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Recent Diophantine results on zeta values: a survey

Michel Waldschmidt *
University of Paris VI, Institut de Mathématiques de Jussieu, France
http://www.math.jussieu.fr/~miw/

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Abstract

After the proof by R. Apéry of the irrationality of $\zeta(3)$ in 1976, a number of articles have been devoted to the study of Diophantine properties of values of the Riemann zeta function at positive integers. A survey has been written by S. Fischler for the Bourbaki Seminar in November 2002 [6].

Here, we review more recent results, including contributions by P. Bundschuh, S. Bruiltet, C. Elsner, S. Fischler, S. Gun, M. Hata, C. Krattenthaler, R. Marcovecchio, R. Murty, Yu.V. Nesterenko, P. Philippon, P. Rath, G. Rhin, T. Rivoal, S. Shimomura, I. Shiokawa, C. Viola, W. Zudilin. We plan also to say a few words on the analog of this theory in finite characteristic, with works of G. Anderson, W.D. Brownawell, M. Pappanikolas, D. Thakur, Chieh-Yu Chang, Jing Yu.

1 Special values of the Riemann zeta function

Several zeta functions exist, including Riemann zeta function, Multizeta functions, Weierstraß zeta function, those of Fibonacci, Hurwitz, Carlitz, Dedekind, Hasse-Weil, Lerch, Selberg, Witten, Milnor and the zeta functions of dynamical systems...

1.1 The Riemann zeta function

We first review the *Riemann zeta function*, which was previously introduced by L. Euler:

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^{s}}$$

for $s\in \mathbb{R}, s\geq 2$. He showed the *Euler product* :

$$\zeta(s)=\prod_{p}\frac{1}{1-p^{-s}}\cdot$$

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This text is available on the web site of the author at the address
http://www.math.jussieu.fr/~miw/articles/pdf/ZetaValuesRIMS2009.pdf

In 1739, Euler calculated the special value of $\zeta(s)$ for s even integers. He found

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \cdots$$

and proved $\pi^{-2k}\zeta(2k) \in \mathbb{Q}$ for $k \geq 1$.

More precisely, he proved that the values of the Riemann zeta function at even integers are related with the Bernoulli numbers by

$$\zeta(2k) = (-1)^{k-1} 2^{2k-1} \frac{B_{2k}}{(2k)!} \pi^{2k}$$

for $k \geq 1$, where B_n is the n-th Bernoulli number, a rational number defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

In 1882, F. Lindemann showed that π is transcendental, hence, the value $\zeta(2k)$ is also transcendental ($k = 1, 2, 3, \cdots$). We have in general:

Theorem 1 (Hermite–Lindemann) *For any non-zero complex number z, one at least of the two numbers z and* $\exp z$ *is transcendental.*

Corollary 2 *Let β be non-zero algebraic (complex) number. Then e^β is also transcendental.*

Corollary 3 *Let α be non-zero algebraic (complex) number. Suppose* $\log \alpha \neq 0$. *Then* $\log \alpha$ *is transcendental.*

The transcendence of π follows from $e^{i\pi} = -1$.

1.2 The values of the Riemann zeta function at odd integers

The next question deals with the values of the Riemann zeta function at positive odd integers $\zeta(2k+1)$ for $k = 1, 2, 3, \ldots$

The following irrationality question is still open:

Conjecture 1 *For all $k \in \mathbb{Z}_{>0}$, the numbers $\zeta(2k+1)$ and $\dfrac{\zeta(2k+1)}{\pi^{2k+1}}$ are both irrational.*

We may ask a more difficult problem:

Conjecture 2 *The numbers*

$$\pi, \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \cdots$$

are algebraically independent.

In particular, the numbers $\zeta(2k+1)$ and $\zeta(2k+1)/\pi^{2k+1}$ for $k \geq 1$ are conjectured to be all transcendental.

The first non-trivial result on $\zeta(2k+1)$ has been obtained by R. Apéry in 1978:

Theorem 4 (Apéry, 1978) *The number*

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1.202\,056\,903\,159\,594\,285\,399\,738\,161\,511\cdots$$

is irrational.

The next breakthrough is due to T. Rivoal [6].

Theorem 5 (Rivoal 2000) *Let $\epsilon > 0$. For any sufficiently large odd integer a, the dimension of the $\mathbb{Q}$-vector space spanned by the numbers 1, $\zeta(3)$, $\zeta(5)$, $\cdots,\zeta(a)$ is at least $\frac{1-\epsilon}{1+\log 2}\log a$.*

Corollary 6 *There are infinitely many $k \in \mathbb{Z}_{>0}$, such that $\zeta(2k+1)$ are irrational.*

W. Zudilin then refined the result to show:

Theorem 7 (Zudilin 2004) *At least one of the 4 numbers*

$$\zeta(5),\quad \zeta(7),\quad \zeta(9),\quad \zeta(11)$$

is irrational.

He also showed that there exists an odd integer $5 \leq j \leq 69$, such that the three numbers $1, \zeta(3), \zeta(j)$ are linearly independent over $\mathbb{Q}$. See the survey [6] on the irrationality of zeta values by S. Fischler.

2 Irrationality measures

Definition 1 (irrationality exponent) *Let $\vartheta \in \mathbb{R}$. Assume that ϑ is irrational. Define $\mu = \mu(\vartheta) > 0$ as the least positive exponent such that for any $\varepsilon > 0$ there exists a constant $q_0 = q_0(\varepsilon) > 0$ for which*

$$\left|\vartheta - \frac{p}{q}\right| \geq \frac{1}{q^{\mu+\epsilon}}$$

holds for all integers p and q with $q > q_0$.

According to Dirichlet's box principle, $\mu(\theta) \geq 2$ for all irrational real numbers θ. On the other hand, $\mu = \mu(\vartheta)$ is finite if and only if ϑ is not a Liouville number.

Here is a table of some of the most recent results.

ϑ	year	author	$\mu(\vartheta) <$
π	2008	V.Kh. Salikhov	7.6063085
$\zeta(2) = \pi^2/6$	1996	G. Rhin and C. Viola	5.441243
$\zeta(3)$	2001	G. Rhin and C. Viola	5.513891
$\log 2$	2008	R. Marcovecchio	3.57455391

It is easy to see that a bound $\mu(\vartheta^2) \leq \kappa$ for some $\vartheta \in \mathbb{R}$ implies $\mu(\vartheta) \leq 2\kappa$. Hence, the result of G. Rhin and C. Viola $\mu(\zeta(2)) \leq 5.441\cdots$ implies only $\mu(\pi) \leq 11.882\cdots$. However, an upper bound for $\mu(\vartheta)$ does not yield any bound for $\mu(\vartheta^2)$.

Historically, the upper estimates for the irrationality exponent of π are as follows:

- K. Mahler (1953) : π is not a Liouville number and $\mu(\pi) \leq 30$.
- M. Mignotte (1974) : $\mu(\pi) \leq 20$
- G.V. Chudnovsky (1984) : $\mu(\pi) \leq 14.5$.
- M. Hata (1992) : $\mu(\pi) \leq 8.0161$.
- V.Kh. Salikhov (2008) : $\mu(\pi) \leq 7.6063$.

For $\zeta(2)$ and $\zeta(3)$, we have the following records.

- R. Apéry (1978), F. Beukers (1979) : $\mu(\zeta(2)) < 11.85$ and $\mu(\zeta(3)) < 13.41$.
- R. Dvornicich and C. Viola (1987) : $\mu(\zeta(2)) < 10.02$ and $\mu(\zeta(3)) < 12.74$.
- M. Hata (1990) : $\mu(\zeta(2)) < 7.52$ and $\mu(\zeta(3)) < 8.83$.
- G. Rhin and C. Viola (1993) : $\mu(\zeta(2)) < 7.39$.
- G. Rhin and C. Viola (1996) : $\mu(\zeta(2)) < 5.44$.
- G. Rhin and C. Viola (2001) : $\mu(\zeta(3)) < 5.51$.

The Hermite–Lindemann Theorem implies the transcendence of π (hence, of $\zeta(2)$) and of $\log 2$. Transcendence measures of $\log 2$ have been investigated by K. Mahler, A. Baker, A.O. Gel'fond, N.I. Fel'dman. As far as the irrationality exponent of $\log 2$ is concerned, the recent results are:

- G. Rhin (1987) : $\mu(\log 2) < 4.07$.
- E.A. Rukhadze (1987) : $\mu(\log 2) < 3.89$.
- R. Marcovecchio (2008) : $\mu(\log 2) < 3.57$.

See the references [10, 12, 13, 14].

In the proof by T. Rivoal of his Theorem 5, a linear independence criterion of Yu.V. Nesterenko was an essential tool. We state here only a qualitative form:

Theorem 8 (Nesterenko, 1985) *Let m be a positive integer and α a positive real number satisfying $\alpha > m-1$. Assume there is a sequence $(L_n)_{n\geq 0}$ of linear forms in $\mathbb{Z}X_0 + \mathbb{Z}X_1 + \ldots + \mathbb{Z}X_m$ of height $\leq e^n$, such that*

$$|L_n(1, \vartheta_1, \ldots, \vartheta_m)| = e^{-\alpha n + o(n)}.$$

Then $1, \vartheta_1, \ldots, \vartheta_m$ are linearly independent over $\mathbb{Q}$.

Recently, S. Fischler and W. Zudilin [7] obtained a refinement of Nesterenko's linear independence criterion 8, which was the source of the paper [2] by A. Chantanasiri, and which they used to prove:

Theorem 9 (Fischler and Zudilin, 2009) *There exist positive odd integers $i \leq 139$ and $j \leq 1961$, such that the numbers 1, $\zeta(3)$, $\zeta(i)$, $\zeta(j)$ are linearly independent over $\mathbb{Q}$.*

Similarly, there exist positive odd integers $i \leq 93$ and $j \leq 1151$, such that the numbers 1, $\log 2$, $\zeta(i)$, $\zeta(j)$ are linearly independent over $\mathbb{Q}$.

Multizeta values are defined by

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

for $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$. The reference [1] provides information on recent developments. Also M. Hoffman's web site

`http://www.usna.edu/Users/math/meh/biblio.html`

is a much valuable source of information. A further reference by J. Blümlein, D.J. Broadhurst and J.A.M. Vermaseren *The Multiple Zeta Value Data Mine* is `arXiv:0907.2557v1`

3 Gamma and Beta values

3.1 Gamma and Beta functions

Let us recall the definition of the *Euler Gamma function*:

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \cdot \frac{dt}{t} = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}.$$

Here, γ is *Euler constant* (also called *Euler-Mascheroni constant*):

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.577\,215\,664\,9\ldots$$

The *Euler Beta function* is defined by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

3.2 Weierstraß functions

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. The canonical product attached to Ω is the *Weierstraß sigma function*

$$\sigma(z) = \sigma_\Omega(z) = z \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{(z/\omega) + (z^2/2\omega^2)}.$$

The logarithmic derivative of the sigma function is the *Weierstraß zeta function* $\dfrac{\sigma'}{\sigma} = \zeta$ and the derivative of ζ is $-\wp$, where $\wp$ is the *Weierstraß elliptic function*:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \quad \wp(z+\omega) = \wp(z) \quad (\omega \in \Omega).$$

The Weierstraß zeta function is quasi–periodic: for any $\omega \in \Omega$ there exists an associated *quasi-period* η such that

$$\zeta(z+\omega) = \zeta(z) + \eta.$$

The first transcendence results on the periods and quasi–periods are due to C.L. Siegel and then to Th. Schneider. Linear independence results over the field of algebraic numbers have been investigated by D.W. Masser. Algebraic independence results are due to G.V. Chudnovskii and Yu.V. Nesterenko.

3.3 Transcendence

In 1934, Th. Schneider showed that the numbers $\Gamma(1/4)^4/\pi^3$ and $\Gamma(1/3)^3/\pi^2$ are transcendental. Indeed, they are not Liouville numbers by means of lower bounds for linear combinations of elliptic logarithms (by A. Baker, J. Coates, M. Anderson in the CM case, by Philippon-Waldschmidt in the general case, refinements are due to N. Hirata-Kohno, S. David, E. Gaudron). S. Lang observed that lower bounds for linear forms in elliptic logarithms are useful for solving Diophantine equations (integer points on elliptic curves).

For a historical survey, see the articles by S. David and N. Hirata-Kohno [3, 4, 5].

Th. Schneider also showed in 1948 that for $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ with $a, b, a+b \notin \mathbb{Z}$, the number $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is transcendental. The proof involves Abelian integrals of higher genus, related with the Jacobian of Fermat curves.

3.4 Algebraic independence

In 1978, G.V. Chudnovsky proved that two at least of the numbers g_2, g_3, ω_1, ω_2, η_1, η_2 are algebraically independent. As a corollary, the numbers π and $\Gamma(1/4) = 3.625\,609\,908\,2\ldots$ are algebraically independent. A transcendence measure for $\Gamma(1/4)$ has been obtained by P. Philippon and refined by S. Bruiltet.

Theorem 10 (Philippon and Bruiltet) *For $P \in \mathbb{Z}[X,Y]$ with degree d and height H, we have*

$$\log|P(\pi,\Gamma(1/4))| > -10^{326}(\log H + d\log(d+1))d^2(\log(d+1))^2.$$

Corollary 11 *The number $\Gamma(1/4)$ is not a Liouville number:*

$$\left|\Gamma(1/4) - \frac{p}{q}\right| > \frac{1}{q^{10^{330}}}.$$

The results due to K.G. Vasil'ev in 1996, P. Grinspan in 2002 show that two at least of the three numbers π, $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent.

The proof by Chudnovsky's method involves a simple factor of dimension 2 of the Jacobian of the Fermat curve $X^5 + Y^5 = Z^5$ which is an Abelian variety of dimension 6.

Further developments are again made by Yu.V. Nesterenko. Let us consider Ramanujan functions:

Definition 2 (Ramanujan functions) *Ramanujan functions are defined for $q \in \mathbb{C}$ with $0 < |q| < 1$ by*

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \quad Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad R(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}.$$

Since Eisenstein series are given for $q \in \mathbb{C}$ with $0 < |q| < 1$ by

$$E_{2k}(q) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n},$$

we have

$$P(q) = E_2(q), \quad Q(q) = E_4(q), \quad R(q) = E_6(q).$$

Here are examples of special values.

When $\tau = i$ we have $q = e^{-2\pi}$ and

$$\omega_1 = \frac{\Gamma(1/4)^2}{\sqrt{8\pi}} = 2.622\,057\,554\,2\ldots$$

In this case

$$P(q) = \frac{3}{\pi}, \quad Q(q) = 3\left(\frac{\omega_1}{\pi}\right)^4, \quad R(q) = 0.$$

When $\tau = \rho$, we have $q = -e^{-\pi\sqrt{3}}$ and

$$\omega_1 = \frac{\Gamma(1/3)^3}{2^{4/3}\pi} = 2.428\,650\,648\ldots$$

In this case

$$P(q) = \frac{2\sqrt{3}}{\pi}, \quad Q(q) = 0, \quad R(q) = \frac{27}{2}\left(\frac{\omega_1}{\pi}\right)^6.$$

Theorem 12 (Nesterenko, 1996) *For any $q \in \mathbb{C}$ with $0 < |q| < 1$, three at least of the four numbers $q, P(q), Q(q), R(q)$ are algebraically independent.*

K. Mahler showed that the functions $P(q), Q(q), R(q)$ are algebraicaly independent over $\mathbb{C}(q)$. An important tool in the proof of Nesterenko's Theorem is that these functions satisfy the following system of differential equations for $D = q \cdot \frac{d}{dq}$:

$$12\frac{DP}{P} = P - \frac{Q}{P}, \quad 3\frac{DQ}{Q} = P - \frac{R}{Q}, \quad 2\frac{DR}{R} = P - \frac{Q^2}{R}.$$

We have the following consequences of Theorem 12:

Corollary 13 *The three numbers π,, e^{π} and $\Gamma(1/4)$ are algebraically independent.*

Corollary 14 *The three numbers π,, $e^{\pi\sqrt{3}}$ and $\Gamma(1/3)$ are algebraically independent.*

Corollary 15 *The following special value of Weierstraß sigma function*

$$\sigma_{\mathbb{Z}[i]}(1/2) = 2^{5/4}\pi^{1/2}e^{\pi/8}\Gamma(1/4)^{-2}$$

is transcendental.

Another consequence of Nesterenko's Theorem 12 concerns the *Fibonacci zeta function*, which is defined for $\Re e(s) > 0$ by

$$\zeta_F(s) = \sum_{n \geq 1} \frac{1}{F_n^s},$$

where $\{F_n\}$ is the Fibonacci sequence: $F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}\ (n \geq 1)$.

Theorem 16 (Elsner, Shimomura, Shiokawa, 2006) *The values $\zeta_F(2), \zeta_F(4), \zeta_F(6)$ are algebraically independent.*

4 Conjectures

4.1 Gamma values

We come back to Gamma values. Here are the three standard relations among the values of the Gamma function.

1) *Translation* :

$$\Gamma(a+1) = a\Gamma(a),$$

2) *Reflexion* :

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}.$$

3) *Multiplication* : for any positive integer n, we have

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)} \Gamma(na).$$

It is expected that these relation among Gamma values are essentially the only ones:

Conjecture 3 (Rohrlich's Conjecture) *Any multiplicative relation*

$$\pi^{b/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{Q}$$

with b and m_a in $\mathbb{Z}$ lies in the ideal generated by the standard relations.

Examples are

$$\Gamma\left(\frac{1}{14}\right)\Gamma\left(\frac{9}{14}\right)\Gamma\left(\frac{11}{14}\right) = 4\pi^{3/2}$$

and

$$\prod_{\substack{1 \le k \le n \\ (k,n)=1}} \Gamma(k/n) = \begin{cases} (2\pi)^{\varphi(n)/2}/\sqrt{p} & \text{if } n = p^r \text{ is a prime power,} \\ (2\pi)^{\varphi(n)/2} & \text{otherwise.} \end{cases}$$

S. Lang suggested a stronger conjecture than Rohrlich's one:

Conjecture 4 (Lang) *Any algebraic dependence relation among the numbers $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ lies in the ideal generated by the standard relations. In other terms, the Gamma function defines a* universal odd distribution.

From the Rohrlich-Lang Conjecture 4, one deduces the following statement:

Consequence
For any $q > 1$, the transcendence degree of the field generated by numbers

$$\pi, \quad \Gamma(a/q) \quad 1 \le a \le q, \ (a,q) = 1$$

is $1 + \varphi(q)/2$.

A variant of the Rohrlich–Lang Conjecture 4 is:

Conjecture 5 (Gun, Murty, Rath, 2009) *For any $q > 1$, the numbers*

$$\log \Gamma(a/q) \quad 1 \le a \le q,\ (a,q) = 1$$

are linearly independent over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

A consequence is that *for any $q > 1$, there is at most one primitive odd character χ modulo q for which $L'(1,\chi) = 0$.*

Other transcendence related results involving zeta and Gamma values are as folows:

Theorem 17 (Bundschuh, 1979) *For $p/q \in \mathbb{Q}$ with $0 < |p/q| < 1$, the number*

$$\sum_{n=2}^{\infty} \zeta(n)(p/q)^n$$

is transcendental.

Further, for $p/q \in \mathbb{Q} \setminus \mathbb{Z}$, the number

$$\frac{\Gamma'}{\Gamma}\left(\frac{p}{q}\right) + \gamma$$

is transcendental.

4.2 Arithmetic nature of the sum of the values of a rational function at the positive integers

An interesting problem is to investigate the arithmetic nature of the numbers of the form

$$\sum_{n \ge 1} \frac{A(n)}{B(n)} \quad \text{when} \quad \frac{A}{B} \in \mathbb{Q}(X).$$

In case B has distinct rational zeroes, by decomposing A/B in simple fractions, one gets linear combinations of logarithms of algebraic numbers; thus we can use Baker's Theorem on the linear independence of logarithms of algebraic numbers. The example $A(X)/B(X) = 1/X^3$ shows that the general case is hard.

Using Nesterenko's Theorem 12, one deduces from the work by P. Bundschuh in 1979 the following result:

Theorem 18 *The number*

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

is transcendental. Hence, the number $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^s - 1}$ is transcendental over $\mathbb{Q}$ for $s = 4$.

Notice that

$$\sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4},$$

since this is a telescoping series:

$$\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

The transcendence of the number $\sum_{n=2}^{\infty} \frac{1}{n^s-1}$ for even integers $s \geq 4$ would follow from Schanuel's Conjecture.

See also the works by S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman in 2001, and by S. Gun, R. Murty and P. Rath in 2009.

4.3 Hurwitz zeta function

Definition 3 (Hurwitz zeta function) *For $z \in \mathbb{C}, z \neq 0$ and $\Re e(s) > 1$, the function*

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$$

is called Hurwitz zeta function.

It generalizes the Riemann zeta function, since $\zeta(s,1) = \zeta(s)$. The following Conjecture by S. Chowla and J.W. Milnor deals with the values of the Hurwitz zeta function.

Conjecture 6 (Chowla-Milnor) *For k and q integers > 1, the $\varphi(q)$ numbers*

$$\zeta(k, a/q) \quad (1 \leq a \leq q, (a,q) = 1)$$

are linearly independent over $\mathbb{Q}$.

The Chowla-Milnor Conjecture 6 for $q = 4$ implies the irrationality of the numbers

$$\zeta(2n+1)/\pi^{2n+1}$$

for $n \geq 1$. A stronger form of Conjecture 6 has been proposed in 2009 by S. Gun, R. Murty and P. Rath:

Conjecture 7 (Strong Chowla-Milnor Conjecture) *For k and q integers > 1, the $1+\varphi(q)$ numbers*

$$1 \quad \textit{and} \quad \zeta(k, a/q) \quad (1 \leq a \leq q, (a,q) = 1)$$

are linearly independent over $\mathbb{Q}$.

For $k > 1$ odd, the number $\zeta(k)$ is irrational if and only if the strong Chowla-Milnor Conjecture 7 holds for this value of k and for at least one of the two values $q = 3$ and $q = 4$.

Hence, the strong Chowla-Milnor Conjecture 7 holds for $k = 3$ (by Apéry) and also for infinitely many k by T. Rivoal's Theorem 5.

4.4 Polylogarithms and digamma functions

The next conjecture, dealing with the values of polylogarithms, has been proposed by S. Gun, R. Murty and P. Rath. We first define:

Definition 4 (Polylogarithms) *The function* $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. *defined for* $k \geq 1$ *and* $|z| < 1$ *is called the* polylogarithm *function.*

By definition, we have $\mathrm{Li}_1(z) = \log(1-z)$ and $\mathrm{Li}_k(1) = \zeta(k)$ for $k \geq 2$.

Conjecture 8 (Polylogarithms Conjecture of Gun, Murty and Rath) *Let* $k > 1$ *be an integer and* $\alpha_1, \dots, \alpha_n$ *algebraic numbers such that* $\mathrm{Li}_k(\alpha_1), \dots, \mathrm{Li}_k(\alpha_n)$ *are linearly independent over* $\mathbb{Q}$*. Then these numbers* $\mathrm{Li}_k(\alpha_1), \dots, \mathrm{Li}_k(\alpha_n)$ *are linearly independent over the field* $\overline{\mathbb{Q}}$ *of algebraic numbers.*

If this Polylog Conjecture 8 is true, then the Chowla-Milnor Conjecture 7 is true for all k and all q.

Definition 5 (Digamma function) *For* $z \in \mathbb{C}, z \neq 0, -1, -2, \dots,$ *the* digamma function *is defined by*

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We have

$$\psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right)$$

and

$$\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}.$$

Some special values of the digamma function are

$$\psi(1) = -\gamma, \quad \psi\left(\frac{1}{2}\right) = -2\log(2) - \gamma,$$

$$\psi\left(2k - \frac{1}{2}\right) = -2\log(2) - \gamma + \sum_{n=1}^{2k-1} \frac{1}{n+1/2},$$

$$\psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - 3\log(2) - \gamma \quad \text{and} \quad \psi\left(\frac{3}{4}\right) = \frac{\pi}{2} - 3\log(2) - \gamma.$$

An example of a linear dependence relation among special values of the digamma function is

$$\psi(1) + \psi(1/4) - 3\psi(1/2) + \psi(3/4) = 0.$$

R. Murty and N. Saradha stated the following conjecture in 2007:

Conjecture 9 *Let* K *be a number field over which the* q*-th cyclotomic polynomial is irreducible. Then the* $\varphi(q)$ *numbers* $\psi(a/q)$ *with* $1 \leq a \leq q$ *and* $\gcd(a,q) = 1$ *are linearly independent over* K.

4.5 Baker periods

R. Murty and N. Saradha defined Baker periods as follows.

Definition 6 (Baker period) *Baker periods are elements of the $\overline{\mathbb{Q}}$-vector space spanned by the logarithms of algebraic numbers.*

Remark 1
By Baker's transcendence Theorem, a Baker period is either zero or else transcendental.

Remark 2
R. Murty and N. Saradha showed that one at least of the two following statements is true:
(1) Euler's Constant γ is not a Baker's period
(2) The $\varphi(q)$ numbers $\psi(a/q)$ with $1 \leq a \leq q$ and $\gcd(a, q) = 1$ are linearly independent over K, whenever K be a number field over which the q-th cyclotomic polynomial is irreducible.

4.6 Euler Constant

Few results concerning the arithmetic nature of Euler's constant γ are known.
(1) Jonathan Sondow showed

$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2\binom{t+k}{k}} dt, \quad \gamma = \lim_{s\to 1+} \sum_{n=1}^\infty \left(\frac{1}{n^s} - \frac{1}{s^n}\right)$$

and

$$\gamma = \int_1^\infty \frac{1}{2t(t+1)}\, {}_3F_2\begin{pmatrix} 1, & 2, & 2 \\ 3, & & t+2 \end{pmatrix} dt.$$

(2) A.I. Aptekarev (2007) obtained approximations to γ.
(3) T. Rivoal (2009) gave an approximation to the function $\gamma + \log x$ (consequently, approximations to γ and to $\zeta(2) - \gamma^2$).

The following problems are open:

Conjecture 10 (1) *Is the number γ irrational? Is it transcendental?*
(2)*(Kontsevich – Zagier) : is γ a "period"?*

5 Finite characteristic

We conclude with a few words concerning the finite characteristic situation, including the *Carlitz zeta values.* Set $A = \mathbb{F}_q[t]$, let A_+ be the subset of monic polynomials in A, P be the subset of prime polynomials in A_+, set $K = \mathbb{F}_q(t)$ and $K_\infty = \mathbb{F}_q((1/t))$.

Definition 7 (Carlitz zeta values) *For $s \in \mathbb{Z}$, define*

$$\zeta_A(s) = \sum_{a\in A_+} \frac{1}{a^s} = \prod_{p\in P}(1 - p^{-s})^{-1} \in K_\infty.$$

Definition 8 (Thakur Gamma function) *For $z \in K_\infty$, set*

$$\Gamma(z) = \frac{1}{z} \prod_{a \in A_+} \left(1 + \frac{z}{a}\right).$$

Independence results on the values of Thakur Gamma function in positive characteristic are known: linear independence of values of Gamma function has been investigated by W.D. Brownawell and M. Papanikolas (2002), algebraic independence results by W.D. Brownawell, M. Papanikolas and Gerg Anderson (2004).

Definition 9 (Carlitz zeta values at even A–integers) *Define*

$$\tilde{\pi} = (t - t^q)^{1/(q-1)} \prod_{n=1}^{\infty} \left(1 - \frac{t^{q^n} - t}{t^{q^{n+1}} - t}\right) \in K_\infty.$$

For m a multiple of $q-1$, the Carlitz - Bernoulli numbers are

$$\tilde{\pi}^{-m} \zeta_A(m) \in A.$$

G. Anderson, D. Thakur and Jing Yu obtained the following theorem.

Theorem 19 (Anderson, Thakur , Yu) *For m a positive integer, $\zeta_A(m)$ is transcendental over K. Moreover, for m a positive integer not a multiple of $q-1$, the quotient $\zeta_A(m)/\tilde{\pi}^m$ is transcendental over K.*

Further results are described in the report [11] by F. Pellarin, as well as in the following related preprints:

- Chieh-Yu Chang, Matthew A. Papanikolas and Jing Yu, *Geometric Gamma values and zeta values in positive characteristic*, in `arXiv:0905.2876`.
- Chieh-Yu Chang, Matthew A. Papanikolas, Dinesh S. Thakur and Jing Yu, *Algebraic independence of arithmetic gamma values and Carlitz zeta values* in `arXiv:0909.0096`.
- Chieh-Yu Chang. *Periods of third kind for rank* 2 *Drinfeld modules and algebraic independence of logarithms* in `arXiv:0909.0101`:

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Michel WALDSCHMIDT
Université P. et M. Curie (Paris VI)
Institut de Mathématiques CNRS UMR 7586
Théorie des Nombres Case 247
F–75005 PARIS
e-mail: `miw@math.jussieu.fr`
URL: `http://www.math.jussieu.fr/~miw/`