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On the Dade -Tasaka correspondence between blocks of finite groups

渡辺アツミ (Atumi Watanabe)

熊本大学大学院自然科学研究科

Graduate School of Science and Technology, Department of Mathematics, Faculty of Science, Kumamoto University

1 Introduction

In this report we state a generalization of Tasaka's isotypy between blocks of finite groups obtained by the Dade character correspondence. Let p be a prime and $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system such that $\mathcal{K}$ is a splitting filed for all finite groups which we consider in this talk. Let $\mathcal{S}$ denote $\mathcal{O}$ or k. For a finite abelian group F, we denote by $\hat{F}$ the character group of F and by $\hat{F}_q$ the subgroup of $\hat{F}$ of order q for $q \in \pi(F)$, where $\pi(F)$ is the set of all primes dividing $|F|$. Let G be a finite group and N be a normal subgroup of G. We denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G and $\mathrm{Irr}^G(N)$ be the set of G-invariant irreducible characters of N. For $\phi \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G|\phi)$ the set of irreducible characters χ of G such that ϕ is a constituent of the restriction χ_N of χ to N.

Hypothesis 1 *G is a finite group which is a normal subgroup of a finite group E such that the factor group $F = E/G$ is a cyclic group of order r. λ is a generator of $\hat{F}$. $E_0 = \{x \in E \mid \bar{x}$ is a generator of $F\}$ where $\bar{x} = xG$. E' is a subgroup of E such that $E'G = E$, $G' = G \cap E'$ and $E'_0 = E' \cap E_0$. Moreover $(E'_0)^\tau \cap E'_0$ is the empty set, for all $\tau \in E - E'$.*

Under the above hypothesis, in [2], E.C. Dade constructed a bijection between $\mathrm{Irr}^E(G)$ and $\mathrm{Irr}^{E'}(G')$ which is a generalization of the cyclic case of the Glauberman correspondence ([3] or, [6], Chap.13).

Theorem 1 ([2], Theorem 6.8, Theorem 6.9) *Assume Hypothesis 1 and $|F| \neq 1$. For each prime $q \in \pi(F)$, we choose some non-trivial character $\lambda_q \in \hat{F}_q$. There is a bijection*

$$\rho(E, G, E', G') : \mathrm{Irr}^E(G) \to \mathrm{Irr}^{E'}(G') \ (\phi \mapsto \phi' = \phi_{(G')})$$

which satisfies the following conditions. If r is odd, then there are a unique integer $\epsilon_\phi = \pm 1$ and a unique bijection $\psi \mapsto \psi_{(E')}$ of $\mathrm{Irr}(E|\phi)$ onto $\mathrm{Irr}(E'|\phi')$ such that

$$(1.1) \qquad \Big(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi\Big)_{E'} = \epsilon_\phi \prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi_{(E')},$$

for any $\psi \in \mathrm{Irr}(E|\phi)$. If r is even, and we choose $\epsilon_\phi = \pm 1$ arbitrarily, then there is a unique bijection $\psi \mapsto \psi_{(E')}$ of $\mathrm{Irr}(E|\phi)$ onto $\mathrm{Irr}(E'|\phi')$ such that (1.1) holds for all $\psi \in \mathrm{Irr}(E|\phi)$. In both cases we have

$$(\lambda\psi)_{(E')} = \lambda\psi_{(E')}$$

for any $\lambda \in \hat{F}$ *and and* $\psi \in \mathrm{Irr}(E|\phi)$. *Furthermore, the resulting bijection is independent of the choice of the non-trivial character* $\lambda_q \in \hat{F}_q$, *for any* $q \in \pi(F)$.

Assume Hypothesis 1. We call $\rho(E,G,E',G')$ the Dade correspondence, where $\rho(E,G,E',G')$ denotes the identity map of $\mathrm{Irr}^E(G)$ when $|F|=1$. Following the notations in [7], for $\phi' \in \mathrm{Irr}^{E'}(G)$, we set $\phi'_{(G)} = \rho(E,G,E',G')^{-1}(\phi')$, and for $\psi' \in \mathrm{Irr}(E'|\phi')$, we set $\psi'_{(E)} = \psi$ if $\psi' = \psi_{(E')}$. From (1.1) ψ' is a constituent of $(\lambda\psi'_{(E)})_{E'}$ for some $\lambda \in \hat{F}$, hence $\phi_{(G')}$ is a constituent of $\phi_{G'}$. In particular if ϕ is the trivial character of G, then $\phi_{(G')}$ is the trivial character of G'.

The Generalized Glauberman case: Let G and A be finite groups such that A is cyclic, A acts on G via automorphism and that $(|C_G(A)|,|A|)=1$. We set $E = G \rtimes A$, $G' = C_G(A)$ and $E' = G' \times A \leq E$. By [2], Lemma 7.5, E, G, E' and G' satisfy Hypothesis 1. Moreover by [2], Proposition 7.8, if $(|A|,|G|)=1$, then $\rho(E,G,E',G')$ coincides with the Glauberman correspondence.

Theorem 2 (Horimoto[4]) *Assume the generalized Glauberman case. Suppose that* $p \nmid |A|$ *and that a Sylow p-subgroup of G is contained in* G'. *Then there is an isotypy between* $b(G)$ *and* $b(G')$ *induced by the Dade correspondence where* $b(G)$ *is the principal block of* G.

Isotypy is a concept introduced in [1].

Hypothesis 2 *Assume Hypothesis 1.* $(p,r)=1$. b *is an* E*-invariant block of* G *covered by* r *distinct blocks of* E.

Hypothesis 3 *Assume Hypothesis 1.* $(p,r)=1$. b' *is an* E'*-invariant block of* G' *covered by* r *distinct blocks of* E'.

Theorem 3 (Tasaka [7], Theorem 5.5) *Assume Hypotheses 2 and 3, and r is a prime power. Moreover assume some* $\phi \in \mathrm{Irr}(b)$, $\phi_{(G')} \in \mathrm{Irr}(b')$. *If r is odd, or* $r=2$, *or b is the principal block of G, then there is an isotypy between b and* b' *induced by the Dade correspondence.*

In this report we state that the arguments in [7] can be extended to the general case (see Theorem 8 below).

2 Dade correspondence and blocks

Let G be a finite group. We denote by $G_0(\mathcal{K}G)$ the Grothendieck group of the group algebra $\mathcal{K}G$. If L is a $\mathcal{K}G$-module, then let $[L]$ denote the element in $G_0(\mathcal{K}G)$ determined by the isomorphism class of L. For $\phi \in \mathrm{Irr}(G)$, we denote by $\check{\phi}$. For a block b of G, we denote by $\mathrm{Irr}(b)$ the set of irreducible characters belonging to b, and by $\mathcal{R}_{\mathcal{K}}(G,b)$ the additive group of generalized characters belonging to b. For other notations, see [5] and [8].

Note that under the Hypothesis 2, any irreducible character in $\mathrm{Irr}(b)$ is E- invariant.

Theorem 4 (see [7], Proposition 3.5)
(i) *Assume Hypothesis 2. Then* $\{\phi_{(G')} | \phi \in \mathrm{Irr}(b)\}$ *is contained in a block* $b_{(G')}$ *of* G'.
(ii) *Assume Hypothesis 3. Then* $\{\phi'_{(G)} | \phi' \in \mathrm{Irr}(b')\}$ *is contained in a block* $b'_{(G)}$ *of* G.

Assume Hypothesis 2. We denote by $\hat{b}_0$ a block of E covering b. For each $\phi \in \mathrm{Irr}(b)$, we denote $\hat{\phi}$ by a unique extension of ϕ which belongs to $\hat{b}_0$. For any $i \in \mathbf{Z}$, we denote by $\hat{b}_i$ be the block of E which contains $\lambda^i\hat{\phi}$ where $\phi \in \mathrm{Irr}(b)$.

Proposition 1 (see [7], Proposition 3.5, (3)) *Assume Hypotheses 2 and 3, and assume* $b' = b_{(G')}$ *using the notation in Theorem 4. Then there exists a block* $(\hat{b}_0)_{(E')}$ *of* E' *such that* $\mathrm{Irr}\ ((\hat{b}_0)_{(E')}) = \{(\hat{\phi})_{(E')} \mid \phi \in \mathrm{Irr}(b)\}$. *If* r *is odd, then* $(\hat{b}_0)_{(E')}$ *is uniquely determined, and if* r *is even, we have exactly two choices for* $(\hat{b}_0)_{(E')}$.

With the notation in the above proposition, we denote by $(\hat{b}_i)_{(E')}$ the block of E' containing $\lambda^i(\hat{\phi})_{(E')}$ ($\phi \in \mathrm{Irr}(b)$). Moreover, when r is even, we fix one of two $(\hat{b}_0)_{(E')}$.

3 Local structure

Lemma 1 ([7], Lemma 3.3)) *Assume* $p \nmid r$. *For a block* b *of* G, b *satisfies Hypothesis 2 if and only if there exists* $s \in E_0$ *such that* $\widehat{C(s)}b$ *is invertible in* $Z(\mathcal{O}Eb)$.

Assume Hypothesis 2. By the above lemma and [7], Lemma 2.4, there exists an element $s \in E'_0$ such that $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$. Hence there exists a defect group D of b centralized by s, and hence contained in G'. Let $P \le D$. Then by [7], Lemma 3.9, $C_E(P)$, $C_G(P)$, $C_{E'}(P)$ and $C_{G'}(P)$ satisfy Hypothesis 1. Here we note $F \cong C_E(P)/C_G(P)$. Let $e \in Bl(C_G(P), b)$. Then we see that $\mathrm{Br}_P^{\mathcal{O}E}(\widehat{C(s)}b)e^* \in (Z(kC_E(P)e^*))^\times$. This implies that e is covered by r blocks of $C_E(P')$. Similarly assume Hypothesis 3. Let D' be a defect group of b' and $e' \in \mathrm{Bl}(C_{G'}(P'), b')$ for a subgroup P' of D'. Then e' is covered by r blocks of $C_{E'}(P')$.

Theorem 5 (see [7], Proposition 3.11) *Using the same notations as in Theorem 4 we have the following.*
(i) *Assume Hypothesis 2. Let* D *be a defect group of* b *obtained in the above and let* $P \le D$. *Let* $e \in \mathrm{Bl}(C_G(P), b)$. *Then* $e_{(C_{G'}(P))} \in \mathrm{Bl}(C_{G'}(P), b_{(G')})$. *In particular,* $b_{(G')}$ *have a defect group containing* D.
(ii) *Assume Hypothesis 3. Let* D' *be a defect group of* b' *and let* $P' \le D'$. *Let* $e' \in \mathrm{Bl}(C_{G'}(P'), b')$. *Then* $e'_{(C_G(P'))} \in \mathrm{Bl}(C_G(P'), b'_{(G)})$. *In particular,* $b'_{(G)}$ *have a defect group containing* D'.

Assume Hypotheses 2 and 3, and $b' = b_{(G')}$. The Dade correspondence $\rho(E, G, E', G')$ gives a bijection between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(b')$ by Theorem 4. By Theorem 5, b and b' have a common defect group D. Let (D, b_D) be a maximal b-Brauer pair. For $P \le D$, let (P, b_P) be a b-Brauer pair contained in (D, b_D). We set

$$(b_P)' = (b_P)_{(C_{G'}(P))}.$$

By the above theorem $(b_P)'$ is associated with b' and $(D, (b_D)')$ is a maximal b'-Brauer pair. The following holds.

Theorem 6 (see [7], Theorem 5.2) *Assume Hypotheses 2 and 3, and assume* $b' = b_{(G')}$. *Then the Brauer categories* $\mathbf{B}_G(b)$ *and* $\mathbf{B}_{G'}(b')$ *are equivalent.*

4 Perfect isometry and isotypy

Assume Hypotheses 2 and 3, and $b' = b_{(G')}$ using the notations in Theorem 4. With the notations in the previous section, we put

$$b_i = \sum_{l=0}^{r-1} (\hat{b}_l)_{(E')} \hat{b}_{l+i} \quad (\forall i \in \mathbf{Z}).$$

Then $(b_i)^2 = b_i$ and $b_i \in (\mathcal{O}Gbb')^{E'}$ for each i. For each prime $q \in \pi(F)$, let $\lambda_q \in \hat{F}_q$ be a non-trivial character as in Theorem 1. Set $l = |\pi(F)|$. Moreover we set for t $(1 \leq t \leq l)$ distinct primes $q_1, q_2, \cdots, q_t \in \pi(F)$

$$\lambda_{q_1} \cdots \lambda_{q_t} = \lambda^{m_{\{q_1,\cdots,q_t\}}} \quad (m_{\{q_1,\cdots,q_t\}} \in \mathbf{Z})$$

where λ is a generator of $\hat{F}$. Then we have

$$\prod_{q \in \pi(F)} (1 - \lambda_q) = 1 + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1,\cdots,q_t\} \subseteq \pi(F)} \lambda^{m_{\{q_1,\cdots,q_t\}}}$$

where $\{q_1, \cdots, q_t\}$ runs over the set of t-element subsets of $\pi(F)$.

Proposition 2 (see [7], Proposition 4.4) *With the above notations we have*

$$[b_0\mathcal{K}G] + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1,\cdots,q_t\} \subseteq \pi(F)} [b_{m_{\{q_1,\cdots,q_t\}}}\mathcal{K}G]$$
$$= \sum_{\phi \in \mathrm{Irr}(b)} \epsilon_\phi [L_{\phi_{(G')}} \otimes_{\mathcal{K}} L_{\check{\phi}}]$$

in $G_0(\mathcal{K}(G' \times G))$.

From the above proposition and [1], Proposition 1.2, we have the following.

Theorem 7 (see [7], Theorem 4.5) *Assume Hypotheses 2 and 3, and that* $b' = b_{(G')}$. *Set* $\mu = \sum_{\phi \in \mathrm{Irr}(b)} \epsilon_\phi \phi_{(G')} \phi$. *Then* μ *induces a perfect isometry* $R_\mu : \mathcal{R}_{\mathcal{K}}(G, b) \to \mathcal{R}_{\mathcal{K}}(G', b')$ *which satisfies* $R_\mu(\phi) = \epsilon_\phi \phi_{(G')}$.

Let D be a common defect group of b and b'. For $P \leq D$, R^P be the perfect isometry between $\mathcal{R}_{\mathcal{K}}(C_G(P), b_P)$ and $\mathcal{R}_{\mathcal{K}}(C_{G'}(P), (b_P)_{(C_{G'}(P))})$ obtained by the Dade correspondence.

Theorem 8 (see [7], Theorem 5.5) *Assume Hypotheses 2 and 3, and assume* $b' = b_{(G')}$. *Then* b *and* b' *are isotypic with the local system* $(R^P)_{\{P(cyclic) \leq D\}}$.

Example Suppose $p = 5$. Let $G = Sz(2^{2n+1})$, the Suzuki group, $A = \langle \sigma \rangle$ where σ is the Frobenius automorphism of G with respect to $\mathrm{GF}(2^{2n+1})/\mathrm{GF}(2)$. Set $G' = Sz(2) = C_G(A)$, $E = G \rtimes A$, $E' = G' \times A$. Suppose that $5 \nmid 2n+1$. Then $(2n+1, |G'|) = 1$. Moreover a Sylow 5-subgroup of G has order 5. By the above theorem, the Dade correspondence gives an isotypy between $b(G)$ and $b(G')$. Moreover, if $5 \mid (2^{2n+1} + 2^{n+1} + 1)$, then $b(G)$ and $b(G')$ are splendidly Morita equivalent.

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