \mathcal{B}(\mathcal{H})$ if $BA \subset AB$ (i.e., for all $\psi \in D(A)$, $B\psi \in D(A)$ and $BA\psi = AB\psi$). For a set $A$ of linear operators on $\mathcal{H}$, we define

\[ A' := \{B \in \mathcal{B}(\mathcal{H})|BA \subset AB, \forall A \in A\}. \quad (5.1) \]
We call \( A' \) the strong commutant of \( A \).

We say that \( A \) is irreducible if \( A' = \{cI|c \in \mathbb{C}\} \) (\( \mathbb{C} \) is the set of complex numbers).

**Lemma 5.1** Let \( S \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) and \( B \in \mathfrak{B}(\mathcal{H}) \) such that \( BS \subset SB \). Then, for all \( t \in \mathbb{R} \), \( Be^{itS} = e^{itS}B \).

**Theorem 5.2** Assume (A) in Subsection 3.2. Let \( \left( \mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n \right) \) be a representation of the CCR with \( n \) degrees of freedom. Suppose that, for each \( j = 1, \ldots, n \), \( Q_j \) and \( P_j \) are essentially self-adjoint on \( \mathcal{D} \) and \( \{Q_j, P_j\}_{j=1}^n \) is irreducible. Then the representation \( \left( \mathcal{H}, \mathcal{D}, \{\bar{Q}_j, \bar{P}_j\}_{j=1}^n \right) \) of \( \text{QPS}_n(\Lambda) \) given by (4.4) is irreducible.

We can apply Theorem 5.2 to the quasi-Schrödinger representation \( \{\bar{q}_j, \bar{p}_j\}_{j=1}^n \) of \( \text{QPS}_n(\Lambda) \) discussed in Section 2.

**Theorem 5.3** Assume (A). Then \( \{\bar{q}_j, \bar{p}_j\}_{j=1}^n \) is irreducible.

# 6 Weyl Representations of \( \text{QPS}_n(\Lambda) \)

## 6.1 Definition and basic facts

As is well known, a Weyl representation of the CCR with \( n \) degrees of freedom on a Hilbert space \( \mathcal{H} \) is defined to be a set \( \{Q_j, P_j\}_{j=1}^n \) of \( 2n \) self-adjoint operators on \( \mathcal{H} \) obeying the Weyl relations:

\[
e^{itQ_j}e^{isP_k} = e^{-ist\delta_{jk}}e^{isP_k}e^{itQ_j}, \quad \text{(6.1)}
\]
\[
e^{itQ_j}e^{isQ_k} = e^{isQ_k}e^{itQ_j}, \quad \text{(6.2)}
\]
\[
e^{itP_j}e^{isP_k} = e^{isP_k}e^{itP_j}, \quad j, k = 1, \ldots, n, s, t \in \mathbb{R}. \quad \text{(6.3)}
\]

Based on an analogy with Weyl representations of CCR, we introduce a concept of Weyl representation of \( \text{QPS}_n(\Lambda) \).

**Definition 6.1** Let \( \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \) be a set of self-adjoint operators on a Hilbert space \( \mathcal{H} \). We say that \( \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \) is a Weyl representation of \( \text{QPS}_n(\Lambda) \) if

\[
e^{it\hat{Q}_j}e^{is\hat{P}_k} = e^{-ist\delta_{jk}}e^{is\hat{P}_k}e^{it\hat{Q}_j}, \quad \text{(6.4)}
\]
\[
e^{it\hat{Q}_j}e^{is\hat{Q}_k} = e^{-ist\theta_{jk}}e^{is\hat{Q}_k}e^{it\hat{Q}_j}, \quad \text{(6.5)}
\]
\[
e^{it\hat{P}_j}e^{is\hat{P}_k} = e^{-ist\eta_{jk}}e^{is\hat{P}_k}e^{it\hat{P}_j}, \quad j, k = 1, \ldots, n, s, t \in \mathbb{R}. \quad \text{(6.6)}
\]

We call these relations the deformed Weyl relations with parameter \( \Lambda \).

For a linear operator \( A \) on a Hilbert space, we denote its spectrum by \( \sigma(A) \).
Proposition 6.2 Let \( \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \) be a Weyl representation of \( QPS_n(\Lambda) \). Then it is a self-adjoint representation of \( QPS_n(\Lambda) \). Moreover, for each \( j = 1, \cdots, n \), \( \hat{Q}_j \) and \( \hat{P}_j \) are purely absolutely continuous with
\[
\sigma(\hat{Q}_j) = \mathbb{R}, \quad \sigma(\hat{P}_j) = \mathbb{R}, \quad j = 1, \cdots, n.
\]

(6.7)

Remark 6.3 The converse of Proposition 6.2 does not hold. Indeed, there exists a self-adjoint representation of \( QPS_n(\Lambda) \) which is not a Weyl one [4].

Proposition 6.4 The set \( \{e^{it\hat{Q}_j}, e^{it\hat{P}_j} | t \in \mathbb{R}, j = 1, \cdots, n\} \) is irreducible if and only if so is \( \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \).

7 Uniqueness Theorems on Weyl Representations of \( QPS_n(\Lambda) \)

For each regular parameter \( \Lambda \), every Weyl representation of \( QPS_n(\Lambda) \) on a separable Hilbert space is unitarily equivalent to a direct sum of a quasi-Schrödinger representation \( \{\overline{\hat{q}}_j, \overline{\hat{p}}_j\}_{j=1}^n \) of \( QPS_n(\Lambda) \):

Theorem 7.1 Assume (A). Let \( \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \) be a Weyl representation of \( QPS_n(\Lambda) \) on a separable Hilbert space \( \mathcal{H} \). Then there exist closed subspaces \( \mathcal{H}_\ell \) such that the following (i)-(iii) hold:

(i) \( \mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell \) (\( N \) is a positive integer or \( \infty \)).

(ii) For each \( j = 1, \cdots, n \), \( \hat{Q}_j \) and \( \hat{P}_j \) are reduced by each \( \mathcal{H}_\ell, \ell = 1, \cdots, N \). We denote by \( \hat{Q}_j^{(\ell)} \) (resp. \( \hat{P}_j^{(\ell)} \)) the reduced part of \( \hat{Q}_j \) (resp. \( \hat{P}_j \)) to \( \mathcal{H}_\ell \).

(iii) For each \( \ell \), there exists a unitary operator \( U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n) \) such that
\[
U_\ell \hat{Q}_j^{(\ell)} U_\ell^{-1} = \overline{\hat{q}}_j, \quad U_\ell \hat{P}_j^{(\ell)} U_\ell^{-1} = \overline{\hat{p}}_j, \quad j = 1, \cdots, n,
\]
where \( \{\overline{\hat{q}}_j, \overline{\hat{p}}_j\}_{j=1}^n \) is the quasi-Schrödinger representation of \( QPS_n(\Lambda) \) defined by (3.11).

Theorem 7.1 tells us that, under the assumption there, every Weyl representation \( \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n \) of \( QPS_n(\Lambda) \) is unitarily equivalent to a direct sum of the quasi-Schrödinger representation \( \{\overline{\hat{q}}_j, \overline{\hat{p}}_j\}_{j=1}^n \), because the operator
\[
U := \oplus_{\ell=1}^N U_\ell : \mathcal{H} \to \oplus N L^2(\mathbb{R}^n),
\]
is unitary and
\[
U \hat{Q}_j U^{-1} = \oplus N \overline{\hat{q}}_j, \quad U \hat{P}_j U^{-1} = \oplus N \overline{\hat{p}}_j.
\]

Remark 7.2 There exist self-adjoint representations of \( QPS_n(\Lambda) \) which are not unitarily equivalent to \( \{\overline{\hat{q}}_j, \overline{\hat{p}}_j\}_{j=1}^n \) [4].
Theorem 7.1 and the irreducibility of the representation \( \{ \tilde{q}_j, \tilde{p}_j \}_{j=1}^n \) immediately lead us to the following fact:

**Corollary 7.3** Assume (A). Let \( \{ \hat{Q}_j, \hat{P}_j \}_{j=1}^n \) be an irreducible Weyl representation of \( \text{QPS}_n(\Lambda) \) on a separable Hilbert space \( \mathcal{K} \). Then there exists a unitary operator \( W : \mathcal{K} \rightarrow L^2(\mathbb{R}^n) \) such that

\[
W \hat{Q}_j W^{-1} = \tilde{q}_j, \quad W \hat{P}_j W^{-1} = \tilde{p}_j, \quad j = 1, \ldots, n.
\]

Applying this corollary to the case where \( \{ \hat{Q}_j, \hat{P}_j \}_{j=1}^n \) is a quasi-Schrödinger representation of \( \text{QPS}_n(\Lambda) \), we obtain the following result:

**Corollary 7.4** Let \( \Lambda \) be regular. Let \( G \) and \( G' \) be two generating matrices of \( \Lambda \): \( G \) is given by (3.7) and

\[
G' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},
\]

where \( A', B', C' \) and \( D' \) are \( n \times n \) real matrices. Let \( \{ \hat{q}_j', \hat{p}_j' \}_{j=1}^n \) be the quasi-Schrödinger representation of \( \text{QPS}_n(\Lambda) \) with generating matrix \( G' \):

\[
\hat{q}' := A'q + B'p, \quad \hat{p}' = C'q + D'p.
\]

Then there exists a unitary operator \( V : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) such that

\[
V \hat{q}_j' V^{-1} = \tilde{q}_j, \quad V \hat{p}_j' V^{-1} = \tilde{p}_j, \quad j = 1, \ldots, n. \tag{7.2}
\]

Corollary 7.4 shows that, for each regular parameter \( \Lambda \), quasi-Schrödinger representations of \( \text{QPS}_n(\Lambda) \) are unique up to unitary equivalences.

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**References**


