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Scaling Limit for the System of Dirac Fields Coupled to Quantized Radiation Fields with Cutoffs.

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Abstract. In this paper the system of the Dirac field interacting with the quantized radiation field is investigated. By introducing ultraviolet cutoffs and spatial cutoffs, it is seen that the total Hamiltonian is a self-adjoint operator on a boson-fermion Fock space. The scaled total Hamiltonian is defined, and its asymptotic behavior is investigated. In the main theorem, it is shown that the effective potential emerges.

This article is devoted to a short review on the obtained results in [13]. We consider the system of the Quantum electrodynamics (QED), which describes the Dirac field coupled to quantized radiation field in the Coulomb gauge. We analyze this system from purely mathematical view point. The state space is defined by the boson-fermion Fock space $\mathcal{F}_{\mathrm{QED}}=\mathcal{F}_{\mathrm{Dirac}}\otimes\mathcal{F}_{\mathrm{rad}}$, where $\mathcal{F}_{\mathrm{Dirac}}$ is the fermion Fock space on $L^2(\mathbf{R}^3;\mathbf{C}^4)$ and $\mathcal{F}_{\mathrm{rad}}$ is the boson Fock space on $L^2(\mathbf{R}^3;\mathbf{C}^2)$. The free Hamiltonian of the Dirac field H_{Dirac} is defined by the second quantization of $\omega_{\mathrm{Dir}}(\mathbf{p})=\sqrt{\mathbf{p}^2+M^2}$ with the rest mass $M>0$. Similarly the free Hamiltonian of the radiation field H_{rad} is defined by the second quantization of $\omega_{\mathrm{rad}}(\mathbf{k})=|\mathbf{k}|$. The field operators of the Dirac field and the radiation field are denoted by $\psi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$, respectively. Here we impose ultraviolet cutoffs on both $\psi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$. The interaction Hamiltonians are given by

$$
\begin{aligned}
H_{\mathrm{I}}' &= \int_{\mathbf{R}^3}\chi(\mathbf{x})\psi^*(\mathbf{x})\alpha\psi(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})d\mathbf{x},\\
H_{\mathrm{II}}' &= \frac{1}{8\pi}\int_{\mathbf{R}^3\times\mathbf{R}^3}\frac{\chi(\mathbf{x})\chi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\psi^*(\mathbf{x})\psi(\mathbf{x})\psi^*(\mathbf{y})\psi(\mathbf{y})d\mathbf{x}\,d\mathbf{y},
\end{aligned}
$$

where $\chi(\mathbf{x})$ denotes the spatial cutoff, and $\psi^*(\mathbf{x})\alpha\psi(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})=\sum_{j=1}^{3}\psi^*(\mathbf{x})\alpha^j\psi(\mathbf{x})A^j(\mathbf{x})$. Then the total Hamiltonian is defined by

$$
H = H_{\mathrm{Dirac}}+H_{\mathrm{rad}}+eH_{\mathrm{I}}' + e^2H_{\mathrm{II}}' \tag{1}
$$

Let us consider the self-adjointness of H. Under sufficient conditions of the ultraviolet cutoff and the spatial cutoff, it is seen that H_{I} is relatively bounded with respect to $H_{\mathrm{rad}}^{1/2}$ and H'_{II} is a bounded operator. Hence the interactions are infinitely small with respect to $H_{\mathrm{Dirac}} + H_{\mathrm{rad}}$. Then the Kato-Rellich theorem shows that H is self-adjoint and essentially self-adjoint any core of H_0 [12]. The spectral properties of H also have been investigated in [2, 12].

Now we introduce the scaled QED Hamiltonian defined by

$$H(\Lambda) \;=\; H_{\mathrm{Dirac}} \;+\; \Lambda^2 H_{\mathrm{rad}} \;+\; e\Lambda H'_{\mathrm{I}} \;+\; e^2 H'_{\mathrm{II}}. \tag{2}$$

We are interested in the asymptotic behavior of $H(\Lambda)$ as $\Lambda \to \infty$. Historilally scaling limits of the Hamiltonians of the form (2) is introduced by E. B. Davies [3]. He investigates the system of paricles coupled to a scalar bose field, and consider the scaled total Hamitonian $H_{\mathrm{p}} + \Lambda\kappa\phi(\mathbf{x}) + \Lambda^2 H_{\mathrm{b}}$ where $H_{\mathrm{p}} = \frac{\mathbf{p}^2}{2M}$ is a shcrödinger operator, $\phi(\mathbf{x})$ is the field operator of the scalar bose field, and H_{b} is the free Hamiltonian. Then an effective Hamiltonian $H_{\mathrm{p}} + \kappa^2 V_{\mathrm{eff}}(\mathbf{x})$ is obtained Then our result can be regarded as a extended model of [3]. In [1], a general theory on scaling limits, which can be applied to a spin-boson model and non-relativistic QED models, is investigated. In [6], by removing ultraviolet cutoffs and taking a scaling limit of the Nelson model *simultaneously*, a Schrödinger operator with the Yukawa potential is derived. Refer to see also [4, 10, 11, 9, 12, 14]. It is noted that the unitary evolution of $H(\Lambda)$ is given by

$$e^{-itH(\Lambda)} \;=\; e^{-it\Lambda^2\left(\frac{1}{\Lambda^2}H_{\mathrm{Dirac}} \;+\; H_{\mathrm{rad}} \;+\; \left(\frac{e}{\Lambda}\right)H'_{\mathrm{I}} \;+\; \left(\frac{e}{\Lambda}\right)^2 H'_{\mathrm{II}}\right)}, \tag{3}$$

and we see that $t\Lambda^2$ is the scaled time and $\frac{e}{\Lambda}$ is the scaled coupling constant. The main theorem is as follows :

Theorem 1 *It follows that for $z \in \mathbf{C}\backslash\mathbf{R}$*

$$s - \lim_{\Lambda\to\infty}\left(H(\Lambda) - z\right)^{-1} = \left(H_{Dirac} + e^2 H'_{II} + e^2 V_{eff} - z\right)^{-1} P_{\Omega_{rad}}, \tag{4}$$

where

$$V_{eff} = -\frac{1}{4}\sum_{j,l}\int_{\mathbf{R}^3\times\mathbf{R}^3} \chi(\mathbf{x})\chi(\mathbf{y})\;\; \psi^*(\mathbf{x})\alpha\psi(\mathbf{x}) \cdot \triangle(\mathbf{x}-\mathbf{y})\psi^*(\mathbf{y})\alpha\psi(\mathbf{y})\;\, d\mathbf{x}d\mathbf{y}, \tag{5}$$

and $\triangle(\mathbf{z}) = (\lambda^{j,l}(\mathbf{z}) + \lambda^{j,l}(-\mathbf{z}))_{j,i=1}^3$ *is the* 3×3 *matrix with a function* $\lambda^{j,l}(\mathbf{z})$ *defined by*

$$\lambda^{j,l}(\mathbf{z}) \;=\; \int_{\mathbf{R}^3} \frac{|\chi_{rad}(\mathbf{k})|^2}{(2\pi)^3|\mathbf{k}|^2}\left(\delta_{j,l} - \frac{k^j k^l}{|\mathbf{k}|^2}\right) e^{-i\mathbf{k}\cdot\mathbf{z}} d\mathbf{k}. \tag{6}$$

By the general theorem ([11], Lemma 2.7), the following corollary immediately follows.

Corollary 2 *It follows that*

$$s-\lim_{\Lambda\to\infty} e^{-itH(\Lambda)} P_{\Omega_{rad}} = e^{-it\left(H_{Dirac}+e^2H_{long}+e^2V_{eff}\right)} P_{\Omega_{rad}}. \tag{7}$$

The outline of the proof of the main theorem is as follows. We consider the unitary transformation , called the dressing transformation, defined by

$$U(\frac{e}{\Lambda}) = e^{-i(\frac{e}{\Lambda})T},$$

where

$$T = \int_{\mathbf{R}^3} \chi(\mathbf{x})\psi^*(\mathbf{x})\alpha\psi(\mathbf{x})\cdot\Pi(\mathbf{x})d\mathbf{x},$$

with the conjugate operator $\Pi(\mathbf{x}) = (\Pi(\mathbf{x})^j)$, $j = 1,2,3$ satisfying $[\Pi(\mathbf{x})^j,\ H_{\mathrm{rad}}] = -iA^j(\mathbf{x})$ and $[\Pi(\mathbf{x})^j,\ A^l(\mathbf{y})] = i\lambda^{j,l}(\mathbf{x}-\mathbf{y})$. Then the Hamiltonian is transformed by

$$U\left(\frac{e}{\Lambda}\right)^{-1} H(\Lambda)U\left(\frac{e}{\Lambda}\right) = \tilde{H}_0(\Lambda) + K(\Lambda),$$

where $\tilde{H}_0(\Lambda) = H_{\mathrm{Dirac}} + e^2H'_{\mathrm{II}} + \Lambda^2H_{\mathrm{rad}}$, and $K(\Lambda)$ is an operator satisfying the following properties :

Proposition 3
(1) For $\varepsilon > 0$, there exists $\Lambda(\varepsilon) \geq 0$ such that for all $\Lambda > \Lambda(\varepsilon)$,

$$\|K(\Lambda)\Psi\| \leq \varepsilon\|\tilde{H}_0(\Lambda)\Psi\| + \nu(\varepsilon)\|\Psi\|. \tag{8}$$

holds, where $\nu(\varepsilon)$ is a constant independent of $\Lambda \geq \Lambda(\varepsilon)$.
(2) For all $z \in \mathbf{C}\backslash\mathbf{R}$, it follows that

$$s-\lim_{\Lambda\to\infty} K(\Lambda)\left(\tilde{H}_0(\Lambda)-z\right)^{-1} = K(H_{Dirac}+e^2H'_{II}-z)^{-1}P_{\Omega_{rad}}, \tag{9}$$

where $K = -\frac{ie^2}{2}[T, H'_I]$.

By Proposition 3 and the general theory ([1] ; Thorem 2.1), it is seen that

$$s-\lim_{\Lambda\to\infty}\left(H(\Lambda)-z\right)^{-1} = \left(H_{\mathrm{Dirac}}+e^2H'_{\mathrm{II}}+K_{\mathrm{rad}}-z\right)^{-1} P_{\Omega_{\mathrm{rad}}},$$

where

$$K_{\mathrm{rad}} = \frac{-ie^2}{2}P_{\Omega_{\mathrm{rad}}}[T,H'_{\mathrm{I}}]P_{\Omega_{\mathrm{rad}}}.$$

By the simple computation of K_{rad}, the main theorem follows.

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