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Comparing Expressiveness of First-Order Modal μ-calculus and First-Order CTL*

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Abstract

In this paper we introduce a first-order extension of propositional modal μ-calculus (first-order modal μ-calculus) and a first-order extension of CTL* (first-order CTL*), and then compare their expressiveness. More precisely we show that first-order CTL* is strictly less expressive than first-order modal μ-calculus. It is known that CTL* is strictly less expressive than propositional modal μ-calculus, hence our result shows that our two first-order extensions of CTL* and PMμ preserve an expressiveness relation.

1 Introduction

Formal verification is an important research area in computer science, and many people verify safety critical systems with formal verification. In formal verification, 1) a verification target is formalized as a theory and a model of a logic, 2) a verification item is formalized as a formula of the logic, 3) and then a verifier checks whether the formalized verification target satisfies the formalized verification item with theorem proving and model checking, which is based on the logic. [Propositional] temporal logics[3, 9] and propositional modal μ-calculus PMμ[7] have been often used for this purpose. As verification targets and verification items are becoming complicated, extensions, in particular first-order extensions, of propositional temporal logics have been introduced [2, 4, 5, 10]. The author and Kashima introduced a first-order extension of PMμ in [6, 8].

There are many propositional temporal logics, linear temporal logic LTL, computation tree logic CTL and CTL* etc., thus their expressiveness results have been proved. For example, it is known that CTL is strictly less expressive than CTL* (see [1]), and that CTL* is strictly less expressive than PMμ (see [3]). PMμ is a yardstick for comparing expressiveness of propositional temporal logics. Then the author thinks that our first-order extension of PMμ can be also a yardstick for comparing expressiveness of first-order extensions of propositional temporal logics.

In this paper we compare expressiveness of a first-order extension of PMμ and a first-order extension of CTL*, more precisely show that the first-order extension of CTL* is strictly less expressive than the first-order extension of PMμ.

2 First-Order Modal μ-calculus

In this section, we define a first-order extension of propositional modal μ-calculus and call the logic *first-order modal μ-calculus* (FOMμ).

2.1 Syntax of FOMμ

Definition 1 *A* signature σ *for first-order modal μ-calculus consists of two infinite sets IVar and PVar, and a set $Pred_n$ for each natural number $n \geq 0$.*

We call elements of *IVar individual variables* and write $x, y, \ldots$ for them, and also call those of *PVar propositional variables* and write $X, Y, \ldots$ for them. We call elements of $\mathcal{P}red_n$ n-ary *predicate symbols* and write $P, Q, \ldots$ for them.

Definition 2 *Let σ be a signature of FOMμ. Then the σ-formulas of FOMμ are the following:*

1. $X \in PVar \Rightarrow X$ *is a σ-formula,*
2. $P \in \mathcal{P}red_n$ *and* $\bar{x} \in IVar \Rightarrow P(\bar{x})$ *is a σ-formula,*
3. φ *and* ψ *are σ-formulas* $\Rightarrow \neg\varphi, \varphi \vee \psi, \Box\varphi$ *are σ-formulas.*
4. $x \in IVar$ *and* φ *is a σ-formula* $\Rightarrow \forall x. \varphi$ *is a σ-formula,*
5. *If $X \in PVar$, φ is a σ-formula and no free occurrence of X in φ is negative* $\Rightarrow \mu X. \varphi$ *is a σ-formula.*

We use usual abbreviations $\varphi \wedge \psi, \varphi \supset \psi, \exists x. \varphi$ and two abbreviations

- $\Diamond\varphi \stackrel{\text{def}}{=} \neg\Box\neg\varphi$,
- $\nu X. \varphi \stackrel{\text{def}}{=} \neg\mu X. \neg\varphi[(\neg X)/X]$

where $\varphi[(\neg X)/X]$ denotes the syntactic substitution of a formula $\neg X$ for free occurrences of a propositional variable X in φ.

2.2 Semantics of FOMμ

We define a structure of FOMμ, and it is the same as structure of first-order modal logic.

Definition 3 *Let σ be a signature of FOMμ. A σ-*structure *is the quadruple $\langle S, R, D, I\rangle$ where*

1. *S is a non-empty set (of* states*),*
2. *D is a non-empty set,*
3. *R is a binary relation on S and*

4. *I is a function of the type $\mathcal{P}red_n \times S \to \wp(D^n)$ for every $n \in \mathbb{N}$.*

A structure $\langle S, R, D, I\rangle$ of FOMμ is an extension of a Kripke frame $\langle S, R\rangle$ of propositional modal μ-calculus, in which a state s represents a first-order structure $\langle D, I(-, s)\rangle$.

FOMμ has two kinds of variables, namely individual variables and propositional variables, hence we define a valuation of FOMμ as the mixture of those of first-order logic and modal μ-calculus.

Definition 4 *Let $\mathcal{A} = \langle S, R, D, I\rangle$ be a σ-structure. A* valuation *in $\mathcal{A}$ is a function V which takes an element $x \in IVar$ to an element of D, and an element $X \in PVar$ to an element of $\wp(S)$.*

In the sequel, we fix a σ-structure $\mathcal{A} = \langle S, R, D, I\rangle$ and a valuation V in $\mathcal{A}$ unless stated otherwise.

Now we prepare functions to define denotations of formulas. Let $x, y \in IVar$, $d \in D$, $X, Y \in PVar$ and $T \in \wp(S)$ of S. Then we define functions $V[d/x]$ and $V[T/X]$ as follows.

$$V[d/x](Y) = V(Y), \quad V[d/x](y) = \begin{cases} d, & \text{if } y = x \\ V(y), & \text{if } y \neq x \end{cases}$$

$$V[T/X](y) = V(y), \quad V[T/X](Y) = \begin{cases} T, & \text{if } Y = X \\ V(Y), & \text{if } Y \neq X \end{cases}$$

Finally we define denotations of formulas. In modal μ-calculus, the denotation of a formula is defined as the set of states at which the formula is true. Therefore we extend that idea to FOMμ.

Definition 5 *Let φ be a formula of FOMμ, then we define the* denotation *$[\![\varphi]\!]^{\mathcal{A}}_V$ $(\in \wp(S))$ of φ as follows:*

1. $[\![X]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} V(X)$ *for* $X \in PVar$,
2. $[\![P(\bar{x})]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} \{\, s \in S \mid V(\bar{x}) \in I(P, s) \,\}$. *where* $P \in \mathcal{P}red_n$ *and* $\bar{x} \in IVar$,
3. $[\![\neg\varphi]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} S \setminus [\![\varphi]\!]^{\mathcal{A}}_V$,
4. $[\![\varphi \vee \psi]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} [\![\varphi]\!]^{\mathcal{A}}_V \cup [\![\psi]\!]^{\mathcal{A}}_V$,
5. $[\![\forall x.\varphi]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} \bigcap_{d \in D} [\![\varphi]\!]^{\mathcal{A}}_{V[d/x]}$,
6. $[\![\Box\varphi]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} \{\, s \in S \mid \forall t \in S.\ R(s,t) \Rightarrow t \in [\![\varphi]\!]^{\mathcal{A}}_V \,\}$,
7. $[\![\mu X.\varphi]\!]^{\mathcal{A}}_V \stackrel{\text{def}}{=} \bigcap \{\, T \subseteq S \mid [\![\varphi]\!]^{\mathcal{A}}_{V[T/X]} \subseteq T \,\}$.

Let $\mathcal{A} = \langle S, R, D, I \rangle$ be a structure, (v, V) a valuation in $\mathcal{A}$ and $s \in S$. Then we define a satisfaction relation $\Vdash$ as follows:

$$\mathcal{A}, V, s \Vdash \varphi \overset{\text{def}}{\iff} s \in [\![\varphi]\!]^{\mathcal{A}}_V$$

Remark 6 *Let φ be a formula, $\mathcal{A}$ a structure and V a valuation in $\mathcal{A}$.*

1. *By the definition of $[\![\mu X.\varphi]\!]^{\mathcal{A}}_V$, the denotation $[\![\mu X.\varphi]\!]^{\mathcal{A}}_V$ is the* lease fixed-point *of the function which takes an element T of $\wp(S)$ to $[\![\varphi]\!]^{\mathcal{A}}_{V[T/X]}$.*
2. *By the definition of the formula $\nu X.\varphi$,*

$$[\![\nu X.\varphi]\!]^{\mathcal{A}}_V = \bigcup\{T \subseteq S \mid [\![\varphi]\!]^{\mathcal{A}}_{V[T/X]} \supseteq T\},$$

hence $[\![\nu X.\varphi]\!]^{\mathcal{A}}_V$ is the greatest fixed-point *of the function which takes an element T of $\wp(S)$ to $[\![\varphi]\!]^{\mathcal{A}}_{V[T/X]}$.*

3 First-Order CTL*

In this section, we define a logic FOCTL* which is a first-order extension of CTL*, i.e., we define syntax and semantics of FOCTL*.

3.1 Syntax of FOCTL*

A signature of FOCTL* is the same as that of FOMμ, and let σ be a signature of FOCTL*.

Definition 7 *We mutually define the* state σ-formulas *and the* path σ-formulas *of FOCTL* as follows:*

1. *$X \in PVar \Rightarrow X$ is a state σ-formula,*
2. *$P \in \mathcal{P}red_n$ and $\bar{x} \in IVar \Rightarrow P(\bar{x})$ is a state σ-formula,*
3. *φ, φ' are state σ-formulas $\Rightarrow \neg\varphi$, $\varphi \vee \varphi'$ are state σ-formulas,*
4. *φ is a state σ-formula and $x \in IVar \Rightarrow \forall x.\varphi$ is a state σ-formula,*
5. *ψ is a path σ-formula $\Rightarrow \mathbf{E}\psi$ is a state σ-formula,*
6. *Every state σ-formula is a path σ-formula,*
7. *ψ. ψ' are path σ-formulas $\Rightarrow \neg\psi$ and $\psi \vee \psi'$ are path σ-formulas.*
8. *ψ is a path σ-formula and $x \in IVar \Rightarrow \forall x.\psi$ is a path σ-formula.*
9. *ψ, ψ' are path σ-formulas $\Rightarrow \mathbf{X}\psi$ and $\psi\mathbf{U}\psi'$ are path σ-formulas.*

We use an abbreviation $\mathbf{A}\psi \overset{\text{def}}{=} \neg\mathbf{E}\neg\psi$ for a path formula ψ. When a signature σ is not important or clear from context, we often write a state (path) formula for state (respectively path) σ-formula. We say that θ is a formula of FOCTL* if it is a state formula or a path formula of FOCTL*.

3.2 Semantics of FOCTL*

In this subsection we define semantics of FOCTL*, i.e., we define structures and a satisfaction relation for FOCTL*. A definition of structure of FOCTL* is the same as that of FOMμ. But a definition of the satisfaction relation is different from that of FOMμ.

Definition 8 *1. A* structure *of FOCTL* is defined in the same way as FOMμ. namely it is a quadruple $\langle S, R, D, I\rangle$ satisfying the conditions in Definition 3.*

2. A valuation *of FOCTL* is defined in the same way as FOMμ, namely it is a function V satisfying the conditions in Definition 4.*

3. For a structure $\langle S, R, D, I\rangle$ of FOCTL, a* full path *in S is a maximal sequence $s_0, s_1 \cdots$, of elements of S such that $(s_i, s_{i+1}) \in R$.*

For a full path $\pi = s_0, s_1, \cdots$ and a natural number i, we write π^i for $s_i, s_{i+1}, \cdots$.

Since there are two kinds of formulas in FOCTL*, namely state formulas and path formulas, a definition of a satisfaction relation for state formulas is different from that for path formulas. Thus we mutually define the satisfaction relation $\Vdash$ for state formulas and path formulas.

Definition 9 *Let $\mathcal{A} = \langle S, R, D, I\rangle$ be a structure of FOCTL*, V a valuation in $\mathcal{A}$, $s \in S$ and π a full path in S. Let X be a propositional variable, P an n-ary predicate symbol and $\bar{x}$ individual variables. Let φ, φ' be state formulas, and ψ, ψ' path formulas. Put Π is the set of full paths in S and $\pi(0)$ denotes the first element of π. Then we define a satisfaction relation $\Vdash$ as follows:*

1. $\mathcal{A}, V, s \Vdash X \overset{\text{def}}{\iff} s \in V(X)$,
2. $\mathcal{A}, V, s \Vdash P(\bar{x}) \overset{\text{def}}{\iff} V(\bar{x}) \in I(P, s)$,
3. $\mathcal{A}, V, s \Vdash \neg\varphi \overset{\text{def}}{\iff} \mathcal{A}, V, s \Vdash \varphi$ *does not hold,*
4. $\mathcal{A}, V, s \Vdash \varphi \vee \varphi' \overset{\text{def}}{\iff} \mathcal{A}, V, s \Vdash \varphi$ *or* $\mathcal{A}, V, s \Vdash \varphi'$,
5. $\mathcal{A}, V, s \Vdash \forall x.\varphi \overset{\text{def}}{\iff} \mathcal{A}, V[d/x], s \Vdash \varphi$ *for any* $d \in D$,
6. $\mathcal{A}, V, s \Vdash \mathbf{E}\psi \overset{\text{def}}{\iff} \exists\pi \in \Pi$ *such that* $\pi(0) = s$ *and* $\mathcal{A}, V, \pi \Vdash \psi$,
7. $\mathcal{A}, V, \pi \Vdash \psi \overset{\text{def}}{\iff} \mathcal{A}, V, \pi(0) \Vdash \psi$ *for a state formula* ψ,
8. $\mathcal{A}, V, \pi \Vdash \neg\psi \overset{\text{def}}{\iff} \mathcal{A}, V, \pi \Vdash \psi$ *does not hold,*
9. $\mathcal{A}, V, \pi \Vdash \psi \vee \psi' \overset{\text{def}}{\iff} \mathcal{A}, V, \pi \Vdash \psi$ *or* $\mathcal{A}, V, \pi \Vdash \psi'$.
10. $\mathcal{A}, V, \pi \Vdash \forall x.\psi \overset{\text{def}}{\iff} \mathcal{A}, V[d/x], \pi \Vdash \psi$ *for any* $d \in D$,

11. $\mathcal{A}, V, \pi \Vdash \mathbf{X}\psi \stackrel{\text{def}}{\Longleftrightarrow} \mathcal{A}, V, \pi^1 \Vdash \psi$,

12. $\mathcal{A}, V, \pi \Vdash \psi \mathbf{U} \psi' \stackrel{\text{def}}{\Longleftrightarrow} \exists j \geq 0. \mathcal{A}, V, \pi^j \Vdash \psi'$ *and* $0 \leq \forall k < j. \mathcal{A}, V, \pi^k \Vdash \psi$

4 Comparison of Expressiveness of FOMμ and FOCTL*

In this section we compare expressiveness of FOMμ and FOCTL*. More precisely we show that FOCTL* is strictly less expressive than FOMμ. On the other hand FOMμ (FOCTL*) is a first-order extension of PMμ (respectively CTL*), and it is known that CTL* is strictly less expressive than PMμ (See Theorem 4.1.4 in [3] which is a linear temporal version of the statement.) Thus FOMμ and FOCTL* preserve a expressiveness relation.

4.1 Definitions of Comparing Expressiveness

For a signature σ of FOCTL*, hence of FOMμ, FOCTL*$[\sigma]$ (FOM$\mu[\sigma]$) denotes the set of σ-formulas of FOCTL* (respectively FOMμ).

Definition 10 *Let σ be a signature.*

1. *We say that a state formula φ of FOCTL*$[\sigma]$* can be expressed in *the set FOM$\mu[\sigma]$ if there is a formula θ of FOM$\mu[\sigma]$ such that $\mathcal{A}', V', s' \Vdash \varphi \Leftrightarrow \mathcal{A}', V', s' \Vdash \theta$ for any structure $\mathcal{A} = \langle S', R', D', I' \rangle$, valuation V' in $\mathcal{A}'$ and an element $s' \in S'$.*

2. *We say that a path formula ψ of FOCTL*$[\sigma]$* can be expressed in *the set FOM$\mu[\sigma]$ if there is a formula θ of FOM$\mu[\sigma]$ such that $\mathcal{A}', V', s' \Vdash \mathbf{A}\psi \Leftrightarrow \mathcal{A}', V', s' \Vdash \theta$ for any structure $\mathcal{A} = \langle S', R', D', I' \rangle$, valuation V' in $\mathcal{A}'$ and an element $s' \in S'$.*

3. *We say that the set FOCTL*$[\sigma]$ is* less expressive than *FOM$\mu[\sigma]$ if any formula of FOCTL*$[\sigma]$ can be expressed in FOM$\mu[\sigma]$. When FOCTL*$[\sigma]$ is less expressive than FOM$\mu[\sigma]$, we write FOCTL*$[\sigma] \leq$ FOM$\mu[\sigma]$.*

4. *We say that FOCTL* is* less expressive than *FOMμ if FOCTL*$[\sigma] \leq$ FOM$\mu[\sigma]$ for any signature σ. When FOCTL* is less expressive than FOMμ, we write FOCTL* $\leq$ FOMμ.*

Proposition 11 *FOCTL* is less expressive than FOMμ.*

Proof We can prove the proposition in a similar way to the fact that CTL* is less expressive than PMμ. (See pp. 367 in [3] for example.) □

Definition 12 *Let σ be a signature.*

1. *We say that a formula θ of the set FOM$\mu[\sigma]$* cannot be expressed in *the set FOCTL*$^*[\sigma]$ *if, for each state formula φ of FOCTL*$^*[\sigma]$*, there is a structure $\mathcal{A}' = \langle S', R', D', I' \rangle$, a valuation V' in $\mathcal{A}'$ and an element $s' \in S'$ such that $\mathcal{A}', V', s' \Vdash \theta \not\Leftrightarrow \mathcal{A}', V', s' \Vdash \varphi$.*

2. *We say that FOCTL*$^*[\sigma]$ *is* strictly less expressive than *FOM$\mu[\sigma]$ (or FOM$\mu[\sigma]$ is* strictly more expressive than *FOCTL*$^*[\sigma]$*) if FOCTL*$^*[\sigma] \leq$ *FOM$\mu[\sigma]$, and there is a formula φ of FOM$\mu[\sigma]$ which cannot be expressed in FOCTL***. We write FOCTL*$^*[\sigma] \lneq$ *FOM$\mu[\sigma]$ when FOCTL*$^*[\sigma]$ *is strictly less expressive than FOM$\mu[\sigma]$.*

3. *We say that FOCTL** *is* strictly less expressive than *FOMμ (or FOMμ is* strictly more expressive than *FOCTL***) if FOCTL** $\leq$ *FOMμ, and FOCTL*$^*[\sigma] \lneq$ *FOM$\mu[\sigma]$ for some signature σ. We write FOCTL** $\lneq$ *FOMμ when FOCTL** *is strictly less expressive than FOMμ.*

4.2 Expressiveness of FOMμ

In this subsection we give an expressiveness result of FOMμ, in particular that of a formula $\nu X.\varphi \wedge \Box\Box X$. In Lemma 14, we introduce a structure $\mathcal{A}_k$ so that we will compare expressiveness of FOMμ and FOCTL* in the next subsection.

Lemma 13 *Let $R = \{(i, i+1) \mid i \in \mathbb{N}\}$, D be a non-empty set, I an interpretation. Let V be a valuation in the structure $\mathcal{A} = \langle \mathbb{N}, R, D, I \rangle$ and φ a formula which does not contain a free propositional variable X. Then*

$$[\![\nu X.\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_V = \{\, i \in \mathbb{N} \mid \mathcal{A}, V, i + 2k \Vdash \varphi \text{ for any } k \in \mathbb{N}\,\}.$$

Proof ($\subseteq$) We show that $i \in [\![\nu X.\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_V \Rightarrow i + 2k \in [\![\varphi]\!]^{\mathcal{A}}_V$ for all $k \in \mathbb{N}$.

Put $[\![\nu X.\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_V = M$. Since M is the [greatest] fixed-point of the function which takes an element $T \in \wp(\mathbb{N})$ to the element $[\![\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_{V[T/X]}$, We have the following equality:

$$M = [\![\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_{V[M/X]} \; (= [\![\varphi]\!]^{\mathcal{A}}_{V[M/X]} \cap [\![\Box\Box X]\!]^{\mathcal{A}}_{V[M/X]}). \quad (*)$$

Assume that $i \in M$. Then $i \in [\![\varphi]\!]^{\mathcal{A}}_{V[M/X]}$ by (*), hence $i \in [\![\varphi]\!]^{\mathcal{A}}_V$. Again by (*), $i \in [\![\Box\Box X]\!]^{\mathcal{A}}_{V[M/X]}$, this implies that $i + 2 \in [\![X]\!]^{\mathcal{A}}_{V[M/X]}$. Thus we have that $i + 2 \in M$. By repeating this argument, we have that $i + 2k \in M$ (in particular $i + 2k \in [\![\varphi]\!]^{\mathcal{A}}_V$) for any $k \in \mathbb{N}$.

($\supseteq$) Put $M' = \{\, i \in \mathbb{N} \mid \mathcal{A}, V, i + 2k \Vdash \varphi$ for all $k \in \mathbb{N}\}$. Since $[\![\nu X.\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_V$ is the [greatest] fixed-point of the function which takes an element $T \in \wp(\mathbb{N})$ to the element $[\![\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_{V[T/X]}$, it is enough to show that $M' \subseteq [\![\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_{V[M'/X]}$.

Assume that $i \in M'$. Then $i + 2k \in [\![\varphi]\!]^{\mathcal{A}}_V$ for any $k \in \mathbb{N}$, in particular $i \in [\![\varphi]\!]^{\mathcal{A}}_V$ when $k = 0$. Hence $i \in [\![\varphi]\!]^{\mathcal{A}}_{V[M'/X]}$. On the other hand,

$$i \in M' \Rightarrow i + 2 \in M' \Leftrightarrow i \in [\![\Box\Box X]\!]^{\mathcal{A}}_{V[M'/X]}.$$

Thus $i \in [\![\varphi \wedge \Box\Box X]\!]^{\mathcal{A}}_{V[M'/X]}$ $(= [\![\varphi]\!]^{\mathcal{A}}_{V[M'/X]} \cap [\![\Box\Box X]\!]^{\mathcal{A}}_{V[M'/X]})$.

□

In the following, we consider a signature which consists of a single unary predicate symbol P.

Lemma 14 *Let $R = \{(i, i+1) \mid i \in \mathbb{N}\}$ and D a non-empty set, and let I_k be an interpretation such that*

$$I_k(P, i) = \begin{cases} \emptyset, & \text{if } k = i, \\ D, & \text{otherwise.} \end{cases}$$

Put $\mathcal{A}_k = \langle \mathbb{N}, R, D, I_k \rangle$ and let V be a valuation in $\mathcal{A}$ such that $V(X) = \mathbb{N}$ for any propositional variable X. Then $\mathcal{A}_k, V, 0 \Vdash \nu X. P(x) \wedge \Box\Box X \Leftrightarrow k$ is odd.

Proof

$$\begin{aligned} & \mathcal{A}_k, V, 0 \Vdash \nu X. P(x) \wedge \Box\Box X \\ \Leftrightarrow\ & 0 \in [\![\nu X. P(x) \wedge \Box\Box X]\!]^{\mathcal{A}_k}_V \\ \Leftrightarrow\ & 0 \in \{ i \in \mathbb{N} \mid \mathcal{A}_k, V, i + 2j \Vdash P(x) \text{ for any } \in \mathbb{N}\} \quad \text{(Lemma13)} \\ \Leftrightarrow\ & k \text{ is odd} \end{aligned}$$

□

We remark that a valuation in a structure $\mathcal{A}_k$ can be a valuation in a structure $\mathcal{A}_l$ for any natural numbers k, l, hence a valuation in $\mathcal{A}_k$'s makes sense.

4.3 FOCTL* is strictly less expressive than FOMμ

In this subsection we show that FOCTL* is strictly less expressive than FOMμ. Similarly it is known that CTL* is strictly less expressive than PMμ.

Our proof of Theorem 18 is a first-order (and branching) extension of Theorem 4.1.4 in [3], hence we must also consider formulas of the form $\forall x.\varphi$ and a valuation V. This requires modification of induction statements in proofs, thus we introduce a binary relation $\sim$ on the set of valuations in a structure.

Definition 15 *Let $\langle S', R', D', I' \rangle$ be a structure and $\mathcal{V}$ the set of valuations in it and $V, V' \in \mathcal{V}$. We define a binary relation $\sim$ on $\mathcal{V}$ as follows: $V' \sim V''$ if there is a finite subset $IVar_0$ of $IVar$ such that*

1. *$V'(x) = V''(x)$ for any $x \in IVar \setminus IVar_0$ and*
2. *$V'(X) = V''(X)$ for any $X \in PVar$.*

Lemma 16 *Let $\mathcal{A}_j = \langle \mathbb{N}, R, D, I_j \rangle$ be the structure for each natural number j and the valuation V defined in Proposition 14.*

1. *For any state formula φ of FOCTL*$[\{P\}]$, natural numbers $k, i \geq 0$ and valuation $V' \sim V$, $\mathcal{A}_k, V', i \Vdash \varphi \Leftrightarrow \mathcal{A}_{k+1}, V', i+1 \Vdash \varphi$.*

2. *For any path formula ψ of FOCTL*$[\{P\}]$, natural numbers $k, i \geq 0$ and valuation $V' \sim V$, $\mathcal{A}_k, V', \pi^i \Vdash \psi \Leftrightarrow \mathcal{A}_{k+1}, V', \pi^{i+1} \Vdash \psi$ where $\pi = 0, 1, 2, \ldots$.*

Proof Let φ be a state formula and ψ a path formula. We prove the lemma by mutual induction on the construction of φ and ψ. Let V' be a valuation with $V' \sim V$.

Case 1 $\varphi \equiv X \in PVar$: By the definition of V', $\mathcal{A}_k, V', i \Vdash X$ for any $k, i \in \mathbb{N}$. In particular $\mathcal{A}_k, V', i \Vdash X \Leftrightarrow \mathcal{A}_{k+1}, V', i+1 \Vdash X$.

Case 2 $\varphi \equiv P(x)$: (Recall that P is the unique predicate symbol.) By the definition of I_k, $\mathcal{A}_k, V', i \Vdash P(x) \Leftrightarrow k \neq i$ for any $k, i \in \mathbb{N}$. Then the following holds for any $k, i \in \mathbb{N}$.

$$\mathcal{A}_k, V', i \Vdash P(x) \Leftrightarrow k \neq i \Leftrightarrow k+1 \neq i+1 \Leftrightarrow \mathcal{A}_{k+1}, V', i+1 \Vdash P(x)$$

Case 3 $\varphi \equiv \neg\varphi'$ or $\varphi' \vee \varphi''$ for some state formulas φ' and φ'': We skip proofs for these cases.

Case 4 $\varphi \equiv \forall x.\varphi'$ for some state formula φ':

$$\begin{array}{ll} & \mathcal{A}_k, V', i \Vdash \forall x.\varphi' \\ \Leftrightarrow & \mathcal{A}_k, V'[d/x], i \Vdash \varphi' \quad \text{for any } d \in D \\ \Leftrightarrow & \mathcal{A}_{k+1}, V'[d/x], i+1 \Vdash \varphi' \quad \text{for any } d \in D \quad (V'[d/x] \sim V, \text{IH}) \\ \Leftrightarrow & \mathcal{A}_{k+1}, V', i+1 \Vdash \forall x.\varphi' \end{array}$$

Since the number of the logical symbol $\forall$ in φ is finite, valuations occurring in the induction must be the form $V'[d_1/x_1]...[d_n/x_n]\,(\sim V)$ for some elements $d_1, \cdots, d_n$ of D and individual variables $x_1, \cdots, x_n$. Thus our induction works.

Case 5 $\varphi \equiv \mathbf{E}\psi$ for some path formula ψ:

$$\begin{array}{rcl} \mathcal{A}_k, V', i \Vdash \mathbf{E}\psi' & \Leftrightarrow & \mathcal{A}_k, V', \pi^i \Vdash \psi' \\ & \Leftrightarrow & \mathcal{A}_{k+1}, V', \pi^{i+1} \Vdash \psi' \quad \text{(IH)} \\ & \Leftrightarrow & \mathcal{A}_{k+1}, V', i+1 \Vdash \mathbf{E}\psi' \end{array}$$

Case 6 ψ is a path formula which is also a state formula: Proofs for these cases are similar to those of the cases 1, 2, 3, 4 and 5.

Case 7 $\psi \equiv \neg\psi'$ or $\psi' \vee \psi''$ for some path formulas ψ' and ψ'': We skip proofs for these cases.

Case 8 $\psi \equiv \forall x.\psi'$ for some path formula ψ': A proof for this case is similar to that of the case 4.

Case 9 $\psi \equiv \mathbf{X}\psi'$ for some path formula ψ':

$$\begin{array}{rcl} \mathcal{A}_k, V', \pi^i \Vdash \mathbf{X}\psi' & \Leftrightarrow & \mathcal{A}_k, V', \pi^{i+1} \Vdash \psi' \\ & \Leftrightarrow & \mathcal{A}_{k+1}, V', \pi^{i+2} \Vdash \psi' \quad \text{(IH)} \\ & \Leftrightarrow & \mathcal{A}_{k+1}, V', \pi^{i+1} \Vdash \mathbf{X}\psi' \end{array}$$

Case 10 $\psi \equiv \psi_1 \mathbf{U} \psi_2$ for some path formulas ψ_1, ψ_2:

$$\begin{array}{ll} & \mathcal{A}_k, V', \pi^i \Vdash \psi_1 \mathbf{U} \psi_2 \\ \Leftrightarrow & \exists i' > i.[\mathcal{A}_k, V', \pi^{i'} \Vdash \psi_2 \,\&\, (i < \forall j < i'.\mathcal{A}_k, V', \pi^j \Vdash \psi_1)] \\ \Leftrightarrow & \exists i' > i.[\mathcal{A}_{k+1}, V', \pi^{i'+1} \Vdash \psi_2 \,\&\, (i < \forall j < i'.\mathcal{A}_{k+1}, V', \pi^{j+1} \Vdash \psi_1)] \quad \text{(IH)} \\ \Leftrightarrow & \exists i'' > i+1.[\mathcal{A}_{k+1}, V', \pi^{i''} \Vdash \psi_2 \,\&\, (i+1 < \forall j < i''.\mathcal{A}_{k+1}, V', \pi^j \Vdash \psi_1)] \\ \Leftrightarrow & \mathcal{A}_{k+1}, V', \pi^{i+1} \Vdash \psi_1 \mathbf{U} \psi_2 \end{array}$$

□

Lemma 17 shows limitations of expressiveness of FOCTL* with respect to the structure $\mathcal{A}_k$.

Lemma 17 *Let $\mathcal{A}_j = \langle \mathbb{N}, R, D, I_j \rangle$ be the structure for each natural number j and the valuation V defined in Proposition 14.*

1. *For every state formula φ of FOCTL*$[\{P\}]$, there is a natural number l such that $\mathcal{A}_k, V', 0 \Vdash \varphi \Leftrightarrow \mathcal{A}_l, V', 0 \Vdash \varphi$ for any natural number $k \geq l$ and valuation $V' \sim V$.*
2. *We write $\pi^i = i, i+1, i+2, \cdots$ for the path $\pi = 0, 1, 2, \cdots$. For every path formula ψ of FOCTL*$[\{P\}]$, there is a natural number l such that $\mathcal{A}_k, V', \pi^0 \Vdash \psi \Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \psi$ for any natural number $k \geq l$ and valuation $V' \sim V$.*

Proof We prove the lemma by mutual induction on the construction of the formula φ and ψ.

Case 1 $\varphi \equiv X$ for some propositional variable X: By the definition of V', we have that $\mathcal{A}_k, V', 0 \Vdash X$ for any $k \geq 0$. Thus put $l = 0$.

Case 2 $\varphi \equiv P(x)$ for some individual variable x: (Recall that P is the unique predicate symbol.) By the definition of I_k, we have that $\mathcal{A}_k, V', 0 \Vdash P(x) \Leftrightarrow k \neq 0$. Thus put $l = 1$.

Case 3 $\varphi \equiv \neg\varphi'$ or $\varphi' \vee \varphi''$ for some state formulas φ' and φ'': We skip proofs for these cases.

Case 4 $\varphi \equiv \forall x.\varphi'$ for some state formula φ': By induction hypothesis, for the formula φ', there is a natural number $l_{\varphi'}$ such that $\mathcal{A}_k, V', 0 \Vdash \varphi' \Leftrightarrow \mathcal{A}_{l_{\varphi'}}, V', 0 \Vdash \varphi'$ for any valuation $V' \sim V$ and natural number $k \geq l_{\varphi'}$. Put $l = l_{\varphi'}$ and $k' \geq l$.

$$\begin{array}{ll} & \mathcal{A}_{k'}, V', 0 \Vdash \forall x.\varphi' \\ \Leftrightarrow & \mathcal{A}_{k'}, V'[d/x], 0 \Vdash \varphi' \text{ for all } d \in D \\ \Leftrightarrow & \mathcal{A}_l, V'[d/x], 0 \Vdash \varphi' \text{ for all } d \in D \quad (k', l \geq l_{\varphi'}, V'[d/x] \sim V) \\ \Leftrightarrow & \mathcal{A}_l, V', 0 \Vdash \forall x.\varphi' \end{array}$$

Case 5 $\varphi \equiv \mathbf{E}\psi$ for some path formula ψ: By induction hypothesis, for the path formula ψ, there is a natural number l_ψ such that $\mathcal{A}_k, V', \pi^0 \Vdash \psi \Leftrightarrow$

$\mathcal{A}_{l_\psi}, V', \pi^0 \Vdash \psi$ for any valuation $V' \sim V$ and natural number $k \geq l_\psi$. Put $l = l_\psi$ and $k' \geq l$.

$$\begin{aligned} \mathcal{A}_{k'}, V', 0 \Vdash \mathbf{E}\psi &\Leftrightarrow \mathcal{A}_{k'}, V', \pi^0 \Vdash \psi \\ &\Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \psi \quad (k', l \geq l_\psi) \\ &\Leftrightarrow \mathcal{A}_l, V', 0 \Vdash \mathbf{E}\psi \end{aligned}$$

Case 6 ψ is a path formula which is also a state formula: Proofs for these cases are similar to those of the cases 1, 2, 3, 4 and 5.

Case 7 $\psi \equiv \neg\psi'$ or $\psi' \vee \psi''$ for some path formulas ψ' and ψ'': We skip proofs for these cases.

Case 8 $\psi \equiv \forall x.\psi'$ for some path formula ψ': A proof for this case is similar to that of the case 4.

Case 9 $\psi \equiv \mathbf{X}\psi'$ for some path formula ψ': By induction hypothesis, for the path formula ψ', there is a natural number $l_{\psi'}$ such that $\mathcal{A}_k, V', \pi^0 \Vdash \psi' \Leftrightarrow \mathcal{A}_{l_{\psi'}}, V', \pi^0 \Vdash \psi'$ for any valuation $V' \sim V$ and natural number $k \geq l_{\psi'}$. Put $l = l_{\psi'} + 1$ and $k \geq l$.

$$\begin{aligned} \mathcal{A}_k, V', \pi^0 \Vdash \mathbf{X}\psi' &\Leftrightarrow \mathcal{A}_k, V', \pi^1 \Vdash \psi' \\ &\Leftrightarrow \mathcal{A}_{k-1}, V', \pi^0 \Vdash \psi' \quad (k \geq 1, \text{Lemma } 16) \\ &\Leftrightarrow \mathcal{A}_{l-1}, V', \pi^0 \Vdash \psi' \quad (k-1, l-1 \geq l_{\psi'}) \\ &\Leftrightarrow \mathcal{A}_l, V', \pi^1 \Vdash \psi' \quad (\text{Lemma } 16) \\ &\Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \mathbf{X}\psi' \end{aligned}$$

Case 10 $\psi \equiv \psi_1 \,\mathbf{U}\, \psi_2$ for some path formulas ψ_1, ψ_2: By induction hypothesis, we can choose a natural number l_{ψ_1} (l_{ψ_2}) for ψ_1 (respectively ψ_2) such that, for every path formula ψ of FOCTL*[$\{P\}$], there is a natural number l such that $\mathcal{A}_k, V', \pi^0 \Vdash \psi_1 \Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \psi_1$ (respectively $\mathcal{A}_k, V', \pi^0 \Vdash \psi_2 \Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \psi_2$) for any natural number $k \geq l$ and valuation $V' \sim V$. Similarly, without referring a natural number $l_{\psi_1 \mathbf{U} \psi_2}$, we can choose a natural number $l_{\mathbf{X}\psi_1}$ ($l_{\mathbf{X}\psi_2}$) for the formula $\mathbf{X}\psi_1$ (respectively $\mathbf{X}\psi_2$). Then we put $l = \max\{l_{\mathbf{X}\psi_1}, l_{\mathbf{X}\psi_2}\}$.

By induction on k, we prove that

$$\mathcal{A}_l, V', \pi^0 \Vdash \psi_1 \,\mathbf{U}\, \psi_2 \Leftrightarrow \mathcal{A}_k, V', \pi^0 \Vdash \psi_1 \,\mathbf{U}\, \psi_2 \text{ for any natural number } k \geq l.$$

Base Case: We prove that $\mathcal{A}_l, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \Leftrightarrow \mathcal{A}_{l+1}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2$.

$$
\begin{array}{ll}
 & \mathcal{A}_{l+1}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \\
\Leftrightarrow & \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \\
\Leftrightarrow & \mathcal{A}_{l+1}, V', \pi^0 \Vdash \mathbf{X}\psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \\
\Leftrightarrow & \mathcal{A}_{l}, V', \pi^0 \Vdash \mathbf{X}\psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \quad (l \geq l_{\mathbf{X}\psi_2}) \\
\Leftrightarrow & \mathcal{A}_{l}, V', \pi^1 \Vdash \psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \\
\Rightarrow & \mathcal{A}_{l}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \\
\Rightarrow & \mathcal{A}_{l}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \mathbf{U} \psi_2 \\
\Leftrightarrow & \mathcal{A}_{l}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \text{ or } \mathcal{A}_{l}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \quad \text{(Lemma 16)} \\
\Leftrightarrow & \mathcal{A}_{l}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2
\end{array}
$$

$$
\begin{array}{ll}
 & \mathcal{A}_l, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_l, V', \pi^i \Vdash \psi_2 \text{ and } 0 < \forall j < i.(\mathcal{A}_l, V', \pi^j \Vdash \psi_1)] \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_{l+1}, V', \pi^{i+1} \Vdash \psi_2 \text{ and} \\
 & \qquad 0 < \forall j < i.(\mathcal{A}_l, V', \pi^j \Vdash \psi_1)] \quad \text{(Lemma 16)} \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_{l+1}, V', \pi^{i+1} \Vdash \psi_2 \text{ and} \\
 & \qquad \mathcal{A}_l, V', \pi^1 \Vdash \psi_1 \text{ and } 0 < \forall j < i.(\mathcal{A}_l, V', \pi^j \Vdash \psi_1)] \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_{l+1}, V', \pi^{i+1} \Vdash \psi_2 \text{ and} \\
 & \qquad \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \text{ and } 0 < \forall j < i.(\mathcal{A}_l, V', \pi^j \Vdash \psi_1)] \quad (l \geq l_{\mathbf{X}\psi_1}) \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_{l+1}, V', \pi^{i+1} \Vdash \psi_2 \text{ and} \\
 & \qquad \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \text{ and } 0 < \forall j < i.(\mathcal{A}_{l+1}, V', \pi^{j+1} \Vdash \psi_1)] \quad \text{(Lemma 16)} \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_{l+1}, V', \pi^{i+1} \Vdash \psi_2 \text{ and} \\
 & \qquad \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \text{ and } 1 < \forall j' < i+1.(\mathcal{A}_{l+1}, V', \pi^{j'} \Vdash \psi_1)] \\
\Leftrightarrow & \exists i > 0.[\mathcal{A}_{l+1}, V', \pi^{i+1} \Vdash \psi_2 \text{ and } 0 < \forall j' < i+1.(\mathcal{A}_{l+1}, V', \pi^{j'} \Vdash \psi_1)] \\
\Rightarrow & \mathcal{A}_{l+1}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2
\end{array}
$$

Induction Step: Assume that $\mathcal{A}_{k'}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2$ for any natural number k' with $l \leq k' \leq k$. Then we prove that $\mathcal{A}_{k+1}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \Leftrightarrow \mathcal{A}_l, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2$.

$$
\begin{array}{ll}
 & \mathcal{A}_{k+1}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \\
\Leftrightarrow & \mathcal{A}_{k+1}, V', \pi^1 \Vdash \psi_2 \text{ or } \mathcal{A}_{k+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \\
\Leftrightarrow & \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_2 \text{ or } \mathcal{A}_{k+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \quad (k+1, l+1 \geq l_{\mathbf{X}\psi_2}) \\
\Leftrightarrow & \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_2 \text{ or } \mathcal{A}_{l+1}, V', \pi^1 \Vdash \psi_1 \wedge (\psi_1 \mathbf{U} \psi_2) \\
 & \quad (k+1, l+1 \geq l_{\mathbf{X}\psi_1}, \text{ Lemma 16 and I.H.}) \\
\Leftrightarrow & \mathcal{A}_{l+1}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \\
\Leftrightarrow & \mathcal{A}_{l}, V', \pi^0 \Vdash \psi_1 \mathbf{U} \psi_2 \quad \text{(Base Case)}
\end{array}
$$

□

Theorem 18 *For a unary predicate symbol P, FOCTL*[$\{P\}$] is strictly less expressive than FOMμ[$\{P\}$]. In particular the formula $\nu X. P(x) \wedge \Box\Box X$ of FOMμ[$\{P\}$] cannot be expressed in FOCTL*[$\{P\}$], namely, for each state formula φ of FOCTL*[σ], there is a structure $\mathcal{A}' = \langle S', R', D', I' \rangle$, a valuation V' in $\mathcal{A}'$ and an element $s' \in S'$ such that $\mathcal{A}', V', s' \Vdash \nu X. P(x) \wedge \Box\Box X \not\Leftrightarrow \mathcal{A}', V', s' \Vdash \varphi$.*

Proof Assume that the formula $\nu X. P(x) \wedge \Box\Box X$ can be expressed as a state formula φ of FOCTL*[$\{P\}$], namely that $\mathcal{A}', V', s' \Vdash \varphi \Leftrightarrow \mathcal{A}', V', s' \Vdash \nu X. P(x) \wedge \Box\Box X$ for any structure $\mathcal{A}' = \langle S', R', D', I' \rangle$, valuation V' in $\mathcal{A}'$ and element $s' \in S'$.

Recall that we have introduced the structure $\mathcal{A}_k = \langle \mathbb{N}, R, D, I_k \rangle$ and the valuation V in Lemma 14. By Lemma 17, there is a natural number l such that, for any valuation $V' \sim V$ and natural numbers $k, k' \geq l$,

$$\begin{aligned} \mathcal{A}_k, V', 0 \Vdash \nu X. P(x) \wedge \Box\Box X &\Leftrightarrow \mathcal{A}_k, V', 0 \Vdash \varphi \\ &\Leftrightarrow \mathcal{A}_l, V', 0 \Vdash \varphi \quad (k \geq l, \text{Lemma 17}) \\ &\Leftrightarrow \mathcal{A}_{k'}, V', 0 \Vdash \varphi \quad (k' \geq l, \text{Lemma 17}) \\ &\Leftrightarrow \mathcal{A}_{k'}, V', 0 \Vdash \nu X. P(x) \wedge \Box\Box X \end{aligned}$$

But this contradicts to the fact (Lemma 14) that

$$\mathcal{A}_k, V', 0 \Vdash \nu X. P(x) \wedge \Box\Box X \Leftrightarrow k \text{ is odd.}$$

Thus the formula $\nu X. P(x) \wedge \Box\Box X$ cannot be expressed in FOCTL*[$\{P\}$], hence FOCTL*[$\{P\}$] is strictly less expressive than FOMμ[$\{P\}$]. □

Corollary 19 *FOCTL* is strictly less expressive than FOMμ.*

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