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ON THE DISTRIBUTION OF PISOT AND CNS POLYNOMIALS

ATTILA PETHŐ

1. INTRODUCTION

This paper is the edited version of my talk, delivered at the RIMS conference "Analytic Number Theory", on 15 October, 2009. I thank the possibility to speak on that event and for the hospitality of RIMS.

Let $d \geq 1$ be an integer and $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Consider the mapping $\tau_r : \mathbb{Z}^d \to \mathbb{Z}^d$; for $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ let

$$\tau_r(a) = (a_2, \ldots, a_d, -\lfloor ra \rfloor),$$

where $ra = r_1a_1 + \cdots + r_da_d$ denotes the inner product. We call $\tau_r$ a shift radix system (SRS for short) if for all $a \in \mathbb{Z}^d$ we can find some $k > 0$ with $\tau_r^k(a) = 0$. This concept was introduced by Akiyama et al. [1]. We proved that it is a common generalization of canonical number systems in residue class rings of polynomial rings (see [8, 10, 12]) as well as of $\beta$-expansions of real numbers, [13].

For the investigation of properties of SRS it turned out convenient to introduce some sets.

For $d \in \mathbb{N}$, $d \geq 1$ let

$$\mathcal{D}_d := \{ r \in \mathbb{R}^d : \forall a \in \mathbb{Z}^d \text{ } (\tau_r^k(a))_{k \geq 0} \text{ is ultimately periodic} \},$$

$$\mathcal{D}_d^0 := \{ r \in \mathbb{R}^d : \forall a \in \mathbb{Z}^d \exists k > 0 : \tau_r^k(a) = 0 \}.$$

It is clear that $\mathcal{D}_d^0 \subset \mathcal{D}_d$ and $r$ is SRS iff $r \in \mathcal{D}_d^0$. In [1] we proved among others that $\mathcal{D}_d, \mathcal{D}_d^0$ are Lebesgue measurable and $\mathcal{D}_d^0$ admits some convexity property. On the other hand the results of [2] showed that the boundary already of $\mathcal{D}_d^0$ is very complicated. Further we proved in [1] that we can embed the discrete sets of Pisot, Salem and CNS polynomials in these continues sets. In [3] and [4] we studied the distribution of Pisot, Salem and CNS polynomials. In the present paper we give a survey about the last mentioned results. Further we present the sketch of the proof one of the main results.

2. PISOT AND SALEM POLYNOMIALS

Let $P(X) = X^d - b_1X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$.

- If all but one root of $P$ is located in the open unit disc then $P$ is called a Pisot polynomial. Its dominant root is called Pisot number.
- If all but one root of $P$ is located in the closed unit disc and at least one of them has modulus 1 then $P$ is called a Salem polynomial. Its dominant root is called Salem number.

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If \( P \) is a Pisot or Salem polynomial, we will denote its dominating root by \( \beta \).

Let \( \text{Fin}(\beta) \) be the set of positive real numbers having finite greedy expansion with respect to \( \beta \). We say that \( \beta > 1 \) has property (F) if

\[
\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty).
\]

It was shown by Frougny and Solomyak [7] that (F) can hold only for Pisot numbers \( \beta \). Analogously to \( \mathcal{D}_d \) and \( \mathcal{D}_d^0 \) define for each \( d \in \mathbb{N}, d \geq 1 \) the sets

\[
\mathcal{B}_d = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : P(X) \text{ is a Pisot or Salem polynomial}\}
\]

and

\[
\mathcal{B}_d^0 = \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : P(X) \text{ is a Pisot polynomial with property (F)}\},
\]

where \( P(X) = X^d - b_1X^{d-1} - \cdots - b_d \). We obviously have \( \mathcal{B}_d^0 \subseteq \mathcal{B}_d \).

If \( (b_1, \ldots, b_d) \in \mathcal{B}_d \) then let \( \beta \) be the dominating root of

\[
P(X) = X^d - b_1X^{d-1} - \cdots - b_d.
\]

Consider the map \( \psi : \mathcal{B}_d \to \mathbb{R}^{d-1} : \psi(b_1, \ldots, b_d) = (r_2, \ldots, r_d) \),

where \( r_2, \ldots, r_d \) are such that

\[
X^d - b_1X^{d-1} - \cdots - b_d = (X - \beta)(X^{d-1} + r_2X^{d-2} + \cdots + r_d).
\]

As \( (b_1, \ldots, b_d) \in \mathcal{B}_d \), the polynomial \( X^{d-1} + r_2X^{d-2} + \cdots + r_d \) has all its roots in the closed unit circle. Thus

\[
\psi(\mathcal{B}_d) \subseteq \overline{\mathcal{D}_{d-1}}.
\]

In [1] we proved:

\[
\psi(\mathcal{B}_d^0) \subseteq \mathcal{D}_{d-1}^0.
\]

This means we can embed the discrete sets \( \mathcal{B}_d \) and \( \mathcal{B}_d^0 \) in the continues sets \( \mathcal{D}_d \) and \( \mathcal{D}_d^0 \) respectively, i.e, SRS can be considered as a generalization of the \( \beta \)-representations.

The sets \( \mathcal{B}_d, \mathcal{B}_d^0 \) are obviously discrete and infinite. To study their distribution we fix the first coordinate. More precisely, for \( M \in \mathbb{N}_{>0} \) we set

\[
\mathcal{B}_d(M) := \{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d \}
\]

and

\[
\mathcal{B}_d^0(M) := \{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} : (M, b_2, \ldots, b_d) \in \mathcal{B}_d^0 \}.
\]

It is clear that \( \mathcal{B}_d^0(M) \subseteq \mathcal{B}_d(M) \), moreover \( \mathcal{B}_d(M) \) is finite. Indeed, as \( M = \beta \) + other roots of \( X^d - MX^{d-1} - b_2X^{d-2} - b_d \) and the roots of \( X^d - MX^{d-1} - b_2X^{d-2} - b_d \) except of \( \beta \) are lying in the unit disc, thus \( |\beta| \leq M + d - 1 \). Hence there are easily computable constants \( c_i(M, d) \) such that \( |b_i| \leq c_i(M, d) \), which ensures the finiteness of \( \mathcal{B}_d(M) \). With these notations we proved in [4] the following theorem.

Theorem 1. We have

\[
|\mathcal{B}_d(M)| \leq \lambda_{d-1}(\mathcal{D}_{d-1}) = O(M^{-d+1+1/d}),
\]

and

\[
\lim_{M \to \infty} \frac{|\mathcal{B}_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_d^0) = \lambda_{d-1}(\mathcal{D}_{d-1}^0).
\]
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where $\lambda_{d-1}$ denotes the $d-1$-dimensional Lebesgue measure and $|A|$ the cardinality of the finite set $A$.

Notice that (2) is weaker than (1). As the boundary of $\mathcal{D}_{d-1}$ is smooth, we were able to estimate accurately the number of images under $\psi$ lying near to the boundary. This was not possible for $\mathcal{D}_{d-1}^0$, because its boundary is quite complicated.

In Theorem 1 and later in Theorem 2 the volume or Lebesgue measure of $\mathcal{D}_d$ appears in the main term. This was calculated by Fam [6]. Using the Barnes G-function we have

$$
\lambda_d(\mathcal{D}_d) = \begin{cases} 
\frac{2^{2n^2+n} \Gamma(n+1) \Gamma(n+1)^4}{\Gamma(2n+2)} & (d = 2n), \\
\frac{2^{2n^2+3n+1} \Gamma(n+2)^4 \Gamma(2n+3)}{\Gamma(n+1)^2} & (d = 2n + 1). 
\end{cases}
$$

Note that for positive integers the Barnes G-function equals the superfactorials: $G(n+2) = \prod_{j=1}^{n} j!$ for $n \in \mathbb{N}$. Moreover, observe that by [6, Formula (2.13)] we have $\lim_{d \to \infty} \lambda_d(\mathcal{D}_d) = 0$. On the other hand the diameter of $\mathcal{D}_d$ tends to infinity with $d$. Indeed, the vector of the coefficients of the $k$-th cyclotomic polynomial $\Phi_k$ belongs to the boundary of $\mathcal{D}_{\phi(k)}$ and by a result of Emma Lehmer [11] the maximum of the absolute value of the coefficients of $\Phi_k$ is not bounded, see also [9].

3. CNS POLYNOMIALS

Assume $P(X) = X^d + p_{d-1} X^{d-1} + \cdots + p_0$ with $p_0 \geq 2$ and set $\mathcal{N} = \{0, 1, \ldots, p_0 - 1\}$. Denote by $x$ the image of $X$ under the canonical epimorphism from $\mathbb{Z}[X]$ to $R := \mathbb{Z}[X]/P(X)\mathbb{Z}[X]$. Each coset of $R$ has a unique element of degree at most $d - 1$, say

$$(3) \quad A(X) = A_{d-1} X^{d-1} + \cdots + A_1 X + A_0 \quad (A_0, \ldots, A_{d-1} \in \mathbb{Z}).$$

Let $\mathcal{G} := \{A(X) \in \mathbb{Z}[X] : \deg A < d\}$ and

$$
T_P(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1}) X^i,
$$

where $A_d = 0$ and $q = [A_0/p_0]$. Then $T_P : \mathcal{G} \to \mathcal{G}$ and

$$
A(X) = (A_0 - qp_0) + XT_P(A), \text{ where } A_0 - qp_0 \in \mathcal{N}.
$$

If for each $A \in \mathcal{G}$ there is a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$ we call $P$ a canonical number system polynomial (CNS polynomial). Let $P(X)$ be a monic irreducible CNS polynomial and denote $\alpha$ one of its roots. Then $\mathcal{G}$ is isomorphic to $\mathbb{Z}[\alpha]$ and $\alpha$ is the base of a canonical number system in $\mathbb{Z}[\alpha]$. Canonical number systems were introduced for quadratic number fields by Kővács and Kovács [8] and for number rings by Kovács and Pethő [10]. You find this general definition in [12, 1].

Similarly to Pisot polynomials, associated to CNS polynomials we define for each $d \in \mathbb{N}$, $d \geq 1$ the sets

$$
\mathcal{C}_d := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } T_P \text{ has only finite orbits}\}
$$

and

$$
\mathcal{C}_d^0 := \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : |p_0| \geq 2 \text{ and } \forall A \in \mathcal{G} \exists k \in \mathbb{N} : T_P^k(A) = 0\},
$$

where $P = X^d + p_{d-1} X^{d-1} + \cdots + p_0$. In [1] we proved that

$$(p_0, p_1, \ldots, p_{d-1}) \in \mathcal{C}_d \text{ (resp. } \mathcal{C}_d^0)$$

where $\lambda_{d-1}$ denotes the $d-1$-dimensional Lebesgue measure and $|A|$ the cardinality of the finite set $A$.
if and only if

$$\left( \frac{1}{p_0}, \frac{p_d-1}{p_0}, \ldots, \frac{p_1}{p_0} \right) \in D_d \text{ (resp. } D_d^0).$$

With other words SRS is a generalization of CNS. Again $C_d$ and $C_d^0$ are infinite discrete sets. To obtain finite portions of them it is enough to fix one coordinate.

For $M \in \mathbb{N}_{>0}$ we set

$$C_d(M) := \{(p_1, \ldots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in C_d\}$$

and

$$C_d^0(M) := \{(p_1, \ldots, p_{d-1}) \in \mathbb{Z}^{d-1} : (M, p_1, \ldots, p_{d-1}) \in C_d^0\}.$$

It is clear that $C_d^0(M) \subseteq C_d(M)$. Moreover $C_d(M)$ is finite. Indeed, it is easy to see (c.f. [1]) that if the coefficients of a polynomial belong to $C_d$ then all roots are lying outside the unit circle. As their product is equal to $M$, their modulus are bounded by $M$, thus $|p_i|, i = 1, \ldots, d-1$ is bounded to.

With the above notations we proved in [3]

\begin{theorem}
We have

$$\lim_{M \rightarrow \infty} \frac{|C_d(M)|}{M^{d-1}} = \lambda_{d-1}(D_{d-1}),$$

and similarly

$$\lim_{M \rightarrow \infty} \frac{|C_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(D_{d-1}^0).$$

\end{theorem}

Notice that in Theorem 2 in contrast to Theorem 1 we were able to establish only the main term in the distribution function. This is natural for $C_d^0(M)$ by the same reason, described after Theorem 1.

4. Sketch of the proof of Theorem 1

In this section we present the main steps of the proof of Theorem 1. You may found the details in [4].

4.1. Properties of two auxiliary mappings. For $M \in \mathbb{Z}$ let the mapping $\chi_M : \mathbb{R}^{d-1} \rightarrow \mathbb{Z}^d$ be such that if $r = (r_d, \ldots, r_2)$ then $\chi_M(r) = b = (b_1, \ldots, b_d)$, where

$$b_1 = M, b_d = \left\lfloor r_d(M + r_2) + \frac{1}{2} \right\rfloor \text{ and }$$

$$b_i = \left\lfloor r_i(M + r_2) - r_{i+1} + \frac{1}{2} \right\rfloor, i = 2, \ldots, d-1.$$

If $b = (b_1, \ldots, b_d) \in B_d$, then $\chi_{b_1}(\psi(b)) = b$, i.e., $\chi_{b_1}$ is a left invers of $\psi$.

To prove Theorem 1 we need some properties of the sets

$$S_d(M) = \chi_M(D_{d-1}) \text{ and } S_d^0(M) = \chi_M(D_{d-1}^0)$$

and

$$S_d = \bigcup_{M \in \mathbb{Z}} S_d(M) \text{ and } S_d^0 = \bigcup_{M \in \mathbb{Z}} S_d^0(M).$$

Our first Lemma shows that if $|M|$ is large enough then the polynomials associated to the elements of $S_d(M)$ behaves in some sense similar as Pisot or Salem polynomials.
Lemma 3. Let \( M \in \mathbb{Z}, (M, b_2, \ldots, b_d) = (b_1, \ldots, b_d) \in S_d(M) \) and \( P(X) = X^d - b_1X^{d-1} - \cdots - b_d \). There exist constants \( c_1 = c_1(d), c_2 = c_2(d) \) such that if \( |M| \) is large enough than \( P(X) \) has a real root \( \beta \) for which the inequalities

\[
|\beta - b_1| < c_1 \\
|\beta - b_1 - \frac{b_2}{b_1}| < \frac{c_2}{|b_1|} + O\left(\frac{1}{b_1^2}\right),
\]

hold.

Now we are in the position to extend the definition of \( \psi \) from the set \( \mathcal{B}_d \) to \( S_d \). If \( (b_1, \ldots, b_d) \in S_d \) and \( |b_1| \) is large enough, then let \( \beta \) be the dominating root of the polynomial

\[ P(X) = X^d - b_1X^{d-1} - \cdots - b_d, \]

which exists by Lemma 3. Then let

\[ \psi(b_1, \ldots, b_d) = (r_d, \ldots, r_2), \]

where the real numbers \( r_2, \ldots, r_d \) are defined in a way that they satisfy the relation

\[ X^d - b_1X^{d-1} - \cdots - b_d = (X - \beta)(X^{d-1} + r_2X^{d-2} + \cdots + r_d). \]

We also introduce an other mapping \( \tilde{\psi} : \mathbb{Z}^d \mapsto \mathbb{Q}^{d-1} \) by

\[ \tilde{\psi}(b_1, \ldots, b_d) = \left( \frac{b_d}{b_1 + b_1^{T}}, \frac{b_{d-1}}{b_1 + b_1^{T}}, \ldots, \frac{b_2}{b_1 + b_1^{T}}, \frac{b_2}{b_1 + b_1^{T}} \right). \]

The next lemma shows that if \( (b_1, \ldots, b_d) \in S_d \) then \( \tilde{\psi}(b_1, \ldots, b_d) \) is a good approximation of \( \psi(b_1, \ldots, b_d) \).

Lemma 4. Let \( (b_1, \ldots, b_d) \in S_d \) and assume that \( |b_1| \) is large enough. Then

\[ \left| \tilde{\psi}(b_1, \ldots, b_d) - \psi(b_1, \ldots, b_d) \right|_{\infty} < \frac{c_3}{b_1^2} + O\left(\frac{1}{|b_1|^3}\right), \]

where \( c_3 \) is depending only on \( d \).

\( B_d \) and \( B_d(M) \) are subsets of a lattice. This nice property does not remain valid after the application of \( \psi \). However, the next lemma shows that the set \( \tilde{\psi}(S_d) \) is lattice like. More precisely we have

Lemma 5. Let \( b = (b_1, \ldots, b_d), b' = (b_1', \ldots, b_d') \in S_d \) such that there exists a \( 1 \leq j \leq d \) with \( b_i = b_i', i \neq j \) and \( b_j' = b_j + 1 \). Then

\[ |\tilde{\psi}(b) - \tilde{\psi}(b')| = \begin{cases} 
0, & \text{if } j > 2 \text{ and } k \neq d - j + 1, d - j + 2 \\
\frac{1}{|b_1|} + O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d - j + 1 \\
O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d - j + 2 \\
|b_{d-k+1}| \frac{1}{b_1} + O(|b_1|^{-3}), & \text{if } j = 1.
\end{cases} \]

\[ \text{or } j = 2, k = d - 1 \]
4.2. **A lemma on the roots of polynomials.** It is well known that the roots of real polynomials are continuous functions of the coefficients. The next lemma is a quantitative version of this fact.

**Lemma 6.** Let $d \in \mathbb{N}$ and $\rho, \epsilon \in \mathbb{R}_{>0}$. Then there exists a constant $c_4 > 0$ depending only on $d$ and $\rho$ with the following property: if all roots $\alpha \in \mathbb{C}$ of the polynomial $P(X) = X^d + p_{d-1}X^{d-1} + \cdots + p_0 \in \mathbb{R}[X]$ satisfy $|\alpha| < \rho$ and $Q(X) = X^d + q_{d-1}X^{d-1} + \cdots + q_0 \in \mathbb{R}[X]$ is chosen such that $|p_i - q_i| < \epsilon$, $i = 0, \ldots, d - 1$ then for each root $\beta$ of $Q(X)$ there exists a root $\alpha$ of $P(X)$ satisfying

$$|\beta - \alpha| < c_4 \epsilon^{1/d}.$$  

In particular, all roots $\beta$ of $Q(X)$ satisfy $|\beta| < \rho + c_4 \epsilon^{1/d}$.

Let

$$E_d(r) := \{(r_1, \ldots, r_d) \in \mathbb{R}^d : X^d + r_dX^{d-1} + \cdots + r_1$$

has only roots $y \in \mathbb{C}$ with $|y| < r\}.$

The next lemma gives a precise estimate for the volume of the strip near to the boundary of $D_d$. It is very important to prove the first part of Theorem 1.

**Lemma 7.** Let $0 < \eta < 1$. Then we have

$$\lambda_d(E_d(1 + \eta) \setminus D_d) \leq 2^{d(d+1)/2} \lambda_d(E_d(1)) \eta$$

and

$$\lambda_d(D_d \setminus E_d(1 - \eta)) \leq 2^{d(d+1)/2} \lambda_d(E_d(1)) \eta.$$

4.3. **Proof of Theorem 1 for $D_d$.** Now we are in the position to finish the first assertion of Theorem 1. Let $M > 0$ and put

$$W(x, s) = \{y \in \mathbb{R}^d : |x - y|_\infty \leq s/2\} \quad (x \in \mathbb{R}^d, s \in \mathbb{R})$$

and

$$W_{d-1}(M) = \bigcup_{x \in B_d(M)} W(\psi(x), M^{-1}).$$

Then we claim

$$\lambda_{d-1}(W_{d-1}(M)) = \frac{|B_d(M)|}{M^{d-1}} \left(1 + O\left(\frac{1}{M}\right)\right).$$

Indeed, let $x, y \in B_d(M)$ such that $x - y = e_j$ for some $j \in \{2, \ldots, d\}$. Then by Lemmata 4 and 5

$$|\psi(x)_k - \psi(y)_k| \leq \left|\psi(x)_k - \tilde{\psi}(x)_k + \tilde{\psi}(x)_k - \tilde{\psi}(y)_k + \tilde{\psi}(y)_k - \psi(y)_k\right|$$

$$\leq \left\{\frac{1}{M} + O\left(\frac{1}{M}\right)\right\}, \quad \text{if } (j, k) = (2, d-1), \text{ or } j > 2, k = d-j+1$$

Thus

$$\lambda_{d-1}(W(\psi(x), M^{-1}) \cap W(\psi(y), M^{-1})) = O\left(\frac{1}{M^d}\right).$$

As $x$ has at most $2^d$ neighbors we get

$$\lambda_{d-1}\left(\bigcup_{x, y \in B_d(M), x \neq y} (W(\psi(x), M^{-1}) \cap W(\psi(y), M^{-1}))\right) = O\left(\frac{|B_d(M)|}{M^d}\right)$$

and the claim is proved.
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Hence in the sequel it is enough to consider $x \in \mathcal{B}_d(M)$.

**Lower estimate for $\lambda_{d-1}(\mathcal{D}_{d-1})$.**

Put $\eta = c_d(2M)^{-1/(d-1)}$. Let $x \in \mathcal{B}_d(M)$ such that $\psi(x) \in \mathcal{E}_{d-1}(\eta) \subseteq \mathcal{D}_{d-1}$. Let $y \in W(\psi(x), M^{-1})$. Then $\rho(\psi(x)) < 1 - \eta$ and as $|\psi(x) - y|_\infty \leq \frac{1}{2M}$ we get $\rho(y) < 1$. Thus

$$\bigcup_{x \in \mathcal{B}_d(M) \atop \rho(\psi(x)) < 1 - \eta} W(\psi(x), M^{-1}) \subseteq \mathcal{D}_{d-1}. \quad (9)$$

By Lemma 7 the measure of the set

$$\mathcal{D}_{d-1} \setminus \mathcal{E}_{d-1}(1 - \eta)$$

is bounded by $O(M^{-1/(d-1)})$. Moreover this set satisfies the conditions of a Theorem of H. Davenport [5]. Thus the number of $x \in \mathcal{B}_d(M)$ such that $1 - \eta \leq \rho(\psi(x)) \leq 1$ is at most $O(M^{d-1-1/(d-1)})$. Combining this with (8) and (9) we obtain

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \geq \frac{|\mathcal{B}_d(M)|}{M^{d-1}} \left( 1 - c_7 M^{-1/(d-1)} \right).$$

**Upper estimate for $\lambda_{d-1}(\mathcal{D}_{d-1})$.**

We construct for every $r = (r_d, \ldots, r_2) \in \mathcal{D}_{d-1}$ and $M$ large enough, an integer vector $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ such that $\psi(b)$ is located near enough to $r$.

Consider

$$\tilde{\psi}(b) = \left( \frac{b_d}{b_1 + \frac{b_d}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_{d-1}}{b_1}}, \ldots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + \frac{b_3}{b_1^2} \right).$$

Set $\eta = 2c_d(2M)^{-1/(d-1)}$. Thus by Lemma 6 we get

$$\rho(\psi(b)) \leq \rho(r) + \eta \leq 1 + \eta.$$  

This means that if $M$ is large enough then all but one root of $X^d - b_1 X^{d-1} - \cdots - b_d$ have absolute value at most $1 + \eta$ and one root is close to $M$.

We have further

$$\mathcal{D}_{d-1} \subseteq \bigcup_{x \in \mathcal{B}_d(M)} W(\psi(x), M^{-1}) \cup \bigcup_{x \in \mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}} W(\psi(x), M^{-1}).$$

We conclude that the volume of the set $\mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}$ is at most $O(M^{-1/(d-1)})$.

As the conditions of the above mentioned Theorem of Davenport [5] hold again we get that the number of $x \in \mathbb{Z}^d$ such that $\psi(x)$ lies in $\mathcal{E}_{d-1}(1+\eta) \setminus \mathcal{D}_{d-1}$ is at most $O(M^{d-1-1/(d-1)})$. Thus there is a constant $c_8 > 0$ such that

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \leq \frac{|\mathcal{B}_d(M)|}{M^{d-1}} \left( 1 + c_8 M^{-1/(d-1)} \right).$$

Combining the lower and upper estimates for $\lambda_{d-1}(\mathcal{D}_{d-1})$ we finish the proof of the first part of Theorem 1.
A. PETHŐ

5. PROBLEM

To fix a coefficient is an unusual way to measure a set of polynomials. Unfortunately, we were not able to prove a to Theorem 1 analogous result for Pisot polynomials with bounded height, i.e., if the maximum modulus of the coefficients is bounded. Therefore we propose the following problem:

For $M \in \mathbb{N}_{>0}$ set

$$B_d'(M) := \{(b_1, b_2, \ldots, b_d) \in \mathbb{Z}^d \cap B_d : \max\{|b_1|, \ldots, |b_d|\} = M\}$$

and

$$B_d^0(M) := \{(b_1, b_2, \ldots, b_d) \in \mathbb{Z}^d \cap B_d^0 : \max\{|b_1|, \ldots, |b_d|\} = M\}.$$  

Do

$$\lim_{M \to \infty} \frac{|B_d'(M)|}{M^{d-1}} \quad \text{and/or} \quad \lim_{M \to \infty} \frac{|B_d^0(M)|}{M^{d-1}}$$

exist?

REFERENCES


A. PETHŐ

FACULTY OF INFORMATICS, UNIVERSITY OF DEBRECEN
NUMBER THEORY RESEARCH GROUP, HUNGARIAN ACADEMY OF SCIENCES AND
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P.O. BOX 12, HUNGARY

E-mail address: pethoe@inf.unideb.hu