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Algebraic types and the number of countable models

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1 Introduction

It is a long-standing conjecture that there is no stable theory with a finite number (>1) of countable models. Tanović [2] proved that if a countable complete theory T with $I(\omega, T)=3$ has infinitely many definable elements then T is unstable and has a dense definable ordering. In this note, we weaken the assumption of the result.

Definition 1 Let $\mathcal{F}=\{\varphi_i(x): i\in I\}$ be a family of consistent formulas over $\emptyset$. We say that $\mathcal{F}$ is a strongly orthogonal family if the following condition is satisfied:

(*) If each σ_i $(i\in I)$ is an elementary permutation of the domain $\varphi_i^{\mathcal{M}}$. Then $\bigcup_{i\in I}\sigma_i$ is an elementary mapping.

Example 2 For each $i\in I$, let $c_i\in \mathrm{dcl}(\emptyset)$. Then $\{\mathrm{tp}(c_i): i\in I\}$ is a strongly orthogonal family.

Example 3 (A modification of Ehrenfeucht's example) Let $L=\{<, U_n\}_{n\in\omega}$. For each $n\in\omega$, let D_n be the convex set $(-\infty, n\sqrt{2})$ of $\mathbb{Q}$. Let T be the theory of $(\mathbb{Q}, <, D_n)_{n\in\omega}$, where the interpretation of U_n is D_n.

In T there is no definable element, since neither U_n nor $\neg U_n$ has end points. Even in a fixed sort of T^{eq}, we don't have infinitely many definable elements. Let $\varphi_n(x)$ be the formula $U_{n+1}(x)\wedge\neg U_n(x)$. Then the set $\mathcal{F}=\{\varphi_n(x): n\in\omega\}$ forms a strongly orthogonal family.

Definition 4 Let $\mathcal{F}$ be a pairwise inconsistent family of L-formulas with free variable x. We say that $p(x) \in \mathrm{S}(\emptyset)$ is an $\mathcal{F}$-limit type if whenever $\varphi(x)$ is a member of $p(x)$ then there are infinitely many formulas $\psi(x) \in \mathcal{F}$ with $\varphi(x) \wedge \psi(x)$ consistent.

Remark 5 Let $\mathcal{F} = \{\varphi_i(x) : i \in \omega\}$ be a set of pairwise inconsistent L-formulas.

1. An $\mathcal{F}$-limit type exists. An $\mathcal{F}$-limit type is a nonprincipal type.

2. Let $p(x)$ be an $\mathcal{F}$-limit type. Then there is an infinite subset $\mathcal{F}_0$ of $\mathcal{F}$ such that (1) $p(x)$ is an $\mathcal{F}_0$-limit type and (2) for every $\varphi(x) \in p(x)$, $\{q(x) \in \mathcal{F}_0 : \varphi(x) \notin q(x)\}$ is finite. Proof: Choose $\varphi_n(x)$ $(n \in \omega)$ such that $p(x)$ is equivalent to $\{\varphi_n(x) : n \in \omega\}$ and that for every $n \in \omega$ $T \vdash \forall x(\varphi_{n+1}(x) \rightarrow \varphi_n(x))$. Let I be the set of all $n \in \omega$ such that $\varphi_n(x) \wedge \neg\varphi_{n+1}(x)$ belongs to some $q \in \mathcal{F}$. First we claim that I is an infinite set. Otherwise, there is $n^* \in \omega$ such that $I \subset \{0, ..., n^*-1\}$. For every $n \geq n^*$ and every $q \in \mathcal{F}$, we have $q(x) \vdash \varphi_n(x) \rightarrow \varphi_{n+1}(x)$. By the definition of $\mathcal{F}$-limit type, there are at least two types $q_0, q_1 \in \mathcal{F}$ such that $\varphi_{n^*}(x) \in q_k(x)$ $(k = 0, 1)$. Then we have $q_0(x) \vdash p$ and $q_1(x) \vdash p$. A contradiction . Thus I is an infinite set and $p(x)$ is equivalent to $\{\varphi_n(x) : n \in I\}$. For each $n \in I$, choose $q_n \in \mathcal{F}$ such that $\varphi_n(x) \wedge \neg\varphi_{n+1}(x) \in q_n(x)$. Then $\mathcal{F}_0 = \{q_n : n \in \omega\}$ has the required properties.

In this paper, T is a countable complete theory formulated in the language L. Since we are interested in theories with a finite number of countable models, throughout we assume that T is a small theory (i.e. $\mathrm{S}(\emptyset)$ is countable). In section 1, we discuss the case where T has a strongly orthogonal infinite family. We show that if T has three countable models then T must be unstable. In section 2, we discuss the case where T has a strongly orthogonal infinite family of algebraic formulas, and show that if T has three countable models then T has the strict order property (and in fact it has a dense tree). Lemmas 9 and 10 can be proved in a similar way as corresponding lemmas in [2].

2 Strongly Orthogonal Family of Isolated Types

In this section, we show the following:

Proposition 6 *Let T be a theory with three countable models. Suppose that there is a strongly orthogonal infinite family of L-formulas. Then T is unstable.*

T is a stable theory with $I(\omega, T) = 3$. We fix a strongly orthogonal infinite family $\mathcal{F} = \{\varphi_i(x) : i \in \omega\}$. Using the fact that T is small, we may assume that each $\varphi_i(x)$ generates a principal type $p_i(x)$. We fix an $\mathcal{F}$-limit type $p^*(x)$. Our aim is to derive a contradiction from these assumptions.

Lemma 7 *Let $q(x)$ be a principal type. Then there are only finitely many types $p_i(y) \in \mathcal{F}$ such that q and p_i are not weakly orthogonal.*

Proof: Suppose otherwise and for simplicity we assume that no $p_i(x)$ is weakly orthogonal to q. For each i choose a formula $\theta_i(x, y)$ witnessing that $q(x)$ and $p_i(y)$ are not weakly orthogonal. Then, by the assumption that q_i is an isolated type, $E_i(u, v) = \forall y\,[p_i(y) \to (\theta_i(u, y) \leftrightarrow \theta_i(v, y))]$ is a $\emptyset$-definable equivalence relation on $q^{\mathcal{M}}$. Moreover E_i has at least two equivalence classes.

Claim A *Let $a \models q$. For any i, the class a_{E_i} is $p_i^{\mathcal{M}}$-definable.*

Let $r = \mathrm{tp}_{\theta_i}(a/p_i^{\mathcal{M}})$. By the stability, r is a definable type. So there is a finite tuple d from $p_i^{\mathcal{M}}$ and a formula $\delta(y, z)$ such that for any $b \models p_i^{\mathcal{M}}$,

$$\theta_i(x, b) \in r \iff \delta(b, d) \text{ holds.}$$

Let $\varphi(x, d)$ be the $L(d)$-formula

$$\forall y(p_i(y) \to (\delta(y, d) \leftrightarrow \theta_i(x, y))).$$

Clearly $\varphi(x, d)$ defines the set $E_i(x, a)$. (End of Proof of Claim A)

Let $\{a_i : i \in \omega\}$ be a set of realizations of q.

Claim B *$\{E_i(x, a_i) : i \in \omega\}$ is consistent.*

By claim A, the class a_{0E_i} is $p_i^{\mathcal{M}}$-definable. Choose elements $b_{i1}, ..., b_{ik_i} \in p_i^{\mathcal{M}}$ and a formula $\varphi_i(x, b_{i1}, ..., b_{ik_i})$ equivalent to $E_i(x, a_0)$. Choose an automorphism σ_i that maps a_0 to a_i. Let τ_i be the restriction of σ_i to the domain $p_i^{\mathcal{M}}$. Then by the strong orthogonality we see that $\bigcup_{i\in\omega} \tau_i$ is an elementary mapping. Since $\{E_i(x, a_0) : i \in \omega\}$ is consistent, $\{\varphi_i(x, \tau_i b_{i1}, ..., \tau_i b_{ik_i}) : i \in \omega\}$ is also consistent. So $\{E_i(x, \sigma_i(a_0)) : i \in \omega\}$ is consistent.

From Claim A, we also know the following.

Claim C *For each $\eta \in 2^\omega$, $q(x) \cup q(y) \cup \{E_i(x, y) : i < n,\ \eta(i) = 1\} \cup \{\neg E_i(x, y) : i < n,\ \eta(i) = 0\}$ is consistent.*

From Claim B, we have continuum many complete types over $\emptyset$. But this is impossible, since $I(\omega, T) < \omega$.

Lemma 8 *Let q be a principal type. Then q and p^* are weakly orthogonal.*

Proof: Suppose otherwise and choose a formula $\theta(x, y)$ such that both $p^*(x) \cup q(y) \cup \{\theta(x, y)\}$ and $p^*(x) \cup q(y) \cup \{\neg\theta(x, y)\}$ are consistent. Let $\chi(y) \in q(y)$ be a formula isolating q. Then the formula

$$\exists y_0 \exists y_1 [\chi(y_0) \wedge \chi(y_1) \wedge \theta(x, y_0) \wedge \neg\theta(x, y_1)]$$

belongs to $p^*(x)$. Since p^* is an $\mathcal{F}$-limit type, this formula belongs to infinitely many p_i's. Among such p_i's, by the previous lemma, there is p_i such that p_i and q are weakly orthogonal. Then we can choose $a \models p_i$ and b_0, b_1 such that

$$\mathcal{M} \models \chi(b_0) \wedge \chi(b_1) \wedge \theta(a, b_0) \wedge \neg\theta(a, b_1)].$$

Since $\chi(y)$ isolates $q(y)$, we have $\mathrm{tp}(b_j) = q$ $(j = 0, 1)$. Thus we have two distinct extensions $\mathrm{tp}(ab_0)$ and $\mathrm{tp}(ab_1)$ of $p_i(x) \cup q(y)$. This contradicts the weak orthogonality of p_i and q.

Lemma 9 *Let $r(x) \in \mathrm{S}(\emptyset)$ be a type with $CB(r) = 1$. Let $b \models r$ and $a_0, a_1 \models p^*$. Suppose that $\mathrm{tp}(a_1/a_0)$ is semi-isolated and that $\mathrm{tp}(b/a_0)$ is not semi-isolated. Then $\mathrm{tp}(a_0 b) = \mathrm{tp}(a_1 b)$.*

Proof: Let $\chi(y)$ be a formula isolating $r(y)$ among the types with CB-rank ≥ 1. By way of contradiction, we assume that the lemma is not true. Choose a formula $\theta(x, y)$ such that $\mathcal{M} \models \theta(a_0, b) \wedge \neg\theta(a_1, b) \wedge \chi(b)$. Choose a formula $\psi(z, a_0)$ witnessing the semi-isolation of $\mathrm{tp}(a_1/a_0)$. Then we have $\mathcal{M} \models \exists z[\theta(a_0, b) \wedge \neg\theta(z, b) \wedge \chi(b) \wedge \psi(z, a_0)]$. Since $\mathrm{tp}(b/a_0)$ is not semi-isolated, we can choose b' and a_1' with the following properties:

1. $\mathcal{M} \models \theta(a_0, b') \wedge \neg\theta(a_1', b') \wedge \chi(b') \wedge \psi(a_1', a_0)$.

2. $\mathrm{tp}(b') \neq \mathrm{tp}(b)$, so $\mathrm{tp}(b')$ is a principal type.

By our choice of $\psi(z, a_0)$, a_1' realizes the type p^*. So $\mathrm{tp}(a_0 b')$ and $\mathrm{tp}(a_1' b')$ are two distinct extensions of $p^*(x) \cup \mathrm{tp}(b')$, contradicting lemma 8.

Lemma 10 *Let* $r = \mathrm{tp}(b)$ *be a type of CB-rank 1. Let* a *be a realization of* p^* *such that* $\mathrm{tp}(a/b)$ *is isolated while* $\mathrm{tp}(b/a)$ *is not semi-isolated. Let* $\psi(x, x')$ *be the formula*

$$\forall y[\chi(y) \rightarrow (\theta(x, y) \rightarrow \theta(x', y))],$$

where $\theta(x, b)$ *is a formula isolating* $\mathrm{tp}(a/b)$*, and* $\chi(y)$ *is a formula isolating* r *among the types with CB-rank* ≥ 1*. Then, for any* $a' \models p^*$*, the following are equivalent:*

1. $\mathrm{tp}(a'/a)$ *is semi-isolated;*

2. $\mathcal{M} \models \psi(a, a')$.

Proof: $1 \Rightarrow 2$: Assume 1. Let b' be any element satisfying $\chi(y)$. First suppose that $\mathrm{tp}(b')$ is principal. Then $\mathrm{tp}(b')$ and p^* are weakly orthogonal by lemma 8, so we have the equivalence of $\theta(a, b')$ and $\theta(a', b')$. Next suppose that $\mathrm{tp}(b')$ is nonprincipal and that $\theta(a, b')$ holds. Now b' realizes $r = \mathrm{tp}(b)$. So we have $\mathrm{tp}(ab') = \mathrm{tp}(ab)$, as $\theta(x, b)$ isolates the type $\mathrm{tp}(b/a)$. In particular, $\mathrm{tp}(b'/a)$ is not semi-isolated.
$2 \Rightarrow 1$: Assume 2. Notice that b satisfies $\chi(y) \wedge \theta(a, y)$. So, by 2, we have $\mathcal{M} \models \theta(a', b)$. From this and the fact that $\theta(x, b)$ isolates $\mathrm{tp}(a/b)$, we have $\mathrm{tp}(a') = \mathrm{tp}(a) = p^*$. Thus, $\mathrm{tp}(a'/a)$ is a semi-isolated type.

Proof of Proposition 6: Since T has exactly three countable models, for any two nonalgebraic types q_i $(i = 1, 2)$ there are $a_i \models q_i$ $(i = 1, 2)$ such that $\mathrm{tp}(a_1/a_2)$ is isolated while $\mathrm{tp}(a_2/a_1)$ is not semi-isolated. This can be shown using the fact that if $I(\omega, T) = 3$ then every type is a powerful type (see [1]). So the assumption of the last lemma 10 is fulfilled. Thus the semi-isolation is definable on $p^{*\mathcal{M}}$. Since the semi-isolation relation is an infinite order, we get a contradiction. So we have shown that T is unstable.

3 Strongly Orthogonal Family of Algebraic Types

Proposition 11 *Let T be a theory with $I(\omega, T) = 3$. Suppose that there is a strongly orthogonal infinite family of algebraic types. Then T has the strict order property.*

We fix a strongly orthogonal infinite family $\mathcal{F} = \{p_i(x) : i \in \omega\}$, where each $p_i(x)$ is an algebraic type.

In section 2 lemma 7, by assuming the stability we proved the weak orthogonality of p_i and q. However if each p_i is an algebraic type, we can prove the same result without assuming the stability.

So let us recall the proof there. We assumed that each p_i and q are not weakly orthogonal. For each i, we defined an equivalence relation $E_i(u, v) = \bigwedge_{d \models p_i}(\theta_i(u, d) \leftrightarrow \theta_i(v, d))$, where $\theta_i(u, v)$ is a witness of the non-weak-orthogonality. It is a $\emptyset$-definable equivalence relation on $q^{\mathcal{M}}$, having at least two equivalence classes. The main task was to show that each class is $p_i^{\mathcal{M}}$-definable. We used the stability at this point. But, if q_i is an algebraic type, the stability assumption is not necessary. The rest can be proven similarly. So we can show that T has the strict order property. The existence of a dense tree can be proved using the argument in [3].

References

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