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ON THE DISTRIBUTION OF PISOT AND CNS POLYNOMIALS

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1. INTRODUCTION

This paper is the edited version of my talk, delivered at the RIMS conference "Analytic Number Theory", on 15 October, 2009. I thank the possibility to speak on that event and for the hospitality of RIMS.

Let \( d \geq 1 \) be an integer and \( \mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d \). Consider the mapping

\[
\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d; \text{ for } \mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d \text{ let }
\]

\[
\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -[\mathbf{r} \cdot \mathbf{a}]),
\]

where \( \mathbf{r} \cdot \mathbf{a} = r_1 a_1 + \cdots + r_d a_d \) denotes the inner product. We call \( \tau_{\mathbf{r}} \) a shift radix system (SRS for short) if for all \( \mathbf{a} \in \mathbb{Z}^d \) we can find some \( k > 0 \) with \( \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \). This concept was introduced by Akiyama et al. [1]. We proved that it is a common generalization of canonical number systems in residue class rings of polynomial rings (see [8, 10, 12]) as well as of \( \beta \)-expansions of real numbers, [13]. For the investigation of properties of SRS it turned out convenient to introduce some sets.

For \( d \in \mathbb{N} \), \( d \geq 1 \) let

\[
\mathcal{D}_d := \{ \mathbf{r} \in \mathbb{R}^d : \forall \mathbf{a} \in \mathbb{Z}^d (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \},
\]

\[
\mathcal{D}_d^0 := \{ \mathbf{r} \in \mathbb{R}^d : \forall \mathbf{a} \in \mathbb{Z}^d \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \}.
\]

It is clear that \( \mathcal{D}_d^0 \subset \mathcal{D}_d \) and \( \mathbf{r} \) is SRS iff \( \mathbf{r} \in \mathcal{D}_d^0 \). In [1] we proved among others that \( \mathcal{D}_d, \mathcal{D}_d^0 \) are Lebesgue measurable and \( \mathcal{D}_d^0 \) admits some convexity property. On the other hand the results of [2] showed that the boundary already of \( \mathcal{D}_d^0 \) is very complicated. Further we proved in [1] that we can embed the discrete sets of Pisot, Salem and CNS polynomials in these continues sets. In [3] and [4] we studied the distribution of Pisot, Salem and CNS polynomials. In the present paper we give a survey about the last mentioned results. Further we present the sketch of the proof one of the main results.

2. PISOT AND SALEM POLYNOMIALS

Let \( P(X) = X^d - b_1X^{d-1} - \cdots - b_d \in \mathbb{Z}[X] \).

- If all but one root of \( P \) is located in the open unit disc then \( P \) is called a Pisot polynomial. Its dominant root is called Pisot number.
- If all but one root of \( P \) is located in the closed unit disc and at least one of them has modulus 1 then \( P \) is called a Salem polynomial. Its dominant root is called Salem number.

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If $P$ is a Pisot or Salem polynomial, we will denote its dominating root by $\beta$.
Let $\text{Fin}(\beta)$ be the set of positive real numbers having finite greedy expansion with respect to $\beta$. We say that $\beta > 1$ has property (F) if

$$\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty).$$

It was shown by Frougny and Solomyak [7] that (F) can hold only for Pisot numbers $\beta$. Analogously to $\mathcal{D}_d$ and $\mathcal{D}_d^0$ define for each $d \in \mathbb{N}$, $d \geq 1$ the sets

$$\mathcal{B}_d = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : P(X) \text{ is a Pisot or Salem polynomial}\}$$

and

$$\mathcal{B}_d^0 = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : P(X) \text{ is a Pisot polynomial with property (F)}\},$$

where $P(X) = X^d - b_1 X^{d-1} - \cdots - b_d$. We obviously have $\mathcal{B}_d^0 \subseteq \mathcal{B}_d$.

If $(b_1, \ldots, b_d) \in \mathcal{B}_d$ then let $\beta$ be the dominating root of

$$P(X) = X^d - b_1 X^{d-1} - \cdots - b_d.$$

Consider the map $\psi : \mathcal{B}_d \to \mathbb{R}^{d-1}$:

$$\psi(b_1, \ldots, b_d) = (r_d, \ldots, r_2),$$

where $r_2, \ldots, r_d$ are such that

$$X^d - b_1 X^{d-1} - \cdots - b_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + \cdots + r_d).$$

As $(b_1, \ldots, b_d) \in \mathcal{B}_d$, the polynomial $X^{d-1} + r_2 X^{d-2} + \cdots + r_d$ has all its roots in the closed unit circle. Thus

$$\psi(\mathcal{B}_d) \subseteq \overline{\mathcal{D}_{d-1}}.$$

In [1] we proved:

$$\psi(\mathcal{B}_d^0) \subseteq \overline{\mathcal{D}_{d-1}^0}.$$

This means we can embed the discrete sets $\mathcal{B}_d$ and $\mathcal{B}_d^0$ in the continues sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ respectively, i.e, SRS can be considered as a generalization of the $\beta$-representations.

The sets $\mathcal{B}_d, \mathcal{B}_d^0$ are obviously discrete and infinite. To study their distribution we fix the first coordinate. More precisely, for $M \in \mathbb{N}_{>0}$ we set

$$\mathcal{B}_d(M) := \{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d\}$$

and

$$\mathcal{B}_d^0(M) := \{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d^0\}.$$

It is clear that $\mathcal{B}_d^0(M) \subseteq \mathcal{B}_d(M)$, moreover $\mathcal{B}_d(M)$ is finite. Indeed, as $M = \beta + \text{other roots of } X^d - MX^{d-1} - b_2 X^{d-2} - b_d$ and the roots of $X^d - MX^{d-1} - b_2 X^{d-2} - b_d$ except of $\beta$ are lying in the unit disc, thus $|\beta| \leq M + d - 1$. Hence there are easily computable constants $c_i(M, d)$ such that $|b_i| \leq c_i(M, d)$, which ensures the finiteness of $\mathcal{B}_d(M)$. With these notations we proved in [4] the following theorem.

**Theorem 1.** We have

$$\left| \frac{|\mathcal{B}_d(M)|}{M^{d-1}} - \lambda_{d-1}(\mathcal{D}_{d-1}) \right| = O(M^{-d+1+1/d}),$$

and

$$\lim_{M \to \infty} \frac{|\mathcal{B}_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_d^0).$$
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where \( \lambda_{d-1} \) denotes the \( d - 1 \)-dimensional Lebesgue measure and \( |A| \) the cardinality of the finite set \( A \).

Notice that (2) is weaker than (1). As the boundary of \( D_{d-1} \) is smooth, we were able to estimate accurately the number of images under \( \psi \) lying near to the boundary. This was not possible for \( D^0_{d-1} \), because its boundary is quite complicated.

In Theorem 1 and later in Theorem 2 the volume or Lebesgue measure of \( D_d \) appears in the main term. This was calculated by Fam [6]. Using the Barnes G-function we have

\[
\lambda_d(D_d) = \begin{cases} 
\frac{2^{2n^2 + n} \Gamma(n+1) G(n+1)^4}{G(2n+2)} & (d = 2n), \\
\frac{2^{2n^2 + 3n+1} G(n+2)^4}{\Gamma(n+1) G(2n+3)} & (d = 2n + 1).
\end{cases}
\]

Note that for positive integers the Barnes G-function equals the superfactorials: \( G(n + 2) = \prod_{j=1}^{n} j! \) for \( n \in \mathbb{N} \). Moreover, observe that by [6, Formula (2.13)] we have \( \lim_{d \to \infty} \lambda_d(D_d) = 0 \). On the other hand the diameter of \( D_d \) tends to infinity with \( d \). Indeed, the vector of the coefficients of the \( k \)-th cyclotomic polynomial \( \Phi_k \) belongs to the boundary of \( D_{\phi(k)} \) and by a result of Emma Lehmer [11] the maximum of the absolute value of the coefficients of \( \Phi_k \) is not bounded, see also [9].

3. CNS POLYNOMIALS

Assume \( P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \) with \( p_0 \geq 2 \) and set \( \mathcal{N} = \{0, 1, \ldots, p_0 - 1\} \). Denote by \( x \) the image of \( X \) under the canonical epimorphism from \( \mathbb{Z}[X] \) to \( R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X] \). Each coset of \( R \) has a unique element of degree at most \( d - 1 \), say

\[
A(X) = A_{d-1}X^{d-1} + \cdots + A_1X + A_0 \quad (A_0, \ldots, A_{d-1} \in \mathbb{Z}).
\]

Let \( \mathcal{G} := \{A(X) \in \mathbb{Z}[X] : \deg A < d \} \) and

\[
T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})X^i,
\]

where \( A_d = 0 \) and \( q = \lfloor A_0/p_0 \rfloor \). Then \( T_P : \mathcal{G} \to \mathcal{G} \) and

\[
A(X) = (A_0 - qp_0) + XT_P(A), \quad \text{where} \quad A_0 - qp_0 \in \mathcal{N}.
\]

If for each \( A \in \mathcal{G} \) there is a \( k \in \mathbb{N} \) such that \( T_P^k(A) = 0 \) we call \( P \) a canonical number system polynomial (CNS polynomial). Let \( P(X) \) be a monic irreducible CNS polynomial and denote \( \alpha \) one of its roots. Then \( \mathcal{G} \) is isomorphic to \( \mathbb{Z} [\alpha] \) and \( \alpha \) is the bases of a canonical number system in \( \mathbb{Z} [\alpha] \). Canonical number systems were introduced for quadratic number fields by Kátai and Kovács [8] and for number rings by Kovács and Pethő [10]. You find this general definition in [12, 1].

Similarly to Pisot polynomials, associated to CNS polynomials we define for each \( d \in \mathbb{N}, \ d \geq 1 \) the sets

\[
\mathcal{C}_d := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } T_P \text{ has only finite orbits}\}
\]

and

\[
\mathcal{C}_{d}^0 := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } \forall A \in \mathcal{G} \exists \ell \in \mathbb{N} : T_P^\ell (A) = 0\},
\]

where \( P = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \). In [1] we proved that

\[
(p_0, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d \text{ (resp. } \mathcal{C}_d^0)\]
if and only if

$$\left( \frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0} \right) \in D_d \text{ (resp. } D_d^0).$$

With other words SRS is a generalization of CNS. Again $C_d$ and $C_d^0$ are infinite discrete sets. To obtain finite portions of them it is enough to fix one coordinate.

For $M \in N_{>0}$ we set

$$C_d(M) := \{(p_1, \ldots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in C_d\}$$

and

$$C_d^0(M) := \{(p_1, \ldots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in C_d^0\}.$$  

It is clear that $C_d^0(M) \subseteq C_d(M)$. Moreover $C_d(M)$ is finite. Indeed, it is easy to see (c.f. [1]) that if the coefficients of a polynomial belong to $C_d$ then all roots are lying outside the unit circle. As their product is equal to $M$, their modulus are bounded by $M$, thus $|p_i|, i = 1, \ldots, d - 1$ is bounded to.

With the above notations we proved in [3]

**Theorem 2.** We have

$$\lim_{M \to \infty} \frac{|C_d(M)|}{M^{d-1}} = \lambda_{d-1}(D_{d-1}),$$

and similarly

$$\lim_{M \to \infty} \frac{|C_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(D_{d-1}^0).$$

Notice that in Theorem 2 in contrast to Theorem 1 we were able to establish only the main term in the distribution function. This is natural for $C_d^0(M)$ by the same reason, described after Theorem 1.

4. Sketch of the proof of Theorem 1

In this section we present the main steps of the proof of Theorem 1. You may found the details in [4].

4.1. Properties of two auxiliary mappings. For $M \in \mathbb{Z}$ let the mapping $\chi_M : \mathbb{R}^{d-1} \to \mathbb{Z}^d$ be such that if $r = (r_d, \ldots, r_2)$ then $\chi_M(r) = b = (b_1, \ldots, b_d)$, where

$$b_1 = M, b_d = \left\lfloor r_d(M + r_2) + \frac{1}{2} \right\rfloor$$

and

$$b_i = \left\lfloor r_i(M + r_2) - r_{i+1} + \frac{1}{2} \right\rfloor, i = 2, \ldots, d - 1.$$

If $b = (b_1, \ldots, b_d) \in B_d$, then $\chi_{b_1}^{\psi}(b) = b$, i.e., $\chi_{b_1}$ is a left invers of $\psi$.

To prove Theorem 1 we need some properties of the sets

$$S_d(M) = \chi_M(D_{d-1}) \quad \text{and} \quad S_d^0(M) = \chi_M(D_{d-1}^0)$$

and

$$S_d = \cup_{M \in \mathbb{Z}} S_d(M) \quad \text{and} \quad S_d^0 = \cup_{M \in \mathbb{Z}} S_d^0(M).$$

Our first Lemma shows that if $|M|$ is large enough then the polynomials associated to the elements of $S_d(M)$ behaves in some sense similar as Pisot or Salem polynomials.
Lemma 3. Let $M \in \mathbb{Z}$, $(M, b_2, \ldots, b_d) = (b_1, \ldots, b_d) \in S_d(M)$ and $P(X) = X^d - b_1 X^{d-1} - \ldots - b_d$. There exist constants $c_1 = c_1(d), c_2 = c_2(d)$ such that if $|M|$ is large enough then $P(X)$ has a real root $\beta$ for which the inequalities

\begin{align}
|\beta - b_1| &< c_1 \\
|\beta - b_1 - \frac{b_2}{b_1}| &< \frac{c_2}{|b_1|} + O\left(\frac{1}{b_1^2}\right),
\end{align}

hold.

Now we are in the position to extend the definition of $\psi$ from the set $\mathcal{B}_d$ to $S_d$. If $(b_1, \ldots, b_d) \in S_d$ and $|b_1|$ is large enough, then let $\beta$ be the dominating root of the polynomial $P(X) = X^d - b_1 X^{d-1} - \ldots - b_d$, which exists by Lemma 3. Then let

$$\psi(b_1, \ldots, b_d) = (r_d, \ldots, r_2),$$

where the real numbers $r_2, \ldots, r_d$ are defined in a way that they satisfy the relation

$$X^d - b_1 X^{d-1} - \ldots - b_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + \ldots + r_d).$$

We also introduce another mapping $\tilde{\psi} : \mathbb{Z}^d \mapsto \mathbb{Q}^{d-1}$ by

$$\tilde{\psi}(b_1, \ldots, b_d) = \left(\frac{b_d}{b_1 + b_2}, \frac{b_{d-1}}{b_1 + b_2}, \ldots, \frac{b_2}{b_1 + b_2}, \frac{b_1}{b_1 + b_2}\right).$$

The next lemma shows that if $(b_1, \ldots, b_d) \in S_d$ then $\tilde{\psi}(b_1, \ldots, b_d)$ is a good approximation of $\psi(b_1, \ldots, b_d)$. We actually prove

Lemma 4. Let $(b_1, \ldots, b_d) \in S_d$ and assume that $|b_1|$ is large enough. Then

$$\left|\tilde{\psi}(b_1, \ldots, b_d) - \psi(b_1, \ldots, b_d)\right|_{\infty} < \frac{c_3}{b_1^2} + O\left(\frac{1}{|b_1|^3}\right),$$

where $c_3$ is depending only on $d$.

$B_d$ and $B_d(M)$ are subsets of a lattice. This nice property does not remain valid after the application of $\psi$. However, the next lemma shows that the set $\tilde{\psi}(S_d)$ is lattice like. More precisely we have

Lemma 5. Let $b = (b_1, \ldots, b_d), b' = (b_1', \ldots, b_d') \in S_d$ such that there exists a $1 \leq j \leq d$ with $b_i = b'_i, i \neq j$ and $b'_j = b_j + 1$. Then

$$|\tilde{\psi}(b)_k - \tilde{\psi}(b')_k| = \begin{cases} 0, & \text{if } j > 2 \text{ and } k \neq d - j + 1, d - j + 2 \\
\frac{1}{|b_1|} + O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d - j + 1 \text{ or } j = 2, k = d - 1 \\
O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d - j + 2 \text{ or } j = 2, k < d - 1 \\
|b_{d-k+1}| \frac{1}{|b_1|} + O(|b_1|^{-3}), & \text{if } j = 1. \end{cases}$$
4.2. A lemma on the roots of polynomials. It is well known that the roots of real polynomials are continues functions of the coefficients. The next lemma is a quantitative version of this fact.

**Lemma 6.** Let $d \in \mathbb{N}$ and $\rho, \varepsilon \in \mathbb{R}_{>0}$. Then there exists a constant $c_4 > 0$ depending only on $d$ and $\rho$ with the following property: if all roots $\alpha \in \mathbb{C}$ of the polynomial $P(X) = X^d + p_d X^{d-1} + \cdots + p_0 \in \mathbb{R}[X]$ satisfy $|\alpha| < \rho$ and $Q(X) = X^d + q_d X^{d-1} + \cdots + q_0 \in \mathbb{R}[X]$ is chosen such that $|p_i - q_i| < \varepsilon, i = 0, \ldots, d - 1$ then for each root $\beta$ of $Q(X)$ there exists a root $\alpha$ of $P(X)$ satisfying

$$|\beta - \alpha| < c_4 \varepsilon^{1/d}.$$ 

In particular, all roots $\beta$ of $Q(X)$ satisfy $|\beta| < \rho + c_4 \varepsilon^{1/d}$.

Let

$$\mathcal{E}_d(r) := \{(r_1, \ldots, r_d) \in \mathbb{R}^d : X^d + r_d X^{d-1} + \cdots + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < r\}.$$ 

The next lemma gives a precise estimate for the volume of the strip near to the boundary of $\mathcal{D}_d$. It is very important to prove the first part of Theorem 1.

**Lemma 7.** Let $0 < \eta < 1$. Then we have

$$\lambda_d (\mathcal{E}_d (1 + \eta) \setminus \mathcal{D}_d) \leq 2^{d(d+1)/2} \lambda_d (\mathcal{E}_d (1)) \eta$$

and

$$\lambda_d (\mathcal{D}_d \setminus \mathcal{E}_d (1 - \eta)) \leq 2^{d(d+1)/2} \lambda_d (\mathcal{E}_d (1)) \eta.$$ 

4.3. **Proof of Theorem 1 for $\mathcal{D}_d$.** Now we are in the position to finish the first assertion of Theorem 1. Let $M > 0$ and put

$$W(x, s) = \{y \in \mathbb{R}^d : |x-y|_\infty \leq s/2\} \text{ for } (x \in \mathbb{R}^d, s \in \mathbb{R})$$

and

$$\mathcal{W}_{d-1}(M) = \bigcup_{x \in \mathcal{B}_d(M)} W(\psi(x), M^{-1}).$$

Then we claim

$$\lambda_{d-1}(\mathcal{W}_{d-1}(M)) = \frac{1}{M^{d-1}} \left| \mathcal{B}_d(M) \right| \left( 1 + O \left( \frac{1}{M} \right) \right).$$

Indeed, let $x, y \in \mathcal{B}_d(M)$ such that $x - y = e_j$ for some $j \in \{2, \ldots, d\}$. Then by Lemmata 4 and 5

$$|\psi(x)_k - \psi(y)_k| \leq |\psi(x)_k - \tilde{\psi}(x)_k + \tilde{\psi}(x)_k + \tilde{\psi}(y)_k - \psi(y)_k| \leq \frac{1}{M^2} + O \frac{1}{M^2}, \quad \text{if } (j, k) = (2, d-1), \text{ or } j > 2, k = d-j+1$$

Thus

$$\lambda_{d-1}(W(\psi(x), M^{-1}) \cap W(\psi(y), M^{-1})) = O \left( \frac{1}{M^d} \right).$$

As $x$ has at most $2^d$ neighbors we get

$$\lambda_{d-1} \left( \bigcup_{x, y \in \mathcal{B}_d(M)} (W(\psi(x), M^{-1}) \cap W(\psi(y), M^{-1})) \right) = O \left( \frac{\left| \mathcal{B}_d(M) \right|}{M^d} \right)$$

and the claim is proved.
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Hence in the sequel it is enough to consider $x \in B_d(M)$.

**Lower estimate for $\lambda_{d-1}(D_{d-1})$.**

Put $\eta = c_4(2M)^{-1/(d-1)}$. Let $x \in B_d(M)$ such that $\psi(x) \in \mathcal{E}_{d-1}(\eta) \subseteq D_{d-1}$. Let $y \in W(\psi(x), M^{-1})$. Then $\rho(\psi(x)) < 1 - \eta$ and as $|\psi(x) - y|_\infty \leq \frac{1}{2M}$ we get $\rho(y) < 1$. Thus

\[ \bigcup_{x \in \mathcal{B}_d(M)} W(\psi(x), M^{-1}) \subseteq D_{d-1}. \tag{9} \]

By Lemma 7 the measure of the set

\[ D_{d-1} \setminus \mathcal{E}_{d-1}(1 - \eta) \]

is bounded by $O(M^{-1/(d-1)})$. Moreover this set satisfies the conditions of a Theorem of H. Davenport [5]. Thus the number of $x \in B_d(M)$ such that $1 - \eta \leq \rho(\psi(x)) \leq 1$ is at most $O(M^{d-1 - 1/(d-1)})$. Combining this with (8) and (9) we obtain

\[ \lambda_{d-1}(D_{d-1}) \geq \frac{|B_d(M)|}{M^{d-1}} \left(1 - c_7M^{-1/(d-1)}\right). \]

**Upper estimate for $\lambda_{d-1}(D_{d-1})$.**

We construct for every $r=(r_d, \ldots, r_2) \in D_{d-1}$ and $M$ large enough, an integer vector $b=(b_1, \ldots, b_d) \in \mathbb{Z}^d$ such that $\psi(b)$ is located near enough to $r$.

Consider

\[ \tilde{\psi}(b) = \left( \frac{b_d}{b_1 + \frac{b_d}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_{d-1}}{b_1}}, \ldots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + b_3 \right). \]

Set $\eta = 2c_4(2M)^{-1/(d-1)}$. Thus by Lemma 6 we get

\[ \rho(\psi(b)) \leq \rho(r) + \eta \leq 1 + \eta. \]

This means that if $M$ is large enough then all but one root of $X^d - b_1X^{d-1} - \cdots - b_d$ have absolute value at most $1 + \eta$ and one root is close to $M$.

We have further

\[ D_{d-1} \subseteq \bigcup_{x \in \mathbb{Z}^d} W(\psi(x), M^{-1}) \]

\[ = \bigcup_{x \in \mathcal{B}_d(M) \setminus \mathcal{E}_{d-1}(1+\eta)} W(\psi(x), M^{-1}) \cup \bigcup_{x \in \mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{E}_{d-1}(1)} W(\psi(x), M^{-1}). \]

We conclude that the volume of the set $\mathcal{E}_{d-1}(1+\eta) \setminus D_{d-1}$ is at most $O(M^{-1/(d-1)})$.

As the conditions of the above mentioned Theorem of Davenport [5] hold again we get that the number of $x \in \mathbb{Z}^d$ such that $\psi(x)$ lies in $\mathcal{E}_{d-1}(1+\eta) \setminus D_{d-1}$ is at most $O(M^{d-1 - 1/(d-1)})$. Thus there is a constant $c_8 > 0$ such that

\[ \lambda_{d-1}(D_{d-1}) \leq \frac{|B_d(M)|}{M^{d-1}} \left(1 + c_8M^{-1/(d-1)}\right). \]

Combining the lower and upper estimates for $\lambda_{d-1}(D_{d-1})$ we finish the proof of the first part of Theorem 1.
A. PETHŐ

5. Problem

To fix a coefficient is an unusual way to measure a set of polynomials. Unfortunately, we were not able to prove a to Theorem 1 analogous result for Pisot polynomials with bounded height, i.e., if the maximum modulus of the coefficients is bounded. Therefore we propose the following problem:

For $M \in \mathbb{N}_{>0}$ set

$$B_d'(M) := \{(b_1, b_2, \ldots, b_d) \in \mathbb{Z}^d \cap B_d : \max\{|b_1|, \ldots, |b_d|\} = M\}$$

and

$$B_d^0(M) := \{(b_1, b_2, \ldots, b_d) \in \mathbb{Z}^d \cap B_d^0 : \max\{|b_1|, \ldots, |b_d|\} = M\}.$$  

Do

$$\lim_{M \to \infty} \frac{|B_d'(M)|}{M^{d-1}}$$

and/or

$$\lim_{M \to \infty} \frac{|B_d^0(M)|}{M^{d-1}}$$

exist?

REFERENCES


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