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RELATIVE GEOMETRIC CONFIGURATIONS

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ABSTRACT. This is a survey of a recent work done by the three authors, in which an analysis of geometric properties of a structure relative to a reduct is initiated. In particular, definable groups and fields in this context are considered. In a relatively 1-based theory every group is definably isogenous to a subgroup of a group definable in the reduct. For relatively CM-trivial theories (which encompass certain Hrushovski's amalgams, such as the fusion of two strongly minimal theories or coloured fields), we prove that every group can be mapped by a homomorphism with central kernel to a group definable in the reduct.

1. INTRODUCTION

Both DCF_0, the theory of differential closed fields in characteristic 0, and the theory ACFA of existentially closed algebraically closed fields equipped with an automorphism satisfy a strong structural condition on definable groups: They can be embedded into an algebraic group. In both cases, the proof reduces to setting up a group configuration diagram from the ambient definable group and from there recovering an algebraic group.

A structure is called 1-*based* if it does not interpret a (complete) pseudo-plane, a specific incidence configuration encapturing the relation between lines and points in the euclidian plane. This has important consequences [11]: a stable 1-based group is abelian-by-finite and every definable set is a boolean combination of cosets of subgroups (which moreover are definable over the algebraic closure of $\emptyset$, so there are only boundedly many such subgroups)

Generalizing the concept of pseudo-plane to higher dimensions, a structure is called *CM-trivial* if it does not interpret a pseudo-space. It has been shown [16] that CM-trivial groups are nilpotent-by-finite.

This article is a survey of recent work [6] by the authors generalizing the previous set-up to the case of a theory relative to a reduct, capturing hence the two examples exhibited at the beginning of the discussion, as well as several exotic structures obtained by the Fraïssé-Hrushovski amalgamation construction, most notably the coloured fields and the fusion of two strongly minimal sets [21, 22, 1, 4, 2, 12, 5]. It is proven that every definable group in a relatively 1-based theory is definably isogenous to a subgroup of a definable group in the reduct, and a definable group in a relatively CM-trivial theory is mapped by a definable homogeny with central kernel to a definable group in the reduct.

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2. Main results

We will consider a stable theory T in a language $\mathcal{L}$ together with a reduct T_0 in a sublanguage $\mathcal{L}_0$. Note that T_0 is again stable. One could (and should in the case of the fusion) work with several reducts simultaneously, but in order to simplify the exposition we shall only consider a single one. We assume that T comes equipped with a finitary closure operator $\langle\rangle$ such that for every real set A we have $A \subseteq \langle A\rangle \subseteq \mathrm{acl}(A)$. Model-theoretical notions such as definable or algebraic closure dcl, acl, types tp, canonical bases Cb or independence $\mathop{\smile\!\!\!\!\!\!|}$ refer to T. If we mean them in the sense of T_0, we will indicate this by the index 0: dcl_0, acl_0, tp_0, Cb_0, $\mathop{\smile\!\!\!\!\!\!|}^0$. Moreover, we will assume that T_0 has geometric elimination of imaginaries, i.e. every T_0-imaginary element is T_0-interalgebraic with a real tuple. Note that this always holds if T_0 is strongly minimal with infinite $\mathrm{acl}_0(\emptyset)$.

Definition 2.1. The theory T is *1-based over* T_0 with respect to $\langle.\rangle$ if for every real algebraically closed sets $A \subseteq B$ and every real tuple c, if

$$\langle Ac\rangle \mathop{\smile\!\!\!\!\!\!|}^0_A B,$$

then the canonical base $\mathrm{Cb}(c/B)$ is algebraic over A (in the sense of T^{eq}).

Definition 2.2. The theory T is *CM-trivial over* T_0 with respect to $\langle.\rangle$ if for every real algebraically closed sets $A \subseteq B$ and every real tuple c, if

$$\langle Ac\rangle \mathop{\smile\!\!\!\!\!\!|}^0_A B,$$

then the canonical base $\mathrm{Cb}(c/A)$ is algebraic over $\mathrm{Cb}(c/B)$ (in the sense of T^{eq}).

Remark 2.3. Every theory is 1-based (*resp.* CM-trivial) over itself with respect to acl. If T is 1-based (*resp.* CM-trivial) over its reduct to equality with respect to acl, then T is 1-based (*resp.* CM-trivial) in the classical sense. The converse holds if T has geometric elimination of imaginaries.

Every relatively 1-based theory is relatively CM-trivial.

Definition 2.4. The theory T is 1-*ample* over T_0 with respect to $\langle.\rangle$ if there are real tuples $\bar{a}$, $\bar{b}$ and $\bar{c}$ such that:

- $\mathrm{acl}(a,\bar{b}) \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{acl}(\bar{a})} \langle \mathrm{acl}(a), c\rangle$.
- $\bar{c} \not\mathop{\smile\!\!\!\!\!\!|}_a \bar{b}$.

Definition 2.5. The theory T is 2-*ample* over T_0 with respect to $\langle.\rangle$ if there are real tuples $\bar{a}$, $\bar{b}$ and $\bar{c}$ such that:

- $\mathrm{acl}(a,\bar{b}) \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{acl}(\bar{a})} \langle \mathrm{acl}(a), c\rangle$.
- $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{b}} \bar{a}\bar{b}$.
- $\bar{c} \not\mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(\bar{a})\cap\mathrm{acl}^{\mathrm{eq}}(\bar{b})} \bar{a}$.

As in the classical setting, it is straightforward to see that:

Remark 2.6. The notions of 1-basedness and not-1-ampleness coincide. So do CM-triviality and non-2-ampleness.

Though the definitions above need not require much in terms of the nature of the closure operator, in order to have these notions stable under adding or removing parameters, we need impose the following extra conditions on $\langle . \rangle$:

(†) If A is algebraically closed and $b \mathop{\smile\!\!\!\!\!\!|}_A c$, then $\langle Abc \rangle \subseteq \mathrm{acl}_0(\langle Ab \rangle, \langle Ac \rangle)$.

(‡) If $\bar{a} \in \mathrm{acl}_0(A)$, then $\langle \mathrm{acl}(\bar{a}), A \rangle \subseteq \mathrm{acl}_0(\mathrm{acl}(\bar{a}), \langle A \rangle)$.

Example 2.7.

- DCF_0 is 1-based over ACF_0 with respect to the differential closure acl_δ, which satisfies (†) and (‡) [27, 15].
- $ACFA$ is 1-based over ACF with respect to the σ-closure acl_σ, which satisfies again (†) and (‡) [8].

Another class of relatively CM-trivial structures are those obtained by means of the Fraïssé-Hrushovski amalgamation construction. Briefly, the method can be described as follows: One considers a specific class of structures equipped with a primitive notion of *(pre-)dimension* satisfying the submodularity inequality, together with the notion of a *self-sufficient* subset $A \leq B$ (the predimension of A is minimal among predimensions of supersets of A in B). One can now apply Fraïssé's amalgamation method to construct a countable strongly homogeneous model; if the conditions are sufficiently definable, it will be saturated for its theory. However, in this structure 0-dimensional sets (in the sense of the predimension) will not be algebraic in general. In order to make them algebraic, one has to impose certain finiteness conditions on the number of realizations of certain minimal 0-dimensional sets; this has to be done uniformly in order to preserve first-order definability. One thus obtains a strongly minimal (or at least finite rank) self-sufficient substructure, known as the *collapse*. By means of this method Hrushovski constructed [13] a strongly minimal counter-example to Zilber's trichotomy conjecture, as well as the fusion of two strongly minimal sets into a single one [12], implying in particular that there is no maximal strongly minimal theory. Other examples include the *coloured fields* as introduced by Poizat (an algebraically closed field with a predicate for either a subset of algebraically independent elements, or a proper non-trivial additive subgroup in positive characteristic, or a proper non-trivial multiplicative subgroup in characteristic zero) [21, 22, 1, 3, 4, 2], or the fusion of two strongly minimal expansions of a common vector space over a finite field [5].

Our goal was to isolate the common features of all known examples of amalgamation, though the construction used in each case is slightly different, in order to obtain the following result.

Proposition 2.8. *All known examples of Fraïssé-Hrushovski's amalgams are CM-trivial over the base theorie(s) with respect to the self-sufficient closure.*

In particular the coloured fields are CM-trivial over the theory of algebraically closed fields.

The key technical result is the following proposition which is valid in a general setting. It allows to pass from the algebraicity relation given by an ambient group law defined in T to a T_0-algebraicity condition after *blowing up* by a countable Morley sequence in the generic type. In general the resulting T_0-algebraicity condition may well be trivial, for example if T_0 is the reduct of T to equality. The relative geometric conditions will ensure that it captures a big part of the original group.

Proposition 2.9. *Let G be a connected group type-definable over $\emptyset$ in T, and consider two generic independent elements a, b of G with $c = ab$. Let D be a countable Morley sequence of the generic type of G over a, b and set*

$$\alpha = \mathrm{acl}_0(\mathrm{acl}(b, D), \mathrm{acl}(c, D)) \cap \mathrm{acl}(a, D)$$
$$\beta = \mathrm{acl}_0(\mathrm{acl}(a, D), \mathrm{acl}(c, D)) \cap \mathrm{acl}(b, D)$$
$$\gamma = \mathrm{acl}_0(\mathrm{acl}(a, D), \mathrm{acl}(b, D)) \cap \mathrm{acl}(c, D).$$

Then α, β and γ are pairwise independent but each one is 0-algebraic over the other two. Moreover, α is 0-interalgebraic with $\mathrm{Cb}_0(\mathrm{acl}(b, D), \mathrm{acl}(c, D))/\mathrm{acl}(a, D))$ and hence

$$\mathrm{acl}(b, D), \mathrm{acl}(c, D) \underset{\alpha}{\downarrow^0} \mathrm{acl}(a, D).$$

Using proposition 2.9, we may now obtain the following theorem by a straightforward application of the group configuration theorem in T_0.

Theorem 2.10. *Let T be a stable theory together with a stable reduct T_0 which has geometric elimination of imaginaries. Every type-definable connected group G in T can be mapped via a type-definable homomorphism ϕ to a T_0-interpretable group H such that for any two generic independent elements g, g' in G we have that $\mathrm{acl}(g), \mathrm{acl}(g') \underset{\phi(g \cdot g')}{\downarrow^0} \mathrm{acl}(gg')$ and $\phi(g \cdot g')$ is 0-interalgebraic with*

$$\mathrm{acl}_0(\mathrm{acl}(g), \mathrm{acl}(g')) \cap \mathrm{acl}(gg').$$

Again, in general H could be trivial. Under the extra assumption that T is 1-based over T_0, we may show that the kernel is finite.

Theorem 2.11. *Let T be a stable theory together with a stable reduct T_0 which has geometric elimination of imaginaries. If T is 1-based over T_0 with respect to a closure operator satisfying (†) and (‡). then every type-definable connected group G in T is allows a definable homomorphism with finite kernel to a T_0-interpretable group H.*

Recall that in the classical setting, CM-trivial groups of finite Morley rank are nilpotent-by-finite since they do not interpret neither infinite fields nor bad groups [16]. We can now prove the following result in the relative CM-trivial case.

Theorem 2.12. *Let T be a stable theory together with a stable reduct T_0 which has geometric elimination of imaginaries. If T is CM-trivial over T_0 with respect to a closure operator satisfying (†) and (‡), then every connected type-definable group G in T can be mapped via a type-definable homomorphism ϕ into a T_0-interpretable group H such that the kernel of ϕ is contained (up to finite index) in the center $Z(G)$ of G.*

Therefore, we obtain the following corollaries.

Corollary 2.13. *Under the same hypotheses as above, every simple group G type-definable in T embeds into a T_0-interpretable group.*

Using the previous corollary, given a field K one can embed $PSL_2(K)$ (which is a simple group) into a T_0-interpretable group; it can be shown that the subgroup $K^+ \rtimes K^\times$ embeds into a T_0-definable group $L^+ \rtimes L^\times$, where L is a T_0-interpretable field. Hence we conclude the following.

Corollary 2.14. *Under the same hypotheses as above, every field K type-definable in T is definably isomorphic to a subfield of a T_0-interpretable one. Moreover, if T has finite Lascar rank, then K is definably isomorphic to a T_0-interpretable field.*

Remark 2.15. Without assuming condition (‡), one can still prove Theorem 2.12 if G is non-abelian. Hence, both corollaries hold without the condition (‡).

Using Poizat(s results on bad linear groups [23], we obtain the following.

Corollary 2.16. *In a coloured field, every infinite simple definable group is linear. No bad field can be defined in a red field. If a green field defines a bad group, then the group consists only of semisimple elements.*

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