数理解析研究所講究録
第 1709 巻 2010 年 68-88

Orbit algebras of repetitive categories

Andrzej Skowroński
Nicolaus Copernicus University

Abstract

We present results concerning the finite dimensional orbit algebras of repetitive categories of algebras over a field and show their importance for the representation theory of selfinjective algebras.

1 Introduction

In the article by an algebra we mean a finite dimensional algebra over a field K which we shall assume (without loss of generality) to be basic and connected. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules, by $\underline{\operatorname{mod}}\, A$ the stable category of $\operatorname{mod} A$ (modulo projectives), and by Γ_A the Auslander-Reiten quiver of A. Two algebras A and Λ are said to be *stably equivalent* if the stable categories $\underline{\operatorname{mod}}\, A$ and $\underline{\operatorname{mod}}\, \Lambda$ are equivalent. An algebra A is said to be *selfinjective* if A_A is an injective module, or equivalently, the projective modules and injective modules in $\operatorname{mod} A$ coincide.

In the representation theory of selfinjective algebras a prominent role is played by the selfinjective algebras A which admit Galois coverings of the form $\widehat{B} \to \widehat{B}/G = A$, where $\widehat{B}$ is the repetitive category of an algebra B and G is an admissible group of automorphisms of $\widehat{B}$. In this theory, the selfinjective orbit algebras $\widehat{B}/G$ given by triangular algebras B (having finite global dimension) and infinite cyclic groups G are of particular interest. Frequently, important selfinjective algebras are socle deformations of such selfinjective orbit algebras, and we may reduce their representation theory to that for the corresponding algebras of finite global dimension. We also mention that, for an algebra B of finite global dimension, the stable module category $\underline{\operatorname{mod}}\, \widehat{B}$ is equivalent, as a triangular category, to the derived category $D^b(\operatorname{mod} B)$ of bounded complexes over $\operatorname{mod} B$ [Ha].

The author acknowledges support from the research grant No. N N201 269135 of the Polish Ministry of Science and Higher Education and the Research Institute for Mathematical Sciences of the Kyoto University.

2 Selfinjective orbit algebras

Let B be an algebra and $D = \operatorname{Hom}_K(-, K) : \operatorname{mod} B \to \operatorname{Mod} B^{\text{op}}$ the standard duality, where B^{op} is the opposite algebra of B. Moreover, let $1_B = e_1 + \cdots + e_n$ be a decomposition of the identity of B into a sum of pairwise orthogonal primitive idempotents. We associate to B a selfinjective locally bounded K-category $\widehat{B}$, called the *repetitive category* of B. The objects of $\widehat{B}$ are $e_{m,i}$, $m \in \mathbb{Z}$, $i \in \{1, \ldots, n\}$, and the morphism spaces are defined as follows

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_j B e_i & , r = m \\ D(e_i B e_j) & , r = m + 1 \\ 0 & , \text{otherwise} \end{cases} .$$

We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}, \text{ for all } (m, i) \in \mathbb{Z} \times \{1, \ldots, n\}.$$

An automorphism φ of the category $\widehat{B}$ is said to be:

- *positive* if, for each pair $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and $j \in \{1, \ldots, n\}$;
- *rigid* if, for each pair $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$;
- *strictly positive* if it is positive but not rigid.

Observe that the powers $\nu_{\widehat{B}}^r$, $r \geq 1$, of the Nakayama automorphism $\nu_{\widehat{B}}$ are strictly positive automorphisms of $\widehat{B}$.

A group G of automorphisms of the repetitive category $\widehat{B}$ of an algebra B is said to be *admissible* if G acts freely on the objects of $\widehat{B}$ and has finitely many orbits. Following Gabriel [Ga] we may then consider the orbit category $\widehat{B}/G$ defined as follows. The objects of $\widehat{B}/G$ are the G-orbits of objects of $\widehat{B}$, and the morphism spaces are given by

$$\left(\widehat{B}/G\right)(a, b) =$$

$$\left\{ (f_{y,x}) \in \prod_{(x,y) \in a \times b} \widehat{B}(x, y) \;\middle|\; g f_{y,x} = f_{gy,gx} \text{ for all } g \in G, x \in a, y \in b \right\}$$

for all objects a, b of $\widehat{B}/G$. Since there are only finitely many G-orbits of objects in $\widehat{B}$, $\widehat{B}/G$ has finitely many objects, and we may identify $\widehat{B}/G$ with the associated finite dimensional algebra (the direct sum $\bigoplus \left(\widehat{B}/G\right)$ of the morphisms spaces $\left(\widehat{B}/G\right)(a, b)$ for all objects a, b of $\widehat{B}/G$). In fact, $\widehat{B}/G$ is a finite dimensional, basic, connected, selfinjective algebra, called the *orbit algebra* of $\widehat{B}$ with respect to the action of G. We have also the canonical *Galois covering functor* $F : \widehat{B} \to \widehat{B}/G$ which assigns to each object x of $\widehat{B}$ its G-orbit Gx, and induces K-linear isomorphisms

$$\bigoplus_{y \in \operatorname{ob} \widehat{B}, Fy = a} \widehat{B}(x, y) \xrightarrow{\sim} (\widehat{B}/G)(Fx, a),$$

$$\bigoplus_{y \in \operatorname{ob} \widehat{B}, Fy = a} \widehat{B}(y, x) \xrightarrow{\sim} (\widehat{B}/G)(a, Fx).$$

For example, for an algebra B and a positive integer r, the infinite cyclic group $(\nu_{\widehat{B}}^r)$ generated by the r-th power $\nu_{\widehat{B}}^r$ of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ is an admissible group of automorphisms of $\widehat{B}$, and the associated selfinjective orbit algebra $\widehat{B}/(\nu_{\widehat{B}}^r)$ is of the form

$$\mathrm{T}(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r) = \left\{ \begin{bmatrix} b_1 & 0 & 0 & & & & \\ f_2 & b_2 & 0 & & & & \\ 0 & f_3 & b_3 & & & & \\ & & \ddots & \ddots & & & \\ & 0 & & f_{r-1} & b_{r-1} & 0 \\ & & & 0 & f_1 & b_1 \end{bmatrix} \;\middle|\; \begin{matrix} \\ b_1, \dots, b_{r-1} \in B,\ f_1, \dots, f_{r-1} \in D(B) \end{matrix} \right\}$$

called the *r-fold trivial extension algebra* of B. In particular, $\mathrm{T}(B)^{(1)} = \widehat{B}/(\nu_{\widehat{B}}^r)$ is the trivial extension $\mathrm{T}(B) = B \ltimes D(B)$ of B by the B-B-bimodule $D(B)$, which is a symmetric algebra.

In fact we have the following result proved by Ohnuki, Takeda and Yamagata [OTY], essential for further considerations.

Theorem 2.1. *Let B be an algebra, φ a positive automorphism of $\widehat{B}$ and $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$. Then A is a symmetric algebra if and only if $A \cong \mathrm{T}(B)$.*

Let B be an algebra, G an admissible group of automorphisms of $\widehat{B}$ and $A = \widehat{B}/G$. The group G acts also on the module category $\widehat{B}$ (identified with the category of contravariant functors from $\widehat{B}$ to

$\operatorname{mod} K$ with finite supports) given by $gM = M \circ g^{-1}$ for any module M in $\operatorname{mod} \widehat{B}$.

Then we have also the *push-down functor* $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ [BG], associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$, such that $F_\lambda(M)(a) = \bigoplus_{x \in a} M(x)$ for M in $\operatorname{mod} \widehat{B}$ and $a \in \operatorname{ob}(\widehat{B}/G)$.

The following special case of a theorem proved by Gabriel [Ga] is fundamental.

Theorem 2.2. *Let B be an algebra, G a torsion-free admissible group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$. Then*

(i) The push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ induces an injection from the set of G-orbits of isomorphism classes of indecomposable modules in $\operatorname{mod} \widehat{B}$ into the set of isomorphism classes of indecomposable modules in $\operatorname{mod} A$.

(ii) The push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ preserves the Auslander-Reiten sequences.

Unfortunately, in general the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}/G$ associated to a Galois covering $F : \widehat{B} \to \widehat{B}/G$ is not dense. A repetitive category $\widehat{B}$ is said to be *locally support-finite* [DS1] if, for any object x of R, the full subcategory of $\widehat{B}$ given by the supports $\operatorname{supp} M$ of all indecomposable modules M in $\operatorname{mod} \widehat{B}$ with $M(x) \neq 0$ is a finite category.

The following special case of the density theorem of Dowbor and Skowroński from [DS1], [DS2] is crucial for our consideration.

Theorem 2.3. *Let B be an algebra, G a torsion-free admissible group of automorphisms of $\widehat{B}$, $A = \widehat{B}/G$, and assume that $\widehat{B}$ is locally support-finite. Then the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$, associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$, is dense. In particular, the Auslander-Reiten quiver Γ_A of A is the orbit translation quiver $\Gamma_{\widehat{B}}/G$ of the Auslander-Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ with respect to the induced action of G.*

3 Criteria for orbit algebras of repetitive categories

Let A be a selfinjective algebra. By a classical result of Nakayama [Na] (see also [Y]) the left socle $\operatorname{soc}_A A$ and the right socle $\operatorname{soc} A_A$ of A coincide, and hence $\operatorname{soc} A := \operatorname{soc}{}_A A = \operatorname{soc} A_A$ is an ideal of A. Two selfinjective algebras A and Λ are said to be *socle equivalent* if the factor algebras $A/\operatorname{soc} A$ and $\Lambda/\operatorname{soc}\Lambda$ are isomorphic.

Let A be a selfinjective algebra and $1 = e_1 + e_2 + \cdots + e_n$ a decomposition of the identity 1_A of A into a sum of pairwise orthogonal idempotents of A. We denote by $\nu = \nu_A$ the *Nakayama permutation* of A (with respect to this decomposition of 1_A) that is the permutation ν of $\{1,\dots,n\}$ such that $\operatorname{top} e_iA \cong \operatorname{soc} e_{\nu(i)}A$ for any $i \in \{1,\dots,n\}$. For a subset X of A, we consider the *left annihilator* $\ell_A(X) = \{a \in A \mid ax = 0\}$ and the *right annihilator* $r_A(X) = \{a \in A \mid xa = 0\}$ of X in A. Let I be an ideal of A, $B = A/I$ and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $e = e_1 + \cdots + e_m$ for some $m \leq n$, and $\{e_i \mid 1 \leq i \leq m\}$ is the set of all idempotents in $\{e_i \mid 1 \leq i \leq n\}$ which are not contained in I. Such an idempotent e is uniquely determined by I up to an inner automorphism of A, and is called a *residual identity* of $B = A/I$. Observe that we have a canonical isomorphism of algebras $eAe/eIe \xrightarrow{\sim} A/I = B$. We also note that if e is an idempotent of A such that $\ell_A(I) = Ie$ or $r_A(I) = eI$, then e is a residual identity of A/I [SY6].

The following proposition proved in [SY1] is essential for further considerations.

Proposition 3.1. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, e a residual identity of B, and assume that $IeI = 0$. The following conditions are equivalent:*

(i) Ie is an injective cogenerator in $\operatorname{mod} B$.

(ii) eI is an injective cogenerator in $\operatorname{mod} B^{\mathrm{op}}$.

(iii) $\ell_A(I) = Ie$.

(iv) $r_A(I) = eI$.

Moreover, under these equivalent conditions, we have $\operatorname{soc} A \subseteq I$ *and* $eIe = \ell_{eIe}(I) = r_{eAe}(I)$.

The following criterion for a selfinjective algebra to be an orbit algebra of the repetitive category of an algebra with respect to action of an infinite cyclic group has been established by Skowroński and Yamagata in [SY3] (sufficency part) and [SY6] (necessity part).

Theorem 3.2. *Let A be a selfinjective algebra over a field K. The following conditions are equivalent:*

(i) A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra over K and φ is a positive automorphism of $\widehat{B}$.

(ii) There is an ideal I of A and an idempotent e of A such that

(1) $r_A(I) = eI$,

(2) the canonical algebra epimorphism $eAe \to eAe/eIe$ *is a retraction.*

Moreover, in this case, B is isomorphic to the factor algebra A/I.

Observe that, by Proposition 3.1, that the condition (ii) (1) is natural and rather easy to check. On the other hand, the condition (ii) (2) is not easy to check and creates problems in applications of Theorem 3.2. In order to deal with this problem socle deformations of selfinjective algebras given by deforming ideals were introduced by Skowroński and Yamagata in [SY1].

For an algebra B, we denote by Q_B the (*valued*) *quiver* of B. Recall that the vertices of Q_B are the numbers $1, \dots, m$ corresponding to the choosen idempotents $e_1, \dots, e_m$ of B with $1_B = e_1 + \dots + e_m$. Further, if $S_1 = \operatorname{top}(e_1 B), \dots, S_m = \operatorname{top}(e_m B)$ are the associated simple B-modules, then there is an arrow from i to j in Q_B if $\operatorname{Ext}_B^1(S_i, S_j) \neq 0$, and to this arrow the valuation

$$\left(\dim_{\operatorname{End}_B(S_i)} \operatorname{Ext}_B^1(S_i, S_j), \dim_{\operatorname{End}_B(S_j)} \operatorname{Ext}_B^1(S_i, S_j)\right)$$

is assigned.

Let A be a selfinjective algebra, I an ideal of A and e a residual identity of A/I. Following [SY1], I is said to be a *deforming ideal* of A if the following conditions are satisfied:

(D1) $\ell_{eAe}(I) = eIe = r_{eAe}(I)$;

(D2) the valued quiver $Q_{A/I}$ of A/I is acyclic.

Assume I is a deforming ideal of A. Then we have a canonical isomorphism of algebras $eAe/eIe \to A/I$ and I can be considered as an (eAe/eIe)-(eAe/eIe)-bimodule. Following [SY1], we denote by $A[I]$ the direct sum of K-vector spaces $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x)(c, y) = (bc,\ by + xc + xy)$$

for $b, c \in eAe/eIe$ and $x, y \in I$. Then $A[I]$ is a K-algebra with the identity $(e + eIe, 1 - e)$, and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$. Moreover, $e = (e + eIe, 0)$ is a residual identity of $A[I]/I = eAe/eIe \xrightarrow{\sim} A/I$, $eA[I]e = (eAe/eIe) \oplus eIe$ and the canonical algebra epimorphism $eA[I]e \to eA[I]e/eIe$ is a retraction.

The following properties of the algebras $A[I]$ have been established in [SY1], [SY2], [SY7].

Theorem 3.3. *Let A be a selfinjective algebra and I a deforming ideal of A. Then the following statements hold.*

(i) *$A[I]$ is a selfinjective algebra with the same Nakayama permutation as A and I is a deforming ideal of $A[I]$.*

(ii) *$A[I]$ is a symmetric algebra if A is a symmetric algebra.*

(iii) *A and $A[I]$ are socle equivalent.*

(iv) *A and $A[I]$ are stably equivalent.*

It follows from Proposition 3.1 that if A is a selfinjective algebra, I an ideal of A, $B = A/I$, e an idempotent of A such that $eI = r_A(I)$, and the valued quiver Q_B of B is acyclic, then I is a deforming ideal of A.

The following theorem has been proved in [SY3].

Theorem 3.4. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$ and e an idempotent of A. Assume that $eI = r_A(I)$ and Q_B is acyclic. Then $A[I]$ is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

As a direct consequence of Theorems 3.3 and 3.4 we obtain the following fact.

Corollary 3.5. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$ and e an idempotent of A. Assume that $eI = r_A I$ and Q_B is acyclic. Then A is socle equivalent and stably equivalent to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

We mention that there are examples of selfinjective algebras A with deforming ideals I such that the algebras A and $A[I]$ are not isomorphic (see [SY3]). The following result from [SY5] describes the situation when the algebras A and $A[I]$ are isomorphic.

Theorem 3.6. *Let A be a selfinjective algebra with a deforming ideal I, $B = A/I$, e be a residual identity of B and ν the Nakayama permutation of A. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e. Then the algebras A and $A[I]$ are isomorphic. In particular, A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

It has been proved in [SY1] that if A is a selfinjective algebra over an algebraically closed field K, I a deforming ideal with a residual identity e, then the second Hochschild cohomology sopace $H^2(eAe/eIe, eIe)$ vanishes. This leads to the following consequence of Theorems 3.2 and 3.4.

Theorem 3.7. *Let A be a selfinjective algebra over an algebraically closed field K. The following conditions are equivalent:*

(i) A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra over K with acyclic quiver Q_B and φ is a positive automorphism of $\widehat{B}$.

(ii) There is an ideal I and an idempotent e of A such that

(1) $r_A(I) = eI$,

(2) the quiver $Q_{A/I}$ of A/I is acyclic.

Moreover, in this case, B is isomorphic to A/I.

4 Selfinjective algebras of canonical type

We exhibit here the class of selfinjective algebras of quasitilted type playing a prominent role in the representation theory of selfinjective algebras.

Following [HRS] an algebra Λ is called *quasitilted* if Λ has global dimension at most two and every indecomposable module in $\operatorname{mod}\Lambda$ has projective or injective dimension at most one. It has been proved in [HRe] that the class of quasitilted algebras consists of the *tilted algebras* (endomorphism algebras of tilting modules over hereditary algebras) [HRi] and the *quasitilted algebras of canonical type* (endomorphism algebras of canonical algebras of Ringel [Rin1], [Rin2]) [LS1].

A selfinjective algebra A over a field K is said to be a *selfinjective algebra of quasitilted type* if A is isomorphic to an orbit algebra $\widehat{B}/G$, where B is a quasitilted algebra over K and G is an admissible torsion-free group of automorphism of $\widehat{B}$.

Theorem 4.1. *Let B be a quasitilted algebra, G an admissible torsion-free group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$ the associated orbit algebra. Then the following statements hold.*

(i) G is an infinite cyclic group generated by a strictly positive automorphism φ of $\widehat{B}$.

(ii) $\widehat{B}$ is locally support-finite.

(iii) The push-down functor $F_\lambda : \operatorname{mod}\widehat{B} \to \operatorname{mod} A$ associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ is dense.

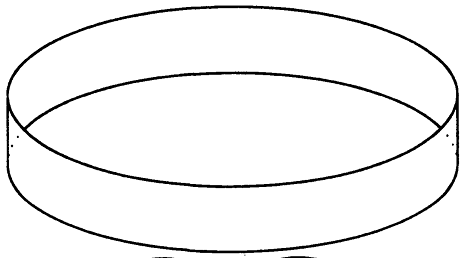
(iv) The Auslander-Reiten quiver Γ_A of A is isomorphic to the orbit quiver $\Gamma_{\widehat{B}}/G$ of the Auslander-Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ with respect to the induced action of G on $\Gamma_{\widehat{B}}$.

Let B be a quasitilted algebra over a field K, G an admissible infinite cyclic group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$ the associated selfinjective algebra of quasitilted type. Then A is said to be a

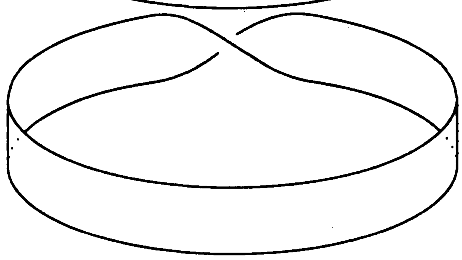
- *selfinjective algebra of tilted type*, if B is a tilted algebra;
- *selfinjective algebra of canonical type*, if B is a quasitilted algebra of canonical type;
- *selfinjective algebra of Dynkin type*, if B is a tilted algebra of Dynkin type;
- *selfinjective algebra of Euclidean type*, if B is a tilted algebra of Euclidean type;
- *selfinjective algebra of wild tilted type*, if B is a tilted algebra of wild type;
- *selfinjective algebra of tubular type*, if B is a tubular algebra;
- *selfinjective algebra of wild canonical type*, if B is a quasitilted algebra of wild canonical type.

In fact, the class of selfinjective algebras of quasitilted type splits into five disjoint classes: selfinjective algebras of Dynkin type, selfinjective algebras of of Euclidean type, selfinjective algebras of wild tilted type, selfinjective algebras of tubular type and selfinjective algebras of wild canonical type. The Auslander-Reiten quivers of selfinjective algebras of quasitilted type are described by the following theorems (for more details see [EKS], [LS2], [Sk1], [Sk4] and [SY9]).

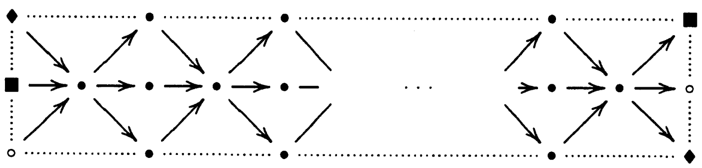
Theorem 4.2. *Let $A = \widehat{B}/G$ be a selfinjective algebra of Dynkin type Δ. Then the stable Auslander-Reiten quiver Γ_A^s of A is isomorphic to the orbit quiver $\mathbb{Z}\Delta/G$ and is of one of the forms*



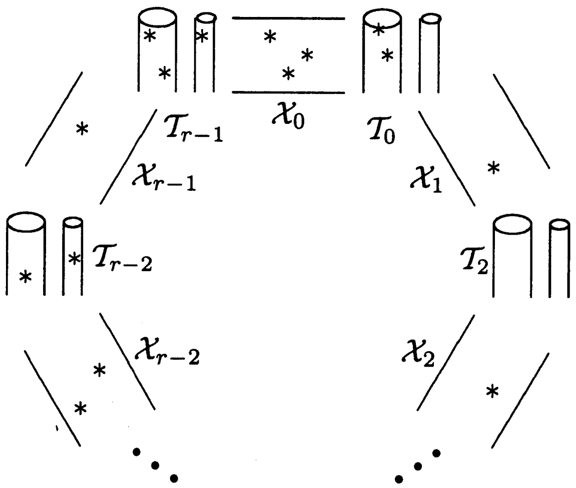
cylinder



Möbius strip

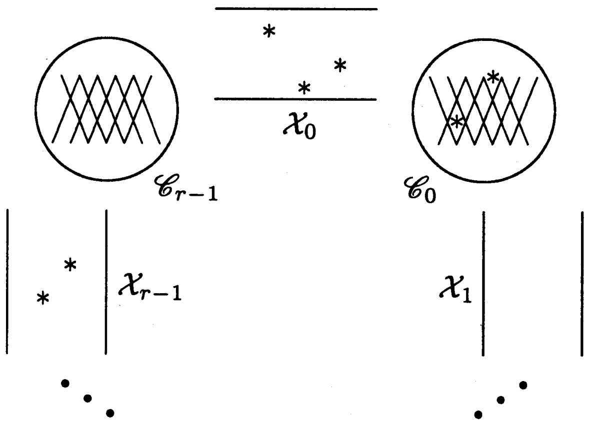


Theorem 4.3. *Let A be a selfinjective algebra of Euclidean type Δ. Then the Auslander-Reiten quiver Γ_A of A is of the form*



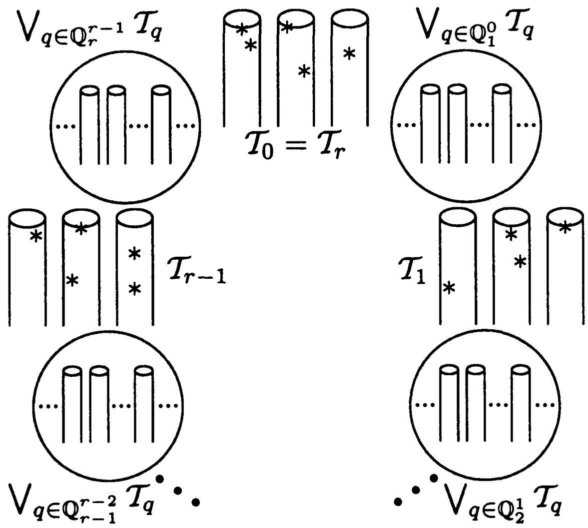
where $*$ denote projective-injective modules, $r \geq 1$, $\mathcal{X}_i^s = \mathbb{Z}\Delta$, $\mathcal{T}_i^s$ is an infinite family of stable tubes, for each $i \in \{0, 1, \ldots, r-1\}$. Moreover, for each $i \in \{0, 1, \ldots, r-1\}$, all but finitely many stable tubes in $\mathcal{T}_i^s$ are of rank one and $\mathcal{X}_i$ contains at least one projective-injective module.

Theorem 4.4. *Let A be a selfinjective algebra of wild tilted type Δ. Then the Auslander-Reiten quiver Γ_A of A is of the form*



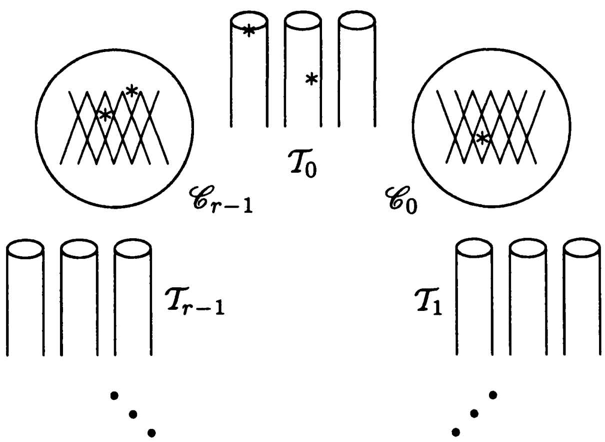
where $*$ denote projective-injective modules, $r \geq 1$, $\mathcal{X}_i^s = \mathbb{Z}\Delta$ and $\mathcal{C}_i^s$ is an infinite family of components of the form $\mathbb{Z}\mathbb{A}_\infty$, for each $i \in \{0, 1, \ldots, r-1\}$.

Theorem 4.5. *Let A be a selfinjective algebra of tubular type. Then the Auslander-Reiten quiver Γ_A of A is of the form*



where $*$ denote projective-injective modules, $r \geq 3$, $\mathcal{T}_i^s$ is an infinite family of stable tubes, for each $i \in \{0, 1, \ldots, r-1\}$, and $\mathcal{T}_q$ is an infinite family of stable tubes, for each $q \in \mathbb{Q}_i^{i-1} = \mathbb{Q} \cap (i-1, i)$. Moreover, for each $q \in \mathbb{Q} \cap [0, r]$, all but finitely many stable tubes in $\mathcal{T}_q^s$ are of rank one.

Theorem 4.6. *Let A be a selfinjective algebra of wild canonical type. Then the Auslander-Reiten quiver Γ_A of A is of the form*



where $*$ denote projective-injective modules, $r \geq 1$, $\mathcal{T}_i^s$ is an infinite family of stable tubes and $\mathscr{C}_i^s$ is an infinite family of components of the form $\mathbb{Z}\mathbb{A}_\infty$, for each $i \in \{0, 1, \ldots, r-1\}$. Moreover, for each $i \in \{0, 1, \ldots, r-1\}$, all but finitely many stable tubes in $\mathcal{T}_i^s$ are of rank one and $\mathscr{C}_i$ contains at least one projective-injective module.

5 Selfinjective algebras of polynomial growth

Throughout this section K will be an algebraically closed field.

Let $K[x]$ be the polynomial algebra in one variable over K. Following Drozd [Dr], an algebra A is said to be *tame* if, for any dimension d, there exists a finite number of $K[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[x]$-modules and all but finitely many isomorphism classes of indecomposable modules in $\operatorname{mod} A$ of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \dots, n_d\}$. Let $\mu_A(d)$ be the least number of $K[x]$-A-bimodules satisfying the above condition for d. Then an algebra A is said to be of *polynomial growth* if there is a positive integer m such that $\mu_A(d) \leq d^m$ for all $d \geq 1$. We note that from the validity of the second Brauer-Thrall conjecture we have $\mu_A(d) = 0$ for all $d \geq 1$ if and only if A is of finite representation type.

The following tame and wild dichotomy theorem proved by Drozd [Dr] is fundamental.

Theorem 5.1. *Every algebra A is either tame or wild, and not both.*

We mention that the representation theory of a wild algebra over a field K comprises the representation theories of all finite dimensional algebras over K, hence a classification of finite dimensional indecomposable modules is only feasible for tame algebras (see [SS, Chapter XIX]).

The following theorem announced in [Sk4] (see also [Sk1]) shows the importance of selfinjective algebras of quasitilted type in the representation theory of tame selfinjective algebras.

Theorem 5.2. *Let A be a nonsimple selfinjective algebra. The following statements are equivalent:*

(i) A is of polynomial growth.

(ii) A is socle equivalent to a selfinjective algebra $\bar{A}$ of Dynkin, Euclidean, or tubular type.

(iii) A is socle equivalent to an orbit algebra $\bar{A} = \widehat{B}/(\varphi)$, where B is a quasitilted algebra with positive semidefinite Euler form χ_B and φ is a strictly positive automorphism of $\widehat{B}$.

We note that the selfinjective algebra $\bar{A} = \widehat{B}/(\varphi)$ of quasitilted type socle equivalent to a nonsimple selfinjective algebra A of polynomial growth is uniquely determined by A (up to isomorphism), and is also a geometric degeneration of A in the affine variety of algebras of dimension $d = \dim_K A$.

We also present the orbit algebras interpretation [Sk4] of the Riedtmann's classification of the selfinjective algebras of finite representation type.

Theorem 5.3. *Let A be a nonsimple selfinjective algebra. The following statements are equivalent:*

(i) A is of finite representation type.

(ii) A is socle equivalent to a selfinjective algebra $\bar{A}$ of Dynkin type.

(iii) A is socle equivalent to a selfinjective algebra $\bar{A} = \widehat{B}/(\varphi)$, where B is a quasitilted algebra with positive definite Euler form χ_B and φ is a strictly positive automorphism of $\widehat{B}$.

We end this section with the orbit algebras interpretation of the Brauer tree algebras playing the fundamental role in the Morita equivalence classification of blocks of group algebras of finite representation type. Recall that the *Brauer tree algebras* are the symmetric algebras $A(T_S^m)$ associated to the Brauer trees T_S^m, which are finite connected trees with a circular ordering of edges converging at each vertex and with one fixed (exceptional) vertex S with a multiplicity $m \geq 1$.

Theorem 5.4. *Let A be a nonsimple selfinjective algebra and m a positive integer. The following statements are equivalent:*

(i) A is isomorphic to a Brauer tree algebra $A(T_S^m)$.

(ii) A is isomorphic to an orbit algebra $\widehat{B}/(\varphi)$, where B is a tilted algebra of Dynkin type $\mathbb{A}_n$ and φ is a strictly positive automorphism of $\widehat{B}$ with $\varphi^m = \nu_{\widehat{B}}$.

In the above theorem, we have $n = me$, where e is the number of edges of the Brauer tree T_S^m.

6 Selfinjective algebras with generalized standard components

Let A be an algebra over an arbitrary field K. By general theory, the Auslander-Reiten quiver Γ_A of A describes essentially "only" the quotient category $\operatorname{mod} A/\operatorname{rad}^{\infty}(\operatorname{mod} A)$, where $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ is the *infinite Jacobson radical* of $\operatorname{mod} A$, that is, the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of Jacobson radical $\operatorname{rad}(\operatorname{mod} A)$ of $\operatorname{mod} A$. In particular, A is of finite representation type if and only if $\operatorname{rad}^{\infty}(\operatorname{mod} A) =$

0 [ARS]. In general, it is important to study the behaviour of the components of Γ_A in the module category $\operatorname{mod} A$. Following Skowroński [Sk2] a component $\mathscr{C}$ of an Auslander-Reiten quiver Γ_A is said to be *generalized standard* if $\operatorname{rad}_A^\infty(X,Y) = 0$ for all indecomposable modules X and Y in $\mathscr{C}$. It has been proved in [Sk2] that every generalized standard component $\mathscr{C}$ of Γ_A is quasiperiodic, that is, all but finitely many $D\,\mathrm{Tr}$-orbits in $\mathcal{C}$ are periodic. Moreover, by a result from [SZ], the additive closure $\operatorname{add}(\mathcal{C})$ of a generalized standard component $\mathcal{C}$ of Γ_A is closed under extensions in $\operatorname{mod} A$.

For a selfinjective algebra A and a generalized standard component $\mathcal{C}$ of Γ_A, the stable part $\mathcal{C}^s$ of $\mathcal{C}$ is one of the three possible forms:

- $\mathbb{Z}\Delta/G$, for a Dynkin quiver (of type $\mathbb{A}_m$, $\mathbb{B}_m$, $\mathbb{C}_m$, $\mathbb{D}_m$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$, $\mathbb{G}_2$) and an infinite cyclic admissible group G of automorphisms of $\mathbb{Z}\Delta$;

- $\mathbb{Z}\Delta$ for a finite valued acyclic quiver, different from a Dynkin quiver;

- stable tube $\mathbb{Z}\Delta/(\tau^r)$, $r \geq 1$.

In this section, we are concerned with the following problem.

Problem 1. Describe the structure of all selfinjective algebras A for which Γ_A admits a generalized standard stable tube.

Observe that the following open problem is a very special case of Problem 1.

Problem 2. Describe the structure of all selfinjective algebras A of finite representation type.

It is expected that a nonsimple selfinjective algebra A is of finite representation type if and only if A is socle equivalent to an orbit algebra $\widehat{B}/(\varphi)$, where B is a tilted algebra of Dynkin type and φ is a strictly positive automorphism of $\widehat{B}$.

The following theorem from [SY3] describes the structure of selfinjective algebras whose Auslander-Reiten quiver admits an acyclic generalized standard right (respectively, left) stable translation subquiver.

Theorem 6.1. *Let A be a selfinjective algebra over a field K. The following statements are equivalent:*

(i) Γ_A admits an acyclic generalized standard right stable full translation subquiver which is closed under successors in Γ_A.

(ii) *A is socle equivalent to an orbit algebra* $\widehat{B}/(\varphi\nu_{\widehat{B}})$, *where* B *is a tilted algebra over* K *not of Dynkin type and* φ *is a positive automorphism of* $\widehat{B}$.

(iii) Γ_A *admits an acyclic generalized standard left stable full translation subquiver which is closed under predecessors in* Γ_A.

Moreover, if K *is an algebraically closed field, we may replace in (ii) "socle equivalent" by "isomorphic".*

In particular, we have the following consequences of the Theorem 6.1 (see also [Sk4]), describing the selfinjective algebras whose Auslander-Reiten quiver admits an acyclic generalized standard component.

Theorem 6.2. *Let* A *be a selfinjective algebra over a field* K. *The following statements are equivalent:*

(i) Γ_A *admits an acyclic regular generalized standard component* $\mathscr{C}$.

(ii) *A is socle equivalent to an orbit algebra* $\widehat{B}/(\varphi\nu_{\widehat{B}})$, *where* B *is a tilted algebra of the form* $\mathrm{End}_H(T)$, *for some hereditary algebra* H *over* K *and a regular tilting* H*-module* T, *and* φ *is a positive automorphism of* $\widehat{B}$.

Moreover, if K *is an algebraically closed field, we may replace in (ii) "socle equivalent" by "isomorphic".*

Theorem 6.3. *Let* A *be a selfinjective algebra over a field* K. *The following statements are equivalent:*

(i) Γ_A *admits an acyclic nonregular generalized standard component* $\mathcal{C}$.

(ii) *A is isomorphic to an orbit algebra* $\widehat{B}/(\varphi\nu_{\widehat{B}})$, *where* B *is a tilted algebra of the form* $\mathrm{End}_H(T)$, *for some hereditary algebra* H *over* K *and a nonregular tilting* H*-module* T, *and* φ *is a strictly positive automorphism of* $\widehat{B}$.

The crucial role in the proof of Theorem 6.1 is played by Theorems 3.2, 3.3, 3.4, Corollary 3.5 and the following result from [SY1].

Theorem 6.4. *Let* A *be a selfinjective algebra over a field* K. *Assume that* Γ_A *contains an acyclic generalized standard right stable (respectively, left stable) full translation subquiver* Σ *which is closed under successors (respectively, predecessors) in* Γ_A. *Let* I *be the annihilator* $\mathrm{ann}_A(\Sigma)$ *of* Σ *in* A, $B = A/I$, *and* e *a residual identity of* B. *Then*

I is a deforming ideal of A such that $eI = r_A(I)$, $Ie = \ell_A(I)$, and B is a tilted algebra of the form $\mathrm{End}_H(T)$, where H is a hereditary algebra over K not of Dynkin type and T is a tilting H-module without nonzero postprojective (respectively, preinjective) direct summands.

The problem of a description of selfinjective algebras whose Auslander-Reiten quiver admits a generalized standard *quasitube* (the stable part is a stable tube) seems to be very difficult. Namely, it has been proved in [Sk3] that, for every finite dimensional algebra B over a field K and a finite dimensional B-module M, there exists a symmetric algebra A such that B is a factor algebra of A, Γ_A admits a generalized standard stable tube $\mathcal{T}$, and M is a subfactor of modules in $\mathcal{T}$.

We have also the following necessity condition for the existence of generalized standard stable tubes in the Auslander-Reiten quivers of symmetric algebras, proved in [BSY].

Theorem 6.5. *Let A be a symmetric algebra over a field K such that Γ_A admits a generalized standard stable tube. Then the Cartan matrix C_A of A is singular.*

We also mention that there are many selfinjective algebras with nonsingular Cartan matrices for which the Auslander-Reiten quivers admit generalized standard stable tubes (see [BSY]).

We end this section with the following recent result established in [SY8].

Theorem 6.6. *Let A be a selfinjective algebra of infinite representation type over a field K. The following statements are equivalent:*

(i) Every component of Γ_A is generalized standard.

(ii) A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a tilted algebra of Euclidean type or a tubular algebra, and φ is a strictly positive automorphism of $\widehat{B}$.

(iii) A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a quasitilted algebra with positive semidefinite Euler form χ_B, and φ is a strictly positive automorphism of $\widehat{B}$.

7 Stable and derived equivalences

We discuss here invariance of the selfinjective algebras of quasitilted type under stable and derived equivalences.

We may associate to an algebra A the *derived category* $D^b(\operatorname{mod} A)$ of bounded complexes of modules from $\operatorname{mod} A$, which is the localization of the homotopy category $K^b(\operatorname{mod} A)$ of bounded complexes of modules from $\operatorname{mod} A$ with respect to quasi-isomorphisms. We note that $D^b(\operatorname{mod} A)$ is a triangulated category (see [Ha]). Two algebras A and B are said to be *derived equivalent* if the derived categories $D^b(\operatorname{mod} A)$ and $D^b(\operatorname{mod} B)$ are equivalent as triangulated categories. The prominent derived equivalences of algebras are induced by tilting modules. Namely, if A is an algebra, T a tilting module in $\operatorname{mod} A$ and $B = \operatorname{End}_A(T)$ the associated tilted algebra, then A and B are derived equivalent. More generally, Rickard proved in [Ric1] his celebrated criterion: two algebras A and B are derived equivalent if and only if B is the endomorphism algebra of a tilting complex over A.

The following result proved by Rickard [Ric2] is fundamental for study of the stable and derived equivalences of selfinjective algebras.

Theorem 7.1. *Let A and Λ be derived equivalent selfinjective algebras. Then A and Λ are stably equivalent.*

The class of all orbit algebras $\widehat{B}/G$ (even all selfinjective algebras of quasitilted type) is not closed under stable equivalences. However, we have the following general result proved in [KSY].

Theorem 7.2. *Let A be a selfinjective algebra over a field K. The following statements are equivalent.*

(i) A is stably equivalent to an orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a quasitilted algebra over K and φ is a strictly positive automorphism of $\widehat{B}$.

(ii) A is isomorphic to an orbit algebra $\widehat{R}/(\psi\nu_{\widehat{R}})$, where R is a quasitilted algebra over K and ψ is a strictly positive automorphism of $\widehat{R}$.

For the orbit algebras $\widehat{B}/(\varphi\nu_{\widehat{B}})$ of tilted type, with φ a positive automorphism of $\widehat{B}$, we have the following theorem from [SY2], [SY7].

Theorem 7.3. *Let A be a selfinjective algebra over a field K and Δ be a finite, connected, acyclic, valued quiver. The following statements are equivalent:*

(i) A is stably equivalent to a selfinjective orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a tilted algebra of type Δ over K and φ is a positive automorphism of $\widehat{B}$.

(ii) A is socle equivalent to a selfinjective orbit algebra $\widehat{R}/(\psi\nu_{\widehat{R}})$, *where R is a tilted algebra of type* Δ *over K and* ψ *is a positive automorphism of* $\widehat{R}$.

We would like to point out that in general we cannot replace in (ii) "socle equivalent" by "isomorphic" (see [SY4, Proposition 4]).

The following problem seems to be interesting.

Problem 3. Is the class of selfinjective algebras stably equivalent to the selfinjective algebras of quasitilted type closed under socle equivalences?

References

[ARS] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge, 1995.

[BSY] J. Białkowski, A. Skowroński and K. Yamagata, Cartan matrices of symmetric algebras having generalized standard stable tubes, *Osaka. J. Math.* **45** (2008), 1–13.

[BG] K. Bongartz and P. Gabriel, Covering spaces in representation theory, *Invent. Math.* **65** (1982), 337–378.

[DS1] P. Dowbor and A. Skowroński, On Galois coverings of tame algebras, *Archiv Math. (Basel)* **44** (1985), 522–529.

[DS2] P. Dowbor and A. Skowroński, Galois coverings of representation-infinite algebras, *Comment. Math. Helv.* **62** (1987), 311–337.

[Dr] Y. A. Drozd, Tame and wild matrix problems. In *Representation Theory II.* Lecture Notes in Mathematics 832, Springer–Verlag, Berlin–Heidelberg, 1980, 242–258.

[EKS] K. Erdmann, O. Kerner and A. Skowroński, Self-injective algebras of wild tilted type, *J. Pure Appl. Algebra* **149** (2000), 127–176.

[Ga] P. Gabriel, The universal cover of a representation-finite algebra. In *Representations of Algebras.* Lecture Notes in Mathematics 903, Springer–Verlag, Berlin–Heidelberg, 1981, 68–105.

[Ha] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras.* London Mathematical Society Lecture Note Series 119, Cambridge University Press, Cambridge, 1988.

[HRe] D. Happel and I. Reiten, Hereditary abelian categories with tilting object over arbitrary basic fields, *J. Algebra* **256** (2002), 414–432.

[HRS] D. Happel, I. Reiten and S. O. Smalø, Tilting in abelian categories and quasitilted algebras, *Memoirs Amer. Math. Soc.* **120** (1996), no. 575.

[HRi] D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 399–443.

[KSY] O. Kerner, A. Skowroński and K. Yamagata, Invariance of selfinjective algebras of quasitilted type under stable equivalences, *Manuscripta Math.* **119** (2006), 359–381.

[LS1] H. Lenzing and A. Skowroński, Quasi-tilted algebras of canonical type, *Colloq. Math.* **71** (1996), 161–181.

[LS2] H. Lenzing and A. Skowroński, Selfinjective algebras of wild canonical type, *Colloq. Math.* **96** (2003), 245–275.

[Na] T. Nakayama, On Frobeniusean algebras I, *Ann. of Math.* **40** (1939), 611–633.

[OTY] Y. Ohnuki, K. Takeda and K. Yamagata, Automorphisms of repetitive algebras, *J. Algebra* **232** (2000), 708–724.

[Ric1] J. Rickard, Morita theory for derived categories, *J. London Math. Soc.* **39** (1989), 436–456.

[Ric2] J. Rickard, Derived categories and stable equivalence, *J. Pure Appl. Algebra* **61** (1989), 303–317.

[Rin1] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms.* Lecture Notes in Mathematics 1099, Springer–Verlag, Berlin–Heidelberg, 1984.

[Rin2] C. M. Ringel, The canonical algebras, with an appendix by W. Crawley-Boevey. In *Topics in Algebra.* Banach Center Publications 26, Part 1, PWN, Warsaw, 1990, 407–432.

[SS] D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras*. London Mathematical Society Student Texts 72, Cambridge University Press, Cambridge, 2007.

[Sk1] A. Skowroński, Selfinjective algebras of polynomial growth, *Math. Ann.* **285** (1989), 177–199.

[Sk2] A. Skowroński, Generalized standard Auslander-Reiten components, *J. Math. Soc. Japan* **46** (1994), 517–543.

[Sk3] A. Skowroński, A construction of complex syzygy periodic modules over symmetric algebras, *Colloq. Math.* **103** (2005), 61–69.

[Sk4] A. Skowroński, Selfinjective algebras: finite and tame type. In *Trends in Representation Theory of Algebras and Related Topics*. Contemporary Mathematics 406, Amer. Math. Soc., Providence, RI, 2006, 169–238.

[SY1] A. Skowroński and K. Yamagata, Socle deformations of selfinjective algebras, *Proc. London Math. Soc.* **72** (1996), 545–566.

[SY2] A. Skowroński and K. Yamagata, Stable equivalence of selfinjective algebras of tilted type, *Archiv Math. (Basel)* **70** (1998), 341–350.

[SY3] A. Skowroński and K. Yamagata, Galois coverings of selfinjective algebras by repetitive algebras, *Trans. Amer. Math. Soc.* **351** (1999), 715–734.

[SY4] A. Skowroński and K. Yamagata, On selfinjective artin algebras having nonperiodic generalized standard Auslander-Reiten components, *Colloq. Math.* **96** (2003), 235–244.

[SY5] A. Skowroński and K. Yamagata, On invariability of selfinjective algebras of tilted type under stable equivalences, *Proc. Amer. Math. Soc.* **132** (2004), 659–667.

[SY6] A. Skowroński and K. Yamagata, Positive Galois coverings of selfinjective algebras, *Advances Math.* **194** (2005), 398–436.

[SY7] A. Skowroński and K. Yamagata, Stable equivalence of selfinjective artin algebras of Dynkin type, *Algebras and Representation Theory* **9** (2006), 33–45.

[SY8] A. Skowroński and K. Yamagata, Selfinjective artin algebras with all Auslander-Reiten components generalized standard, Preprint 2008.

[SY9] A. Skowroński and K. Yamagata, Selfinjective algebras of quasitilted type, In *Trends in Representation Theory of Algebras and Related Topics.* European Mathematical Society Series of Congress Reports, European Math. Soc. Publ. House, Zürich, 2008, 639–708.

[SZ] A. Skowroński and G. Zwara, Degeneration-like orders on the additive categories of generalized standard Auslander-Reiten components, *Arch. Math. (Basel)* **74** (2000), 11–21.

[Y] K. Yamagata, Frobenius Algebras. In *Handbook of Algebra 1*, Elsevier, North-Holland, Amsterdam, 1996, 841–887.

A. Skowroński: Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, Toruń, POLAND
`skowron@mat.uni.torun.pl`