数理解析研究所講究録
第 1705 巻 2010 年 5-12

On a universal framework of the homogenization problems for infinite dimensional diffusions

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April 19, 2010

Abstract

By restricting the universal frame work of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process, that corresponds to the original infinite dimensional diffusion for which the homogenization problem is considered, a sufficient condition for the mapping between these processes under which the ergodic process is a unique Markov process that corresponds to a unique Markovian extension of a closable symmetric bilinear form is considered.

1 Introduction

In This note, by restricting the universal frame work of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process denoted by $(Y_\theta(t))_{t\geq 0}$, that corresponds to $(X^\theta(t))_{t\geq 0}$, the original infinite dimensional diffusion, for which the homogenization problem is considered, we discuss a sufficient condition for the mapping between these processes (denoted by $T_{\mathbf{x}}(\theta)$) under which the ergodic process is the one that corresponds to a unique Markovian extension of a closable symmetric bilinear form. Since, the present announcement plays a part of introduction of our subsequent researches on this subject, we give here a statement in a rough style without proof. All the exact and new results on this concern will be found in forthcoming papers.

2 Probability space $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$, the ergodic flow and the core

Suppose that we are given the following:
$\{(\Theta_{\mathbf{k}}, \mathcal{B}_{\mathbf{k}}, \lambda_{\mathbf{k}})\}_{\mathbf{k}\in\mathbb{Z}^d}$: a system of complete probability (*resp. measure*) spaces,

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where d is a given natural number. (*resp. for each k, λ_k is a σ-finite measure.*)
$(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$: the probability (*resp. complete measure*) space that is the completion of $(\prod_{\mathbf{k}} \Theta_{\mathbf{k}}, \bigotimes_{\mathbf{k}} \mathcal{B}_{\mathbf{k}}, \prod_{\mathbf{k}} \lambda_{\mathbf{k}})$, i.e., the completion of the direct product probability (*resp. complete measure*) space.
$(\Theta, \overline{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows:
for $\forall D \subset\subset \mathbb{Z}^d$ and for any bounded measurable function φ defined on $\prod_{\mathbf{k}\in D'} \Theta_{\mathbf{k}}$ with some $\forall D' \subset\subset \mathbb{Z}^d$, μ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu), \tag{2.1}$$

where

$$\begin{aligned}(\mathbb{E}^D \varphi)(\theta) &\equiv \int_\Theta \varphi(\theta'_D \cdot \theta_{D^c}) \mathbb{E}^D(d\theta' | \theta_{D^c}) \\ &\equiv \int_\Theta \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \overline{\lambda}(d\theta'),\end{aligned} \tag{2.2}$$

and

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{\mathbf{k}\in D^+} U_{\mathbf{k}}, \tag{2.3}$$

$$\Theta \ni \theta \longmapsto \theta_D \in \prod_{\mathbf{k}\in D} \Theta_{\mathbf{k}}$$

is the natural projection,
$\theta'_D \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta''_D = \theta'_D, \qquad \theta''_{D^c} = \theta_{D^c},$$

$$D^+ = \{\mathbf{k}' | \, \text{support of } U_{\mathbf{k}'} \cap D \neq \emptyset\},$$

also, for each $\mathbf{k} \in \mathbb{Z}^d$, $U_{\mathbf{k}}$ is a given bounded measurable function of which support is in $\prod_{|\mathbf{k}'-\mathbf{k}|\le L} \Theta_{\mathbf{k}'}$, where the number L (the range of interactions) does not depend on $\mathbf{k}$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

On $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ we are given a *measure preserving map* $T_{\mathbf{x}}$ (which is also a map on $(\Theta, \overline{\mathcal{B}}, \mu)$, but is not a measure preserving map on it *an ergodic flow*) as follows:
Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{Z}^d \quad \text{there exists a } d_{\mathbf{k}} \text{ such that} \quad d_{\mathbf{k}} \le M_1. \tag{2.4}$$

For each $\mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}}$ such that $\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ with $\mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}})$ the map $T_{\mathbf{x}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ is defined by
i)

$$T_{\mathbf{x}} : \Theta \longrightarrow \Theta$$

that is a measure preserving transformation with respect to the measure $\overline{\lambda}$;
ii)

$$T_{\mathbf{0}} = \text{the identity},$$

$$\text{for } \mathbf{x},\ \mathbf{x}' \in \mathbf{x} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \qquad T_{\mathbf{x}+\mathbf{x}'} = T_{\mathbf{x}} \circ T_{\mathbf{x}'},$$

where

$$\mathbf{x} + \mathbf{x}' \equiv (\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d},$$

with

$$\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}} = (x_1^{\mathbf{k}} + x'^{\mathbf{k}}_1, \dots, x_{d^{\mathbf{k}}}^{\mathbf{k}} + x'^{\mathbf{k}}_{d^{\mathbf{k}}}),$$

for

$$\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \dots, x_{d_{\mathbf{k}}}^{\mathbf{k}}),$$

$$\mathbf{x}' = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}'^{\mathbf{k}} = (x'^{\mathbf{k}}_1, \dots, x'^{\mathbf{k}}_{d_{\mathbf{k}}}),$$

and

$$\mathbf{0} \equiv (\mathbf{0}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{0}^{\mathbf{k}} = (0, \dots, 0) \in \mathbb{R}^{d_{\mathbf{k}}};$$

iii)

$$(\mathbf{x}, \theta) \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \Theta \longrightarrow T_{\mathbf{x}}(\theta) \in \Theta$$

is $\mathcal{B}(\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}}$–measurable, where $\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is assumed to be the topological space with the direct product topology;
iv) A function which is $T_{\mathbf{x}}$ invariant for all $\mathbf{x} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is a constant function on $(\Theta, \overline{\mathcal{B}}, \mu)$;
v) For $D \subset \mathbb{Z}^d$, let

$$\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \ni \mathbf{x} \longmapsto \mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$$

be the natural *projection*. If $\mathbf{x}_{D^c} = \mathbf{0}_{D^c}$, then

$$(T_{\mathbf{x}}(\theta))_{D^c} = \theta_{D^c}, \qquad \forall \theta \in \Theta, \quad \forall D \subset\subset \mathbb{Z}^d.$$

□

We assume that an existence of a *core* $\mathcal{D}^\Theta$. Namely, there exists $\mathcal{D}^\Theta$ which is a dense subset of both $L^2(\mu)$ and $L^1(\mu)$, and $\forall\varphi \in \mathcal{D}^\Theta$ satisfies
($\mathcal{D}$-1) φ is a bounded measurable function having only a finite number of variables θ_D for some $D \subset\subset \mathbb{Z}^d$,
($\mathcal{D}$-2)

$$\varphi(T_{\mathbf{x}_D}(\theta)) \in C^\infty(\prod_{\mathbf{k}\in D} \mathbb{R}^{d_\mathbf{k}} \to \mathbb{R}), \quad \forall\theta \in \Theta,$$

(cf. v) in the previous section) where we identify $\mathbf{x}_D \in \prod_{\mathbf{k}\in D}\mathbb{R}^{d_\mathbf{k}}$ with an $\mathbf{x} \in (\prod_{\mathbf{k}\in\mathbb{Z}^d}\mathbb{R}^{d_\mathbf{k}})$ of which projection to $\prod_{\mathbf{k}\in D}\mathbb{R}^{d_\mathbf{k}}$ is $\mathbf{x}_D$,
($\mathcal{D}$-3) in ($\mathcal{D}$-2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T_\cdot(\theta))$ (with the variables $\mathbf{x}_D$) are bounded and

$$\forall\varphi \in \mathcal{D},\ \exists M < \infty; \quad |\nabla_\mathbf{k}\varphi(T_\mathbf{x}(\theta))| < M, \quad \forall\theta \in \Theta,\ \forall\mathbf{x},\ \forall\mathbf{k} \in \mathbb{Z}^d, \tag{2.5}$$

where

$$\nabla_\mathbf{k} = (\frac{\partial}{x_1^\mathbf{k}}, \dots, \frac{\partial}{x_{d_\mathbf{k}}^\mathbf{k}}).$$

□

3 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and the processes

Suppose that we are given a system of family of functions $a_{ij}^\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^d$, $1 \le i, j \le d_\mathbf{k}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $\mathbf{k} \in \mathbb{Z}^d$ and each $1 \le i, j \le d_\mathbf{k}$, $a_{ij}^\mathbf{k}$ is a measurable function on $\Theta_\mathbf{k}$ and there exists $M_2 \in (0, \infty)$ and

$$M_2^{-1}\,\|\mathbf{x}\|^2 \le \sum_{1\le i,j\le d_\mathbf{k}} a_{ij}^\mathbf{k}(\theta_\mathbf{k}) x_i\, x_j \le M_2\,\|\mathbf{x}\|^2, \quad \forall\mathbf{k} \in \mathbb{Z}^d,\ \forall\theta_\mathbf{k} \in \Theta_\mathbf{k},$$
$$\forall\mathbf{x} = (x_1, \dots, x_{d_\mathbf{k}}) \in \mathbb{R}^{d_\mathbf{k}}, \tag{3.1}$$

also

$$a_{ij}^\mathbf{k}(\cdot) = a_{ji}^\mathbf{k}(\cdot).$$

We assume that

$$U_\mathbf{k},\ a_{ij}^\mathbf{k} \in \mathcal{D}^\Theta, \quad \mathbf{k} \in \mathbb{Z}^d,\ 1 \le i, j \le d_\mathbf{k}.$$

Also, we assume that there exists a common $M < \infty$ by which the evaluation (2.5) holds for all $a^{\mathbf{k}}_{i,j}$ and $U_{\mathbf{k}}$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtoration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbb{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

Now, for each $\theta \in \Theta$, let

$$X^{\theta} \equiv \{(X^{\theta,\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbb{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

be the unique solution of

$$\begin{aligned} X^{\theta,\mathbf{k},i}(t) = X^{\theta,\mathbf{k},i}(0) + \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \Big\{ \frac{\partial}{\partial x^{\mathbf{k}}_j} a^{\mathbf{k}}_{ij}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \\ - a^{\mathbf{k}}_{ij}(T_{X^{\theta,\mathbf{k}}(s)}(\theta))\Big(\frac{\partial}{\partial x^{\mathbf{k}}_j}\Big(\sum_{\mathbf{k}'\in\{\mathbf{k}\}^+} U_{\mathbf{k}'}(T_{X^{\theta}(s)}(\theta))\Big)\Big)\Big\} ds \\ + \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \sigma^{\mathbf{k}}_{ij}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) dB^{\mathbf{k},j}(s), \qquad t \geq 0, \end{aligned} \tag{3.2}$$

where, *as the matrix sense*,

$$(\sigma^{\mathbf{k}}_{ij}) = (2a^{\mathbf{k}}_{ij})^{\frac{1}{2}},$$

and

$$X^{\theta,\mathbf{k}}(t) = (X^{\theta,\mathbf{k},1}(t), \dots, X^{\theta,\mathbf{k},d_{\mathbf{k}}}(t)), \quad \{\mathbf{k}\}^+ = \{\mathbf{k}' |\text{ support of } U_{\mathbf{k}'} \cap \{\mathbf{k}\} \neq \emptyset\},$$

also, by $X^{\theta}(t)$ we denote the vector

$$(X^{\theta,\mathbf{k}}(t))_{\mathbf{k}\in\mathbb{Z}^d} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}.$$

To get the unique solution for (3.2) we assume the following:

Assumption 1. *All the coefficients appeared in (3.2) are uniformly bounded and equi-continuous for all $1 \leq i, j \leq d_{\mathbf{k}}$ and $\mathbf{k} \in \mathbb{Z}^d$.*

□

Proposition 3.1 *Under Assumption 1, for each $\theta \in \Theta$ the SDE (3.2) has a unique solution, and the random variable X^θ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in*

$$C([0,\infty) \to \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}).$$

□

Definition 3.1 *For $\theta \in \Theta$, let $(X_0^\theta(t))_{t\geq 0}$ be the stochastic process defined by (3.2) with the initial condition $X_0^\theta(0) = \mathbf{0}$. By using $(X_0^\theta(t))_{t\geq 0}$ and the map $T_{\mathbf{x}}(\cdot)$ we define a Θ-valued process $(Y_\theta(t))_{t\geq 0}$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ as follows:*

$$(Y_\theta(t))_{t\geq 0} = (X_0^\theta(t))_{t\geq 0}.$$

□

4 A homeomorhism

The problem of homogenization of the process $(X_0^\theta(t))_{t\geq 0}$ is described as follows:

Problem. For each $\theta \in \Theta$, $\mu - a.s.$, we are concerning the scaling limit of $(X_0^\theta(t))_{t\geq 0}$ such that

$$\lim_{\epsilon\downarrow 0}\{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t\geq 0} \tag{4.1}$$

More precisely, we consider the weak convergence of (4.1), where the sequence of the processes $\{\epsilon X_0^\theta(\frac{t}{\epsilon^2})\}_{t\geq 0}$ is understood as the sequence of random variables on $(\Omega \times \Theta, \mathcal{F} \times \overline{\mathcal{B}}, P \times \overline{\mu}; \mathcal{F}_t \times \{\Theta, \emptyset\})$ taking values in the direct product space $\prod_{\mathbf{k}\in\mathbb{Z}^d} C([0,\infty) \to \mathbb{R}^{d_{\mathbf{k}}})$ equipped with the direct product topology.

□

In order to prove the weak convergence of (4.1), the ergodicity of the process $(Y_\theta(t))_{t\geq 0}$ plays a crucial role (cf. [ABRY 1,2,3] and [AY]). Hence, for a concrete analysis on this problem, in any lale, we have to characterize both the probabilistic and analytic properties of $(Y_\theta(t))_{t\geq 0}$. In this report, assuming in particular that $\Theta_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d$, are topological spaces, and then we consider a sufficient condition under which $(Y_\theta(t))_{t\geq 0}$ is a process corresponding to a unique Markovian extension of a symmetric quadratic form.

Definition 4.1 *For each* $\mathbf{k} \in \mathbb{Z}^d$ *and* $i = 1, \ldots, d_{\mathbf{k}}$*, define an operator* $D^{\mathbf{k},i} : \mathcal{D}^{\Theta} \to \mathcal{D}^{\Theta}$ *such that*

$$(D^{\mathbf{k},i}\varphi)(\theta) \equiv \frac{\partial}{\partial x_i^{\mathbf{k}}}\varphi(T_{\mathbf{x}}(\theta))|_{\mathbf{x}=0}, \qquad \varphi \in \mathcal{D}^{\Theta}, \quad \theta \in \Theta.$$

Also, define a quadratic form $\mathcal{E}$ *on* $L^2(\mu)$ *such that*

$$\mathcal{E}(\varphi, \psi) \equiv \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{1\leq i,j\leq d_{\mathbf{k}}} \int_{\Theta} (D^{\mathbf{k},i}\varphi)(\theta)\, a_{i,j}^{\mathbf{k}}(\theta)(D^{\mathbf{k},j}\psi)(\theta)\, \mu(d\theta), \qquad \varphi, \psi \in \mathcal{D}^{\Theta}.$$

□

Theorem 4.1 *Let* $\prod_{\mathbf{k}\in\mathbb{Z}^d}\mathbb{R}^{d_{\mathbf{k}}}$ *be the topological space with the direct product topology, and for each* $M > 0$ *let* $C^{X,M}$ *be the space of continuous functions with the uniform convergence topology such that*

$$C^{X,M} \equiv \{\mathbf{x}(\cdot)\,|\,\mathbf{x}(\cdot) \in C([0, M] \to \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \text{ with } \mathbf{x}(0) = \mathbf{0}\}.$$

Suppose that for each $\mathbf{k} \in \mathbb{Z}^d$, $\Theta_{\mathbf{k}}$ *is a topological space and let* $\mathcal{B}_{\mathbf{k}}$ *be its Borel* σ*-field, also* $\boldsymbol{\Theta} = \prod_{\mathbf{k}} \Theta_{\mathbf{k}}$ *be the direct product space with the direct product topology. for each* $\theta \in \Theta$ *and* $M > 0$ *let* $C^{\theta,Y,M}$ *be the space of continuous functions with the uniform convergence topology such that*

$$C^{\theta,Y,M} \equiv \{\mathbf{y}(\cdot)\,|\,\mathbf{y}(\cdot) \in C([0, M] \to \Theta) \text{ with } \mathbf{y}(0) = \theta\}.$$

For any $\theta \in \Theta$ *and* $M > 0$ *if the map* f *defined by*

$$f \,:\, C^{X,M} \ni \mathbf{x}(\cdot) \longmapsto T_{\mathbf{x}(\cdot)}(\theta) \in C^{\theta,Y,M}$$

is a continuous onto one to one map of which inverse map f^{-1} *is also continuous (i.e.* $C^{X,M}$ *and* $C^{\theta,Y,M}$ *are homeomorhic), then the probability law of the process* $(Y_{\theta}(t))_{t\geq 0}$ *is identical with the probability law of the Markov process which corresponds to a unique Markovian extension of the quadratic form* $\mathcal{E}(\varphi, \psi)$ *defined by Definition 4.1.*

□

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