SPECIAL GENERIC MAPS ON OPEN 4-MANIFOLDS

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ABSTRACT. We characterize those smooth 1-connected open 4-manifolds with certain finite type properties which admit proper special generic maps into 3-manifolds. As a corollary, we show that a smooth 4-manifold homeomorphic to \(\mathbb{R}^4\) admits a proper special generic map into \(\mathbb{R}^3\) if and only if it is diffeomorphic to \(\mathbb{R}^4\). We also characterize those smooth 4-manifolds homeomorphic to \(L \times \mathbb{R}\) for some closed orientable 3-manifold \(L\) which admit proper special generic maps into \(\mathbb{R}^3\).

1. INTRODUCTION

A special generic map \(f : M \to N\) between smooth manifolds is a smooth map with at most definite fold singularities, which have the normal form
\[
(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, x_n^2 + x_{n+1}^2 + \cdots + x_m^2),
\]
where \(m = \dim M \geq \dim N = n\). For some typical examples of special generic maps, refer to Fig. 1. Note also that the map \(\mathbb{R}^m \to \mathbb{R}^n\) defined by (1.1) is itself a proper special generic map, where a continuous map is proper if the inverse image of a compact set is always compact. Submersions are also considered special generic maps.

It has been known as the Reeb Theorem [19] that if a smooth connected closed \(m\)-dimensional manifold admits a special generic map into \(\mathbb{R}\), then it is homeomorphic to the \(m\)-sphere \(S^m\). In [20, 21], the author has shown that a smooth connected closed \(m\)-dimensional manifold \(M\) admits a special generic map into \(\mathbb{R}^n\) for every \(n\) with \(1 \leq n \leq m\) if and only if \(M\) is diffeomorphic to the standard \(m\)-sphere \(S^m\). In [23, 24] Sakuma and the author found some pairs of homeomorphic smooth closed 4-manifolds such that one of them admits a special generic map into \(\mathbb{R}^3\), while the other does not. These show that special generic maps are sensitive to detecting distinct differentiable structures on a given topological manifold.

On the other hand, it has been known that a smooth \(m\)-dimensional manifold is homeomorphic to \(\mathbb{R}^m\) if and only if it is diffeomorphic to the standard \(\mathbb{R}^m\), provided \(m \neq 4\) (see [15, 26]), while for \(m = 4\), there exist uncountably many...

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distinct differentiable structures on $\mathbb{R}^4$ (for example, see [4, 6, 8, 27]). In fact, it is known that most open 4-manifolds admit infinitely (and very often, uncountably) many distinct differentiable structures [1, 3, 5, 7].

In this paper, we characterize those smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds, using the solution to the Poincaré Conjecture in dimension three (see [16, 17, 18] or [14], for example). Here, an open 4-manifold is of finite type if its homology is finitely generated and it has only finitely many ends, whose associated fundamental groups are stable and finitely presentable. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbb{R}^4$ is diffeomorphic to the standard $\mathbb{R}^4$ if and only if it admits a proper special generic map into $\mathbb{R}^3$.

Furthermore, we show that if a smooth 4-manifold $M$ is homeomorphic to $L \times \mathbb{R}$ for some connected closed orientable 3-manifold $L$ and if $M$ admits a proper special generic map into $\mathbb{R}^3$, then $M$ is diffeomorphic to $L \times \mathbb{R}$ and the 3-manifold $L$ admits a special generic map into $\mathbb{R}^2$.

All these results claim that among the (uncountably or infinitely) many distinct differentiable structures on a certain open topological 4-manifold, there is at most one smooth structure that allows the existence of a proper special generic map into a 3-manifold.

Throughout the paper, manifolds and maps between them are differentiable of class $C^\infty$ unless otherwise indicated. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

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2. PRELIMINARIES

Let us first recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and the points $x$ and $x'$ belong to the same connected component of $f^{-1}(y)$. We define
$W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation, and denote by $q_f : M \to W_f$ the quotient map. Then we see easily that there exists a unique continuous map $\bar{f} : W_f \to N$ that makes the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{q_f} & & \uparrow{\bar{f}} \\
W_f & & 
\end{array}
$$

commutative. The above diagram is called the Stein factorization of $f$ (see [13]). Refer to Fig. 2 for an example.

The Stein factorization is a very useful tool for studying topological properties of special generic maps. In fact, we can prove the following, which is folklore (for example, see [2, 20]).

**Proposition 2.2.** Let $f : M \to N$ be a proper special generic map between smooth manifolds with $m = \dim M > \dim N = n$. Then we have the following.

1. The set of singular points $S(f)$ of $f$ is a regular submanifold of $M$ of dimension $n - 1$, which is closed as a subset of $M$.
2. The quotient space $W_f$ has the structure of a smooth $n$-dimensional manifold possibly with boundary such that $\bar{f} : W_f \to N$ is an immersion.
3. The quotient map $q_f : M \to W_f$ restricted to $S(f)$ is a diffeomorphism onto $\partial W_f$.
4. If $M$ is connected, then the quotient map $q_f$ restricted to $M \setminus S(f)$ is a smooth fiber bundle over $\text{Int} W_f$. Furthermore, if $S(f) \neq \emptyset$, then the fiber is the standard $(m - n)$-sphere $S^{m-n}$.

See Fig. 3 for an illustrative explanation.

Using the above proposition, the author proved the following [20].

**Theorem 2.3** (Disk bundle theorem). Let $f : M \to N$ be a proper special generic map between smooth connected manifolds with $\dim M = m$ and $\dim N = n$. If $m - n = 1, 2, 3$ and $S(f) \neq \emptyset$, then $M$ is diffeomorphic to the boundary of a $D^{m-n+1}$-bundle over $W_f$ with $O(m - n + 1)$ as structure group.
In the following, we recall several notions concerning ends of manifolds. For details, the reader is referred to Siebenmann's thesis [25].

**Definition 2.4.** Let $X$ be a Hausdorff space. Consider a collection $\epsilon$ of subsets of $X$ with the following properties.

(i) Each $G \in \epsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,

(ii) If $G, G' \in \epsilon$, then there exists $G'' \in \epsilon$ with $G'' \subset G \cap G'$,

(iii) $\bigcap_{G \in \epsilon} \overline{G} = \emptyset$.

Adding to $\epsilon$ every connected open non-empty set $H \subset X$ with compact frontier such that $G \subset H$ for some $G \in \epsilon$, we produce a collection satisfying (i), (ii) and (iii), which we call the *end* of $X$ determined by $\epsilon$.

An *end* of a Hausdorff space $X$ is a collection $\epsilon$ of subsets of $X$ which is maximal with respect to the properties (i), (ii) and (iii) above.

A *neighborhood* of an end $\epsilon$ is any set $N \subset X$ that contains some member of $\epsilon$. (See Fig. 4.)

**Definition 2.5.** Let $\epsilon$ be an end of a topological manifold $X$. The fundamental group $\pi_1$ is *stable* at $\epsilon$ if there exists a sequence of path connected neighborhoods of $\epsilon, X_1 \supset X_2 \supset \cdots, \text{with} \bigcap X_i = \emptyset$ such that (with base points and base paths chosen) the sequence

$$\pi_1(X_1) \xrightarrow{f_1} \pi_1(X_2) \xrightarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\text{Im}(f_1) \xrightarrow{\cong} \text{Im}(f_2) \xrightarrow{\cong} \cdots$$

The following lemma is proved in [25].
Lemma 2.6. If $\pi_1$ is stable at $\varepsilon$ and $Y_1 \supset Y_2 \supset \cdots$ is any path connected sequence of neighborhoods of $\varepsilon$ such that $\bigcap Y_i = \emptyset$, then for any choice of base points and base paths, the inverse sequence

$$G: \pi_1(Y_1) \leftarrow^{g_1} \pi_1(Y_2) \leftarrow^{g_2} \cdots$$

induced by the inclusions is stable, i.e. there exists a subsequence

$$\pi_1(Y_{i_1}) \leftarrow^{h_1} \pi_1(Y_{i_2}) \leftarrow^{h_2} \cdots$$

inducing isomorphisms

$$\text{Im}(h_1) \cong \text{Im}(h_2) \cong \cdots,$$

where each $h_j$ is a suitable composition of $g_i$'s.

Definition 2.7. When $\pi_1$ is stable at an end $\varepsilon$, we define $\pi_1(\varepsilon)$ to be the projective limit $\lim \mathcal{G}$ for some fixed system $\mathcal{G}$ as above. According to [25], $\pi_1(\varepsilon)$ is well defined up to isomorphism.

Let us introduce the following definition.

Definition 2.8. An open manifold $M$ is of finite type if

(i) $M$ has finitely many ends,

(ii) for each end $\varepsilon$, $\pi_1$ is stable at $\varepsilon$ with $\pi_1(\varepsilon)$ being finitely presentable, and

(iii) $H_*(M; \mathbb{Z}_2)$ is finitely generated.

We will need the following result due to Husch–Price [11, 12].

Lemma 2.9 (Husch–Price, 1970). Let $W$ be an open orientable 3-manifold of finite type. Then there exists a compact orientable 3-manifold $\overline{W}$ and an embedding $h : W \rightarrow \overline{W}$ such that $h(\text{Int} W) = \text{Int} \overline{W}$.
3. OPEN 4-MANIFOLDS THAT ADMIT SPECIAL GENERIC MAPS

In the following, a manifold is open if it has no boundary and each of its component is non-compact, while a manifold is closed if it has no boundary and is compact.

**Theorem 3.1.** Let $M$ be a smooth 1-connected open 4-manifold of finite type. Then there exists a proper special generic map $f : M \to N$ into a smooth 3-manifold $N$ with $S(f) \neq \emptyset$ if and only if $M$ is diffeomorphic to the connected sum of a finite number of copies of the following 4-manifolds:

1. $\mathbb{R}^4$,
2. the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$,
3. the total space of a 2-plane bundle over $S^2$,
4. the total space of an $S^2$-bundle over $S^2$,

where at least one manifold of the form (1), (2) or (3) should appear in the connected sum.

**Sketch of proof.** Let $f : M \to N$ be a proper special generic map into a 3-manifold $N$. Then we can prove that the quotient space $W_f$ in the Stein factorization of $f$ is an open 3-manifold of finite type. Since $M$ is 1-connected, so is $W_f$. By the solution to the Poincaré Conjecture together with the Husch–Stein Lemma (Lemma 2.9), we see that $W_f \cong D^3 \setminus F$ or $\mathfrak{h}^k(S^2 \times [0,1]) \setminus F$, where $F$ is a compact surface (possibly with boundary) contained in the boundary. On the other hand, $M$ is diffeomorphic to the boundary of a $D^2$-bundle over $W_f$ by the Disk bundle theorem, Theorem 2.3. Then we easily get the desired conclusion.

Conversely, it is easy to construct explicitly a proper special generic map into a 3-manifold for each 4-manifold in the list. □

**Remark 3.2.** Every 4-manifold as in Theorem 3.1 admits infinitely many (or uncountably many) distinct smooth structures. Theorem 3.1 implies that among them there is exactly one structure that allows the existence of a proper special generic map into a 3-manifold.

In particular, we have the following.

**Corollary 3.3.** Let $M$ be a smooth 4-manifold homeomorphic to $\mathbb{R}^4$. Then there exists a proper special generic map $f : M \to \mathbb{R}^3$ if and only if $M$ is diffeomorphic to the standard $\mathbb{R}^4$.

We also have the following\(^1\).

**Theorem 3.4.** Let $L$ be a smooth connected closed orientable 3-manifold. A smooth 4-manifold $M$ homeomorphic to $L \times \mathbb{R}$ admits a proper special generic map into $\mathbb{R}^3$ if and only if $M$ is diffeomorphic to $L \times \mathbb{R}$ and $L$ is a smooth closed 3-manifold that admits a special generic map into $\mathbb{R}^2$.

\(^1\)Theorem 3.4 was first conjectured by Kazuhiro Sakuma to whom the author would like to express his sincere gratitude.
Sketch of proof. Suppose $M$ is homeomorphic to $L \times \mathbb{R}$ and let $f : M \to N$ be a proper special generic map into a 3-manifold $N$. Then one can show that $W_f$ is of finite type and has exactly two ends $F_i \times [0, \infty)$, $i = 1, 2$, for some surfaces $F_i$. Furthermore, the inclusions $F_i \times \{0\} \hookrightarrow W_f$ induce isomorphisms of fundamental groups. By the standard theory of 3-manifolds together with the solution to the Poincaré Conjecture and the Husch–Price Lemma, we see that $W_f \cong (F_i \times \mathbb{R})\#(\#^kD^3)$ (for example, see [10]). Since $M$ is homeomorphic to $L \times \mathbb{R}$, we see that $W_f \cong F_i \times \mathbb{R}$. Therefore, $M$ is diffeomorphic to $L' \times \mathbb{R}$ for some 3-manifold $L'$. Note that $\pi_1(L') \cong \pi_1(L)$ is free. Therefore, $L' \cong L \cong \#^i(S^1 \times S^2)$, and hence there exists a special generic map $g : L \to \mathbb{R}^2$ by a result of Burlet–de Rham [2].

Conversely, if $L$ admits a special generic map $g : L \to \mathbb{R}^2$, then

$$g \times \text{id}_{\mathbb{R}} : L \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$$

is a proper special generic map, where $\text{id}_{\mathbb{R}}$ denotes the identity map of $\mathbb{R}$. □

Conjecture 3.5. Let $M$ be a topological 4-manifold. Then there exists at most one smooth structure on $M$ that allows the existence of a proper special generic map into $\mathbb{R}^3$.

Remark 3.6. In the above conjecture, the properness of the special generic map is essential. Let $f : M \to N$ be a special generic map of an open 4-manifold and assume that $M'$ is homeomorphic to $M$. Then there exists a "formal solution" over $M'$ on the jet level for the open differential relation corresponding to special generic maps. Therefore, $M'$ admits a special generic map by the Gromov $h$-principle for open manifolds [9]. Note that even if $f$ is proper, the resulting special generic map on $M'$ may not be proper.

Compare this with the following: if a smooth 4-manifold $M$ is homeomorphic to $\mathbb{R}^4$, then there exists a proper special generic map $g : M \to \mathbb{R}^4$. In the equidimensional case, the $C^0$ dense $h$-principle holds and the properness can be preserved (see [9]).

REFERENCES


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