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On a universal framework of the homogenization problems for infinite dimensional diffusions

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Abstract

By restricting the universal framework of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process, that corresponds to the original infinite dimensional diffusion for which the homogenization problem is considered, a sufficient condition for the mapping between these processes under which the ergodic process is a unique Markov process that corresponds to a unique Markovian extension of a closable symmetric bilinear form is considered.

1 Introduction

In this note, by restricting the universal framework of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process denoted by \((Y_\theta(t))_{t \geq 0}\), that corresponds to \((X_\theta(t))_{t \geq 0}\), the original infinite dimensional diffusion, for which the homogenization problem is considered, we discuss a sufficient condition for the mapping between these processes (denoted by \(T_x(\theta)\)) under which the ergodic process is the one that corresponds to a unique Markovian extension of a closable symmetric bilinear form. Since, the present announcement plays a part of introduction of our subsequent researches on this subject, we give here a statement in a rough style without proof. All the exact and new results on this concern will be found in forthcoming papers.

2 Probability space \((\Theta, \overline{\mathcal{B}}, \overline{\mu})\), the ergodic flow and the core

Suppose that we are given the following:

\[\{(\Theta_k, \mathcal{B}_k, \lambda_k)\}_{k \in \mathbb{Z}^d}: \text{ a system of complete probability (resp. measure) spaces,}\]

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where $d$ is a given natural number. (resp. for each $k$, $\lambda_k$ is a $\sigma$-finite measure.)

$\Theta, \overline{B}, \overline{\lambda}$: the probability (resp. complete measure) space that is the completion of $(\prod_k \Theta_k, \otimes_k B_k, \prod_k \lambda_k)$, i.e., the completion of the direct product probability (resp. complete measure) space.

$\Theta, \overline{B}, \mu$: a complete probability space (corresponding to a Gibbs state) defined as follows:

for $\forall D \subset \subset \mathbb{Z}^d$ and for any bounded measurable function $\varphi$ defined on $\prod_{k\in D'} \Theta_k$ with some $\forall D' \subset \subset \mathbb{Z}^d$, $\mu$ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu), \quad (2.1)$$

where

$$(\mathbb{E}^D \varphi)(\theta) \equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta'_{D^c}) \mathbb{E}^D(d\theta'|\theta_{D^c}) \quad (2.2)$$

$$\equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta'_{D^c}) m_D(\theta'_D \cdot \theta'_{D^c}) \overline{\lambda}(d\theta'),$$

and

$$m_D(\theta'_D \cdot \theta'_{D^c}) \equiv \frac{1}{Z_D(\theta'_{D^c})} e^{-U_D(\theta'_D \cdot \theta'_{D^c})}, \quad U_D \equiv \sum_{k \in D^+} U_k, \quad (2.3)$$

$\Theta \ni \theta \mapsto \theta_D \in \prod_{k \in D} \Theta_k$ is the natural projection,

$\theta'_D \cdot \theta'_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta'_D = \theta'_D, \quad \theta''_{D^c} = \theta'_{D^c},$$

$$D^+ = \{k' | \text{support of } U_{k'} \cap D \neq \emptyset\},$$

also, for each $k \in \mathbb{Z}^d$, $U_k$ is a given bounded measurable function of which support is in $\prod_{|k' - k| \leq L} \Theta_{k'}$, where the number $L$ (the range of interactions) does not depend on $k$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

On $(\Theta, \overline{B}, \overline{\lambda})$ we are given a measure preserving map $T_x$ (which is also a map on $(\Theta, \overline{B}, \mu)$, but is not a measure preserving map on it an ergodic flow) as follows:

Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall k \in \mathbb{Z}^d \quad \text{there exists a } d_k \text{ such that } d_k \leq M_1. \quad (2.4)$$
For each $x \in \prod_{k} \mathbb{R}^{d_{k}}$ such that $x = (x^{k})_{k \in \mathbb{Z}^{d}}$ with $x^{k} = (x_{1}^{k}, \ldots, x_{d_{k}}^{k})$
the map $T_{x}$ on $(\Theta, \overline{B}, \overline{\lambda})$ is defined by

i) $T_{x} : \Theta \longrightarrow \Theta$

that is a measure preserving transformation with respect to the measure $\overline{\lambda}$;

ii) $T_{0} =$ the identity,

for $x, x' \in x \in \prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}$ $T_{x + x'} = T_{x} \circ T_{x'},$

where

$x + x' \equiv (x^{k} + x'^{k})_{k \in \mathbb{Z}^{d}},$

with

$x^{k} + x'^{k} = (x_{1}^{k} + x_{1}'^{k}, \ldots, x_{d_{k}}^{k} + x_{d_{k}}'^{k}),$

for

$x = (x^{k})_{k \in \mathbb{Z}^{d}}, \quad x^{k} = (x_{1}^{k}, \ldots, x_{d_{k}}^{k}),$

$x' = (x^{k})_{k \in \mathbb{Z}^{d}}, \quad x'^{k} = (x_{1}'^{k}, \ldots, x_{d_{k}}'^{k}),$

and

$0 \equiv (0^{k})_{k \in \mathbb{Z}^{d}}, \quad 0^{k} = (0, \ldots, 0) \in \mathbb{R}^{d_{k}};

iii)$

$(x, \theta) \in (\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}) \times \Theta \longrightarrow T_{x}(\theta) \in \Theta$

is $\mathcal{B}(\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}) \times \overline{B}/\overline{B} -$ measurable, where $\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}$ is assumed to be the topological space with the direct product topology;

iv) A function which is $T_{x}$ invariant for all $x \in \prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}$ is a constant function on $(\Theta, \overline{B}, \mu);$  

v) For $D \subset \mathbb{Z}^{d},$ let

$$\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}} \ni x \longmapsto x_{D} \in \prod_{k \in D} \mathbb{R}^{d_{k}}$$

be the natural projection. If $x_{D^c} = 0_{D^c},$ then

$$(T_{x}(\theta))_{D^c} = \theta_{D^c}, \quad \forall \theta \in \Theta, \quad \forall D \subset \subset \mathbb{Z}^{d}.$$
We assume that an existence of a core $\mathcal{D}^{\Theta}$. Namely, there exists $\mathcal{D}^{\Theta}$ which is a dense subset of both $L^2(\mu)$ and $L^1(\mu)$, and $\forall \varphi \in \mathcal{D}^{\Theta}$ satisfies

$\varphi$ is a bounded measurable function having only a finite number of variables $\theta_{D}$ for some $D \subset \subset \mathbb{Z}^d$,

$\mathcal{D}^{\Theta}$ which is a dense subset of both $L^2(\mu)$ and $L^1(\mu)$, and $\forall \varphi \in \mathcal{D}^{\Theta}$ satisfies

$\varphi(T_{x_D}(\theta)) \in C^\infty(\prod_{k \in D} \mathbb{R}^{d_k} \to \mathbb{R}), \forall \theta \in \Theta$,

(cf. v) in the previous section) where we identify $x_D \in \prod_{k \in D} \mathbb{R}^{d_k}$ with an $x \in (\prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k})$ of which projection to $\prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}$ is $x_D$,

in (D-2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T(\theta))$ (with the variables $x_D$) are bounded and

$\forall \varphi \in \mathcal{D}, \exists M < \infty; |\nabla_k \varphi(T_{x}(\theta))| < M, \forall \theta \in \Theta, \forall x, \forall k \in \mathbb{Z}^d,$

(2.5) where

$\nabla_k = (\frac{\partial}{x_1^k}, \ldots, \frac{\partial}{x_{d_k}^k}).$

\rule{1cm}{0.4pt}

3 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and the processes

Suppose that we are given a system of family of functions $a_{ij}^k$, $k \in \mathbb{Z}^d$, $1 \leq i, j \leq d_k$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $k \in \mathbb{Z}^d$ and each $1 \leq i, j \leq d_k$, $a_{ij}^k$ is a measurable function on $\Theta_k$ and there exists $M_2 \in (0, \infty)$ and

$M_2^{-1} \|x\|^2 \leq \sum_{1 \leq i, j \leq d_k} a_{ij}^k(\theta_k)x_i x_j \leq M_2 \|x\|^2, \forall k \in \mathbb{Z}^d, \forall \theta_k \in \Theta_k,$

(3.1) also

$a_{ij}^k(\cdot) = a_{ji}^k(\cdot).$

We assume that

$U_k, a_{ij}^k \in \mathcal{D}^{\Theta}, k \in \mathbb{Z}^d, 1 \leq i, j \leq d_k.$
Also, we assume that there exists a common $M < \infty$ by which the evaluation (2.5) holds for all $a_{i,j}^k$ and $U_k$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{k,i}(t))_{t \geq 0}\}_{k \in \mathbb{Z}^d, 1 \leq i \leq d_k}.$$ 

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^\theta,k,i(t))_{t \geq 0}\}_{k \in \mathbb{Z}^d, 1 \leq i \leq d_k}$$

be the unique solution of

$$X^\theta,k,i(t) = X^\theta,k,i(0) + \int_0^t \sum_{1 \leq j \leq d_k} \left\{ \frac{\partial}{\partial x_{j'}^k} a_{ij}^k(T_{X^\theta,k}(s)(\theta)) - a_{ij}^k(T_{X^\theta,k}(s)(\theta)) \left( \frac{\partial}{\partial x_j^k} \left( \sum_{k' \in \{k\}^+} U_{k'}(T_{X^\theta}(s)(\theta)) \right) \right) \right\} ds$$

$$+ \int_0^t \sum_{1 \leq j \leq d_k} \sigma_{ij}^k(T_{X^\theta,k}(s)(\theta)) dB^{k,j}(s), \quad t \geq 0,$$  \hspace{0.5cm} (3.2)

where, as the matrix sense,

$$(\sigma_{ij}^k) = (2a_{ij}^k)^{\frac{1}{2}},$$

and

$$X^{\theta,k}(t) = (X^{\theta,k,1}(t), \ldots, X^{\theta,k,d_k}(t)), \quad \{k\}^+ = \{k' \mid \text{support of } U_{k'} \cap \{k\} \neq \emptyset\},$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,k}(t))_{k \in \mathbb{Z}^d} \in \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}.$$ 

To get the unique solution for (3.2) we assume the following:

**Assumption 1.** All the coefficients appeared in (3.2) are uniformly bounded and equi-continuous for all $1 \leq i, j \leq d_k$ and $k \in \mathbb{Z}^d$. 

\[\square\]
Proposition 3.1 Under Assumption 1, for each $\theta \in \Theta$ the SDE (3.2) has a unique solution, and the random variable $X^\theta$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in

$$C([0, \infty) \rightarrow \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}).$$

Definition 3.1 For $\theta \in \Theta$, let $(X^\theta_0(t))_{t \geq 0}$ be the stochastic process defined by (3.2) with the initial condition $X^\theta_0(0) = 0$. By using $(X^\theta_0(t))_{t \geq 0}$ and the map $T_x(\cdot)$ we define a $\Theta$-valued process $(Y^\theta(t))_{t \geq 0}$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ as follows:

$$(Y^\theta(t))_{t \geq 0} = (X^\theta_0(t))_{t \geq 0}.$$

4 A homeomorhism

The problem of homogenization of the process $(X^\theta_0(t))_{t \geq 0}$ is described as follows:

Problem. For each $\theta \in \Theta$, $\mu$-a.s., we are concerning the scaling limit of $(X^\theta_0(t))_{t \geq 0}$ such that

$$\lim_{\epsilon \downarrow 0} \{\epsilon X^\theta_0(\frac{t}{\epsilon^2})\}_{t \geq 0} \quad (4.1)$$

More precisely, we consider the weak convergence of (4.1), where the sequence of the processes $\{\epsilon X^\theta_0(\frac{t}{\epsilon^2})\}_{t \geq 0}$ is understood as the sequence of random variables on $(\Omega \times \Theta, \mathcal{F} \times \overline{\mathcal{B}}, P \times \overline{\mu}; \mathcal{F}_t \times \{\Theta, \emptyset\})$ taking values in the direct product space

$$\prod_{k \in \mathbb{Z}^d} C([0, \infty) \rightarrow \mathbb{R}^{d_k})$$

qipped with the direct product topology.

In order to prove the weak convergence of (4.1), the ergodicity of the process $(Y^\theta(t))_{t \geq 0}$ plays a crucial role (cf. [ABRY 1,2,3] and [AY]). Hence, for a concrete analysis on this problem, in any lale, we have to characterize both the probabilistic and analytic properties of $(Y^\theta(t))_{t \geq 0}$. In this report, assuming in particular that $\Theta_k, k \in \mathbb{Z}^d$, are topological spaces, and then we consider a sufficient condition under which $(Y^\theta(t))_{t \geq 0}$ is a process corresponding to a unique Markovian extension of a symmetric quadratic form.
Definition 4.1 For each \( k \in \mathbb{Z}^d \) and \( i = 1, \ldots, d_k \), define an operator \( D^{k,i} : \mathcal{D}^\Theta \to \mathcal{D}^\Theta \) such that
\[
(D^{k,i}\phi)(\theta) \equiv \frac{\partial}{\partial x_i^{k}}\phi(T_x(\theta))|_{x=0}, \quad \phi \in \mathcal{D}^\Theta, \quad \theta \in \Theta.
\]

Also, define a quadratic form \( \mathcal{E} \) on \( L^2(\mu) \) such that
\[
\mathcal{E}(\phi, \psi) \equiv \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq i,j \leq d_k} \int_{\Theta} (D^{k,i}\phi)(\theta) a_{i,j}^k(\theta) (D^{k,j}\psi)(\theta) \mu(d\theta), \quad \phi, \psi \in \mathcal{D}^\Theta.
\]

Theorem 4.1 Let \( \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k} \) be the topological space with the direct product topology, and for each \( M > 0 \) let \( C^{X,M} \) be the space of continuous functions with the uniform convergence topology such that
\[
C^{X,M} \equiv \{x(\cdot)|x(\cdot) \in C([0, M] \rightarrow \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}) \text{ with } x(0) = 0\}.
\]
Suppose that for each \( k \in \mathbb{Z}^d \), \( \Theta_k \) is a topological space and let \( B_k \) be its Borel \( \sigma \)-field, also \( \Theta = \prod_k \Theta_k \) be the direct product space with the direct product topology. For each \( \theta \in \Theta \) and \( M > 0 \) let \( C^{\theta,Y,M} \) be the space of continuous functions with the uniform convergence topology such that
\[
C^{\theta,Y,M} \equiv \{y(\cdot)|y(\cdot) \in C([0, M] \rightarrow \Theta) \text{ with } y(0) = \theta\}.
\]
For any \( \theta \in \Theta \) and \( M > 0 \) if the map \( f \) defined by
\[
f : C^{X,M} \ni x(\cdot) \longmapsto T_{x(\cdot)}(\theta) \in C^{\theta,Y,M}
\]
is a continuous onto one to one map of which inverse map \( f^{-1} \) is also continuous (i.e. \( C^{X,M} \) and \( C^{\theta,Y,M} \) are homeomorhic), then the probability law of the process \( (Y_\theta(t))_{t \geq 0} \) is identical with the probability law of the Markov process which corresponds to a unique Markovian extension of the quadratic form \( \mathcal{E}(\phi, \psi) \) defined by Definition 4.1.

References

