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Algebraic types and the number of countable models

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1 Introduction

It is a long-standing conjecture that there is no stable theory with a finite number (> 1) of countable models. Tanović [2] proved that if a countable complete theory $T$ with $I(\omega, T) = 3$ has infinitely many definable elements then $T$ is unstable and has a dense definable ordering. In this note, we weaken the assumption of the result.

Definition 1 Let $\mathcal{F} = \{\varphi_i(x) : i \in I\}$ be a family of consistent formulas over $\emptyset$. We say that $\mathcal{F}$ is a strongly orthogonal family if the following condition is satisfied:

(*) If each $\sigma_i$ ($i \in I$) is an elementary permutation of the domain $\varphi_i^M$. Then $\bigcup_{i \in I} \sigma_i$ is an elementary mapping.

Example 2 For each $i \in I$, let $c_i \in \text{dcl}(\emptyset)$. Then $\{\text{tp}(c_i) : i \in I\}$ is a strongly orthogonal family.

Example 3 (A modification of Ehrenfeucht’s example) Let $L = \{<, U_n\}_{n \in \omega}$. For each $n \in \omega$, let $D_n$ be the convex set $(-\infty, n\sqrt{2})$ of $\mathbb{Q}$. Let $T$ be the theory of $(\mathbb{Q}, <, D_n)_{n \in \omega}$, where the interpretation of $U_n$ is $D_n$.

In $T$ there is no definable element, since neither $U_n$ nor $\neg U_n$ has end points. Even in a fixed sort of $T^{eq}$, we don’t have infinitely many definable elements. Let $\varphi_n(x)$ be the formula $U_{n+1}(x) \land \neg U_n(x)$. Then the set $\mathcal{F} = \{\varphi_n(x) : n \in \omega\}$ forms a strongly orthogonal family.
Definition 4 Let $\mathcal{F}$ be a pairwise inconsistent family of $L$-formulas with free variable $x$. We say that $p(x) \in S(\emptyset)$ is an $\mathcal{F}$-limit type if whenever $\varphi(x)$ is a member of $p(x)$ then there are infinitely many formulas $\psi(x) \in \mathcal{F}$ with $\varphi(x) \land \psi(x)$ consistent.

Remark 5 Let $\mathcal{F} = \{\varphi_i(x) : i \in \omega\}$ be a set of pairwise inconsistent $L$-formulas.

1. An $\mathcal{F}$-limit type exists. An $\mathcal{F}$-limit type is a nonprincipal type.

2. Let $p(x)$ be an $\mathcal{F}$-limit type. Then there is an infinite subset $\mathcal{F}_0$ of $\mathcal{F}$ such that (1) $p(x)$ is an $\mathcal{F}_0$-limit type and (2) for every $\varphi(x) \in p(x)$, $\{q(x) \in \mathcal{F}_0 : \varphi(x) \not\in q(x)\}$ is finite. Proof: Choose $\varphi_n(x) (n \in \omega)$ such that $p(x)$ is equivalent to $\{\varphi_n(x) : n \in \omega\}$ and that for every $n \in \omega$ $T \vdash \forall x(\varphi_{n+1}(x) \rightarrow \varphi_n(x))$. Let $I$ be the set of all $n \in \omega$ such that $\varphi_n(x) \land \neg \varphi_{n+1}(x)$ belongs to some $q \in \mathcal{F}$. First we claim that $I$ is an infinite set. Otherwise, there is $n^* \in \omega$ such that $I \subset \{0, \ldots, n^*-1\}$. For every $n \geq n^*$ and every $q \in \mathcal{F}$, we have $q(x) \vdash \varphi_n(x) \rightarrow \varphi_{n+1}(x)$. By the definition of $\mathcal{F}$-limit type, there are at least two types $q_0, q_1 \in \mathcal{F}$ such that $\varphi_{n}^*(x) \in q_k(x)$ ($k = 0, 1$). Then we have $q_0(x) \vdash p$ and $q_1(x) \vdash p$. A contradiction. Thus $I$ is an infinite set and $p(x)$ is equivalent to $\{\varphi_n(x) : n \in I\}$. For each $n \in I$, choose $q_n \in \mathcal{F}$ such that $\varphi_n(x) \land \neg \varphi_{n+1}(x) \in q_n(x)$. Then $\mathcal{F}_0 = \{q_n : n \in \omega\}$ has the required properties.

In this paper, $T$ is a countable complete theory formulated in the language $L$. Since we are interested in theories with a finite number of countable models, throughout we assume that $T$ is a small theory (i.e. $S(\emptyset)$ is countable). In section 1, we discuss the case where $T$ has a strongly orthogonal infinite family. We show that if $T$ has three countable models then $T$ must be unstable. In section 2, we discuss the case where $T$ has a strongly orthogonal infinite family of algebraic formulas, and show that if $T$ has three countable models then $T$ has the strict order property (and in fact it has a dense tree). Lemmas 9 and 10 can be proved in a similar way as corresponding lemmas in [2].
2 Strongly Orthogonal Family of Isolated Types

In this section, we show the following:

Proposition 6 Let $T$ be a theory with three countable models. Suppose that there is a strongly orthogonal infinite family of $L$-formulas. Then $T$ is unstable.

$T$ is a stable theory with $I(\omega, T) = 3$. We fix a strongly orthogonal infinite family $\mathcal{F} = \{\varphi_i(x) : i \in \omega\}$. Using the fact that $T$ is small, we assume that each $\varphi_i(x)$ generates a principal type $p_i(x)$. We fix an $\mathcal{F}$-limit type $p^*(x)$. Our aim is to derive a contradiction from these assumptions.

Lemma 7 Let $q(x)$ be a principal type. Then there are only finitely many types $p_i(y) \in \mathcal{F}$ such that $q$ and $p_i$ are not weakly orthogonal.

Proof: Suppose otherwise and for simplicity we assume that no $p_i(x)$ is weakly orthogonal to $q$. For each $i$ choose a formula $\theta_i(x, y)$ witnessing that $q(x)$ and $p_i(y)$ are not weakly orthogonal. Then, by the assumption that $q_i$ is an isolated type, $E_i(u, v) = \forall y [p_i(y) \rightarrow (\theta_i(u, y) \leftrightarrow \theta_i(v, y))]$ is a $\emptyset$-definable equivalence relation on $q^M$. Moreover $E_i$ has at least two equivalence classes.

Claim A Let $a \models q$. For any $i$, the class $a_{E_i}$ is $p_i^M$-definable.

Let $r = \operatorname{tp}_{\varphi_i}(a/p_i^M)$. By the stability, $r$ is a definable type. So there is a finite tuple $d$ from $p_i^M$ and a formula $\delta(y, z)$ such that for any $b \models p_i^M$,

$$\theta_i(x, b) \in r \iff \delta(b, d)$$

Let $\varphi(x, d)$ be the $L(d)$-formula

$$\forall y (p_i(y) \rightarrow (\delta(y, d) \leftrightarrow \theta_i(x, y))).$$

Clearly $\varphi(x, d)$ defines the set $E_i(x, a)$. (End of Proof of Claim A)

Let $\{a_i : i \in \omega\}$ be a set of realizations of $q$.

Claim B $\{E_i(x, a_i) : i \in \omega\}$ is consistent.
By claim A, the class \( a_{0E_{i}} \) is \( p_{i}^{M} \)-definable. Choose elements \( b_{i1}, \ldots, b_{ik_{i}} \in p_{i}^{M} \) and a formula \( \varphi_{i}(x, b_{i1}, \ldots, b_{ik_{i}}) \) equivalent to \( E_{i}(x, a_{0}) \). Choose an automorphism \( \sigma_{i} \) that maps \( a_{0} \) to \( a_{i} \). Let \( \tau_{i} \) be the restriction of \( \sigma_{i} \) to the domain \( p_{i}^{M} \). Then by the strong orthogonality we see that \( \bigcup_{i \in \omega} \tau_{i} \) is an elementary mapping. Since \( \{ E_{i}(x, a_{0}) : i \in \omega \} \) is consistent, \( \{ \varphi_{i}(x, \tau_{i}b_{i1}, \ldots, \tau_{i}b_{ik_{i}}) : i \in \omega \} \) is also consistent. So \( \{ E_{i}(x, \sigma_{i}(a_{0})) : i \in \omega \} \) is consistent.

From Claim A, we also know the following.

**Claim C** For each \( \eta \in 2^{\omega}, q(x) \cup q(y) \cup \{ E_{i}(x, y) : i < n, \eta(i) = 1 \} \cup \{ \neg E_{i}(x, y) : i < n, \eta(i) = 0 \} \) is consistent.

From Claim B, we have continuum many complete types over \( \emptyset \). But this is impossible, since \( I(\omega, T) < \omega \).

**Lemma 8** Let \( q \) be a principal type. Then \( q \) and \( p^{*} \) are weakly orthogonal.

**Proof:** Suppose otherwise and choose a formula \( \theta(x, y) \) such that both \( p^{*}(x) \cup q(y) \cup \{ \theta(x, y) \} \) and \( p^{*}(x) \cup q(y) \cup \{ \neg \theta(x, y) \} \) are consistent. Let \( \chi(y) \in q(y) \) be a formula isolating \( q \). Then the formula

\[
\exists y_{0}\exists y_{1}[\chi(y_{0}) \land \chi(y_{1}) \land \theta(x, y_{0}) \land \neg \theta(x, y_{1})]
\]

belongs to \( p^{*}(x) \). Since \( p^{*} \) is an \( F \)-limit type, this formula belongs to infinitely many \( p_{i} \)'s. Among such \( p_{i} \)'s, by the previous lemma, there is \( p_{i} \) such that \( p_{i} \) and \( q \) are weakly orthogonal. Then we can choose \( a \models p_{i} \) and \( b_{0}, b_{1} \) such that

\[
\mathcal{M} \models \chi(b_{0}) \land \chi(b_{1}) \land \theta(a, b_{0}) \land \neg \theta(a, b_{1})].
\]

Since \( \chi(y) \) isolates \( q(y) \), we have \( \text{tp}(b_{j}) = q \) \((j = 0, 1)\). Thus we have two distinct extensions \( \text{tp}(ab_{0}) \) and \( \text{tp}(ab_{1}) \) of \( p_{i}(x) \cup q(y) \). This contradicts the weak orthogonality of \( p_{i} \) and \( q \).

**Lemma 9** Let \( r(x) \in S(\emptyset) \) be a type with \( CB(r) = 1 \). Let \( b \models r \) and \( a_{0}, a_{1} \models p^{*} \). Suppose that \( \text{tp}(a_{1}/a_{0}) \) is semi-isolated and that \( \text{tp}(b/a_{0}) \) is not semi-isolated. Then \( \text{tp}(a_{0}b) = \text{tp}(a_{1}b) \).

**Proof:** Let \( \chi(y) \) be a formula isolating \( r(y) \) among the types with \( CB \)-rank \( \geq 1 \). By way of contradiction, we assume that the lemma is not true. Choose a formula \( \theta(x, y) \) such that \( \mathcal{M} \models \theta(a_{0}, b) \land \neg \theta(a_{1}, b) \land \chi(b) \). Choose a formula \( \psi(z, a_{0}) \) witnessing the semi-isolation of \( \text{tp}(a_{1}/a_{0}) \). Then we have \( \mathcal{M} \models \exists z[\theta(a_{0}, b) \land \neg \theta(z, b) \land \chi(b) \land \psi(z, a_{0})] \). Since \( \text{tp}(b/a_{0}) \) is not semi-isolated, we can choose \( b' \) and \( a'_{1} \) with the following properties:
1. \( \mathcal{M} \models \theta(a_0, b') \wedge \neg \theta(a_1', b') \wedge \chi(b') \wedge \psi(a_1', a_0) \).

2. \( \text{tp}(b') \neq \text{tp}(b) \), so \( \text{tp}(b') \) is a principal type.

By our choice of \( \psi(z, a_0) \), \( a_1' \) realizes the type \( p^* \). So \( \text{tp}(a_0b') \) and \( \text{tp}(a_1'b') \) are two distinct extensions of \( p^*(x) \cup \text{tp}(b') \), contradicting lemma 8.

**Lemma 10** Let \( r = \text{tp}(b) \) be a type of \( CB \)-rank 1. Let \( a \) be a realization of \( p^* \) such that \( \text{tp}(a/b) \) is isolated while \( \text{tp}(b/a) \) is not semi-isolated. Let \( \psi(x, x') \) be the formula
\[
\forall y [\chi(y) \rightarrow (\theta(x, y) \rightarrow \theta(x', y))],
\]
where \( \theta(x, b) \) is a formula isolating \( \text{tp}(a/b) \), and \( \chi(y) \) is a formula isolating \( r \) among the types with \( CB \)-rank \( \geq 1 \). Then, for any \( a' \models p^* \), the following are equivalent:

1. \( \text{tp}(a'/a) \) is semi-isolated;

2. \( \mathcal{M} \models \psi(a, a') \).

**Proof:** 1 \( \Rightarrow \) 2: Assume 1. Let \( b' \) be any element satisfying \( \chi(y) \). First suppose that \( \text{tp}(b') \) is principal. Then \( \text{tp}(b') \) and \( p^* \) are weakly orthogonal by lemma 8, so we have the equivalence of \( \theta(a, b') \) and \( \theta(a', b') \). Next suppose that \( \text{tp}(b') \) is nonprincipal and that \( \theta(a, b') \) holds. Now \( b' \) realizes \( r = \text{tp}(b) \). So we have \( \text{tp}(ab') = \text{tp}(ab) \), as \( \theta(x, b) \) isolates the type \( \text{tp}(b/a) \). In particular, \( \text{tp}(b'/a) \) is not semi-isolated.

2 \( \Rightarrow \) 1: Assume 2. Notice that \( b \) satisfies \( \chi(y) \wedge \theta(a, y) \). So, by 2, we have \( \mathcal{M} \models \theta(a', b) \). From this and the fact that \( \chi(y) \) isolates \( \text{tp}(a/b) \), we have \( \text{tp}(a') = \text{tp}(a) = p^* \). Thus, \( \text{tp}(a'/a) \) is a semi-isolated type.

**Proof of Proposition 6:** Since \( T \) has exactly three countable models, for any two nonalgebraic types \( q_i \) (\( i = 1, 2 \)) there are \( a_i \models q_i \) (\( i = 1, 2 \)) such that \( \text{tp}(a_1/a_2) \) is isolated while \( \text{tp}(a_2/a_1) \) is not semi-isolated. This can be shown using the fact that if \( I(\omega, T) = 3 \) then every type is a powerful type (see [1]). So the assumption of the last lemma 10 is fulfilled. Thus the semi-isolation is definable on \( p^* \mathcal{M} \). Since the semi-isolation relation is an infinite order, we get a contradiction. So we have shown that \( T \) is unstable.
3 Strongly Orthogonal Family of Algebraic Types

Proposition 11 Let $T$ be a theory with $I(\omega, T) = 3$. Suppose that there is a strongly orthogonal infinite family of algebraic types. Then $T$ has the strict order property.

We fix a strongly orthogonal infinite family $\mathcal{F} = \{p_i(x) : i \in \omega\}$, where each $p_i(x)$ is an algebraic type.

In section 2 lemma 7, by assuming the stability we proved the weak orthogonality of $p_i$ and $q$. However if each $p_i$ is an algebraic type, we can prove the same result without assuming the stability.

So let us recall the proof there. We assumed that each $p_i$ and $q$ are not weakly orthogonal. For each $i$, we defined an equivalence relation $E_i(u, v) = \bigwedge_{d \models p_i} (\theta_i(u, d) \leftrightarrow \theta_i(v, d))$, where $\theta_i(u, v)$ is a witness of the non-weak-orthogonality. It is a $\emptyset$-definable equivalence relation on $q^M$, having at least two equivalence classes. The main task was to show that each class is $p_i^M$-definable. We used the stability at this point. But, if $q_i$ is an algebraic type, the stability assumption is not necessary. The rest can be proven similarly. So we can show that $T$ has the strict order property. The existence of a dense tree can be proved using the argument in [3].

References


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