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A Generalization of Automorphism Classification of Cellular Automata —an Extended Abstract—

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Abstract

In this paper we make a generalization (called g-automorphism) of the automorphism of cellular automata (CA for short) introduced by H. Nishio in 2009. At defining g-automorphism, we consider both permutations of the position and the value of the arguments of the local function with relevant permutation of the neighborhood. We prove that the g-automorphisms constitutes a group under a rule of the semi-direct product. The group acts on the local function and naturally induces a classification of CA. Every CA in a class has the same global property *up to permutation.* For explaining the idea we preferably use a computation universal rule 110. We give the g-automorphism classification of 256 local functions of ECA into 11 classes. To be specific we show that 48 functions are g-automorphic with f_{110} and therefore universal up to permutation.

1 Introduction

In the history of the cellular automaton (CA for short), most studies first assume some standard neighborhood (von Neumann, Moore) and then investigate the global behaviors and mathematical properties or look for a local function that would solve a given problem, say, the self-reproduction, the Game of Life and so on. One could, however, ask a question: What happens if the neighborhood is changed from the standard.

Suppose that CA is defined by a 4 tuple $(\mathbb{Z}^d, Q, f, \nu)$, where $\mathbb{Z}^d$ is the d-dimensional Euclidean cellular space, Q is the set of cell states, f is the local function and ν is the neighborhood, which is a mapping from $\{1, ..., n\}$ to $\mathbb{Z}^d$. The i-th neighbor $\nu(i),\ 1 \leq i \leq n$ is connected to the i-th argument of f. When the space $\mathbb{Z}^d$ and the state set Q are understood, the global behavior of CA is determined by its *local structure* (f, ν). Two local structures are called *equivalent* if and only if they induce the same global functions. As for equivalence we particularly proved a basic theorem: *Two CA are equivalent if and only if their local structures are permutation of each other* [7].

Based on this theory of the permutation equivalence of local structures, we defined the automorphism for local structures and investigated the automorphism classification of the local functions [4, 5] . We defined the automorphism: (f', ν') is called *automorphic* with (f, ν) if and only if there is a pair of permutations ν and φ such that $(f', \nu') = (\varphi^{-1} f^{\pi} \varphi, \nu^{\pi})$. For example, if we permute $(f_{110}, (-1, 0, 1))$ with π^2 and $\varphi = (1, 2)$ (transposition of states 0 and 1) [2], we have $(f_{161}, (0, -1, 1))$. Therefore f_{161} is also universal *up to permutation.* In what follows we often omit the suffix *up to permutation.*

Now we generalize the automorphism of CA in such a way that every argument of f is permuted independently. The local function is expressed by a polynomial in n variables $f(\mathrm{x}_n) = f(x_1, ..., x_n)$ over finite field $GF(q)$ and the set of such polynomials is denoted $\mathcal{P}_{n,q},\ 1 \leq n, 2 \leq q$. We are going to define the g-automorphism for $\mathcal{P}_{n,q}$. For two CA A and A', $A' = (f', \nu')$ is called *g-automorphic* with $A = (f, \nu)$ denoted $A \cong_g A'$, if and only if there is a 3-tuple of permutations $(\pi, \psi, \varphi(n))$ such that $(f', \nu') = (f^{\pi}, \psi f^{\pi} \varphi(n))$, where $\varphi(n) = (\varphi_1, ..., \varphi_n)$ and $\varphi_i, 1 \leq i \leq n$ permutes the value of the i-th

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[2] $\varphi^{-1} f \varphi$ is called *conjugation*, see Subsection 2.3

argument.

The set of automorphisms $G_{n,q} = \{(\pi, \psi, \varphi(n)) | \pi \in S_n, \psi \in S_q, \varphi(n) \in S_q^n\}$ is proved a group under the group operation of semi-direct product. The g-automorphism group acts on $\mathcal{P}_{n,q}$ and induces a classification of CA such that every CA in a class has the same global property *up to permutation*. For explaining the idea we preferably use rule 110 a computation universal ECA. To be specific we show that there are 48 functions which are universal up to permutation. This is compared with 6 ECA which are automorphic with f_{110} [4, 5].

Finally we show the g-automorphism classification of ELFs in the form of a table, where every g-automorphism class (GN class for short) is expressed by a union of several NW classes obtained by H. Nishio and Th. Worsch(2009)[6]. It is seen that 256 ELF are classified into 11 GN classes, which is compared with 46 NW classes.

This work has been inspired by the past mathematical works about the logical circuits made by C. Shannon [8],D. Slepian [9] and M. Harrison [2] during 1950s the dawn of the computer science. Specifically they formulated and generally solved the problem of counting the number of the equivalent or symmetry classes of Boolean functions by use of the Pólya's counting theory. Mathematically speaking, their theory is exclusively concerned with the Boolean functions ($\mathcal{P}_{n,2}$) and even afterward, as far as I know, has not been generalized to arbitrary functions ($\mathcal{P}_{n,q}$).

2 Preliminaries

The definitions and previous results are briefly restated, of which details will be found in [7, 4, 5].

2.1 CA and local structures

A cellular automaton is defined by a 4-tuple $(\mathbb{Z}^d, Q, f, \nu)$, where $\mathbb{Z}^d$ is a d-dimensional Euclidean space, Q is a finite set of *cell states*, $f : Q^n \to Q$ is a *local function* and ν is a *neighborhood*.

- **[neighborhood]** *A neighborhood* is a mapping $\nu : \mathbb{N}_n \to \mathbb{Z}^d$, where $\mathbb{N}_n = \{1, \dots, n\}$ and $n \in \mathbb{N}$. This can equivalently be seen as a list ν with n components $(\nu_1, \dots, \nu_n)$, where $\nu_i = \nu(i), 1 \leq i \leq n$, is called the i-th *neighbor*. The i-th argument of f is connected to the i-th neighbor.

- **[local structure]** A pair (f, ν) is called a *local structure* of CA. We call n the *arity* of the local structure. When the space $\mathbb{Z}^d$ and the state set Q are understood, CA is often identified with its local structure.

- **[global function]** A local structure uniquely induces a *global function* $F : Q^{\mathbb{Z}^d} \to Q^{\mathbb{Z}^d}$, which is defined by
$$F(c)(x) = f(c(x + \nu_1), ..., c(x + \nu_n)), \tag{1}$$
for any *global configuration* $c \in Q^{\mathbb{Z}^d}$, where $c(x)$ is the state of cell $x \in \mathbb{Z}^d$ in c.

Remark 1 *In the previous paper [7] the definition of local structures was more general, but in this paper we assume, without loss of generality, a restricted but most usual case of* reduced *local structures, see the following definition and Lemma 1.*

2.2 Previous results on the equivalence of local structures

Here we extract from the previous papers some basic results on the equivalence of local structures, which entail the present work on the generalized automorphism.

Definition 1 *[reduced local structure] A local structure is called* reduced, *if and only if*

- *ν is injective, i.e. $\nu_i \neq \nu_j$ for $i \neq j$ in the list of neighborhood ν and*
- f depends on all arguments.

Lemma 1 *For each local structure (f, ν) there is an equivalent local structure (f', ν') which is reduced.*

Definition 2 *[equivalence] Two local structures (f, ν) and (f', ν') are called* equivalent, *if and only if they induce* the same global function. *In that case we write $(f, \nu) \approx (f', \nu')$.*

Definition 3 *[permutation of local structure] For $\pi \in S_n$ we define the permutation of the local function and neighborhood by*

$$f^{\pi}(x_1, ..., x_n) = f(x_{\pi(1)}, ..., x_{\pi(n)}) \tag{2}$$

and

$$\nu^{\pi} = (\nu_1^{\pi}, ..., \nu_n^{\pi}), \textit{where } \nu_{\pi(i)}^{\pi} = \nu_i, \ 1 \leq i \leq n. \tag{3}$$

Then we have the basic properties of the permutation of local structures.

Lemma 2 *(f, ν) and (f^{π}, ν^{π}) are equivalent for any permutation π.*

Theorem 1 *[permutation-equivalence of local structures]*
If (f, ν) and (f', ν') are two reduced local structures which are equivalent, then there is a permutation π such that $(f^{\pi}, \nu^{\pi}) = (f', \nu')$.

2.3 Polynomial expression of local functions and S_3

The local function is expressed by a polynomial in n variables $f(\mathbf{x}_n) = f(x_1, ..., x_n)$ over finite field $GF(q)$ and the set of such polynomials will be denoted $\mathcal{P}_{n,q}$, $n \geq 1, q \geq 2$. $\mathcal{P}_{n,q}$ is a polynomial ring over $GF(q)$ mod $(x_1^q - x_1) \cdots (x_n^q - x_n)$. Obviously $|\mathcal{P}_{n,q}| = q^{q^n}$. For small n and q, f is written as follows.

The local function of an ECA is called the *elementary local function* denoted ELF, which is generally expressed by a polynomial $f(x_1, x_2, x_3)$ over $GF(2)$ as shown below.

$$\begin{aligned} f(x_1, x_2, x_3) = u_0 + u_1x_1 + u_2x_2 + u_3x_3 \\ + u_4x_1x_2 + u_5x_1x_3 + u_6x_2x_3 + u_7x_1x_2x_3, \\ \text{where } u_i \in GF(2) = \{0, 1\}, 0 \leq i \leq 7. \end{aligned} \tag{4}$$

Note that for $f \in \mathcal{P}_{3,2}$, the polynomial expression is equivalently transformed to the Boolean expression by $a + b + ab$ (polynomial) $= a \vee b$ (Boolean), ab (polynomial) $= a \wedge b$ (Boolean) and $a + 1$ (polynomial) $= \overline{a}$ (Boolean). Conjugation $f' = \varphi_1^{-1} f \varphi_1 = f(x_1 + 1, x_2 + 1, x_3 + 1) + 1$.

In the sequel, every ELF is numbered by a so called Wolfram number such as $f_{110} = x_1x_2x_3 + x_2x_3 + x_2 + x_3$. The Java program **catest106d** made by C.Lode [3] contains a useful tool for conversion between the Boolean, the polynomial and the Wolfram number.

Permutations of 3 objects are usually expressed by a symmetric group $S_3 = \{\pi_i, 0 \le i \le 5\}$ as is shown below.

$$\pi_0 = \mathbf{1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \ \pi_1 = (23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ \pi_2 = (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\pi_3 = (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ \pi_4 = (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \ \pi_5 = (13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Note that S_3 is not commutative: $\pi_2\pi_1 = (12)(23) = (123) = \pi_3$ but $\pi_1\pi_2 = (23)(12) = (132) = \pi_4$. The neighborhood $(-1, 0, 1)$ of ECA is called the elementary neighborhood (ENB for short). Then $ENB^{\pi_1} = (-1, 1, 0), ENB^{\pi_2} = (0, -1, 1)$ and so on.

3 (n, q)-permutation of local functions

We define two kinds of permutations called *n-permutation* and *q-permutation* of the local function and then unify them as *(n,q)-permutation* of f.

1. **Definition 4** *[n-permutation of f] The permutation of f defined in Definition 3 is essentially related to a permutation of the neighborhood and called hereafter the* n-permutation *of f.*

$$f^{\pi}(x_1, ..., x_n) = f(x_{\pi(1)}, ..., x_{\pi(n)})$$

Example 1 *The n-permutations of* $f_{110} = x_1x_2x_3 + x_2x_3 + x_2 + x_3$ *are*

$$f_{110}^{\pi_0} = f_{110}^{\pi_1} = x_1x_2x_3 + x_2x_3 + x_2 + x_3.$$
$$f_{110}^{\pi_2} = f_{110}^{\pi_4} = x_1x_2x_3 + x_1x_3 + x_1 + x_3 = f_{122}.$$
$$f_{110}^{\pi_3} = f_{110}^{\pi_5} = x_1x_2x_3 + x_1x_2 + x_1 + x_2 = f_{124}.$$

2. **Definition 5** *[q-permutation of f] For an argument x of f which takes a value from Q, define a permutation $\varphi \in S_q$ as a bijection $x^{\varphi} : Q \to Q$. Then consider a list of permutations $\varphi(n) = (\varphi_1, ..., \varphi_n)$ where $\varphi_i \in S_q$, $1 \le i \le n$ or $\varphi(n) \in S_q^n = S_q \times \cdots \times S_q$ (direct product of n copies of S_q) . Now we define the* q-permutation *of f by*

$$f\varphi(n)(\mathbf{x}_n) = f(x_1^{\varphi_1}, ..., x_n^{\varphi_n}). \tag{5}$$

Example 2 *For the binary case $Q = \{0, 1\}$ the permutations $\varphi(n)$ is expressed by a binary word $\varphi(a_1 \cdots a_n)$ which* operates *on $\mathbf{x}_n$ such that $x_i^{a_i} = x_i$ if $a_i = 0$ and $x_i^{a_i} = x_i{+}1$ if $a_i = 1$ (Boolean negation). For example $f_{110}\varphi(100) = (x_1+1)x_2x_3+x_2x_3+x_2+x_3 = x_1x_2x_3+x_2+x_3 = f_{230}$, $f_{110}\varphi(110) = f_{185}$ and so on. In general for a prime number of states $Q = \{0, 1, ..., p-1\} = GF(p)$, the permutation of Q is expressed by an addition modulo p such that $x + a, a \in Q$.*

3. **Definition 6** *[(n, q)-permutation of f] Combining n-permutation and q-permutation with an additional permutation $\psi : Q \to Q$ of the function value, we finally define a* unified *permutation of f called* (n, q)-permutation *of f which is expressed by a 3-tuple of permutations $(\pi, \psi, \varphi(n))$.*

$$(\pi, \psi, \varphi(n))f(\mathbf{x}_n) = \psi f^{\pi}\varphi(n)(\mathbf{x}_n) = \psi f(x_{\pi(1)}^{\varphi_1}, ..., x_{\pi(n)}^{\varphi_n}). \tag{6}$$

Example 3

$$\begin{aligned} (\pi_2, (1, 2), \varphi(100))f_{110} &= f_{110}(x_2^1, x_1^0, x_3^0) + 1 \\ &= (x_2 + 1)x_1x_3 + x_1x_3 + x_1 + x_3 + 1 \\ &= x_1x_2x_3 + x_1 + x_3 + 1 \\ &= f_{37}. \end{aligned}$$

All (n, q)-permutations of f_{110} are given in Example 5.

4 Generalized automorphism of CA

In this section, using the (n, q)-permutation of f, we define a generalized automorphism called *g-automorphism* of CA and prove that the set of the g-automorphisms constitutes a group under a rule of the semi-direct product.

Definition 7 *For two CA $A = (f, \nu)$ and $A' = (f', \nu')$, A is called* g-automorphic *with A' denoted $A \cong_g A'$, if and only if there is an (n, q)-permutation $(\pi, \psi, \varphi(n))$ such that the following equation holds.*

$$(f', \nu') = (\psi f^{\pi} \varphi(n), \nu^{\pi}). \tag{7}$$

Remarks 1 *If for any $\varphi \in S_q, \varphi_i = \varphi$, $1 \leq i \leq n$, then by taking $\psi = \varphi^{-1}$, g-automorphism becomes the original automorphism [4, 5].*

We show here that the set of the 3-tuples of permutations

$$G_{n,q} = \{(\pi, \psi, \varphi(n)) \mid \pi \in S_n, \psi \in S_q, \varphi(n) \in S_q^n\}$$

is a group. The order of $G_{n,q}$ is $n!q^{n+1}$.

Theorem 2 *Let $g = (\pi, \psi, \varphi(n)) \in G_{n,q}$ and $g' = (\pi', \psi', \varphi'(n)) \in G_{n,q}$. Then $G_{n,q}$ is a group under the rule of* semi-direct product;

$$g'g = (\pi', \psi', \varphi'(n))(\pi, \psi, \varphi(n)) = (\pi'\pi, \psi'\psi, \varphi'(n)^{\pi}\varphi(n)), \tag{8}$$

where $\varphi'(n)^{\pi}\varphi(n) = (\varphi'_{\pi(1)}\varphi_1, \ldots, \varphi'_{\pi(n)}\varphi_n)$ is the componentwise group operation of the direct product S_q^n.

Proof 1 *The proof is done in the same way as the proof given by M. Harrison for Boolean functions, see page 822 of [2]. He utilizes Theorem 6.5.1, page 88, Section 6.5 of the text book by M. Hall [1], where the semi-direct product $K \rtimes_{\varphi} H$ of K by H is defined by the rule*

$$[h_1, k_1] \cdot [h_2, k_2] = [h_1 h_2, k_1^{h_2} k_2], \tag{9}$$

where $h_1, h_2 \in H, k_1, k_2 \in K$ and the automorphism φ^3 of K is defined by for any $h \in H, k \rightleftharpoons k^h$ for all $k \in K$. The product rule (9) is shown well defined: (1) associative, (2) the identity is $[\mathbf{1}, \mathbf{1}]$ and (3) a left inverse $[h, k]^{-1}$ of $[h, k]$ is $[h^{-1}, (k^{-1})^{h^{-1}}]$.

At applying this standard rule of the semi-direct product to the 3-tuples in Equation (8), first consider the semi-direct product $S_n \rtimes_{\varphi} S_q^n$ and then combine $\psi \in S_q$ as a direct product.

The following example will help understanding the semi-direct product of $G_{n,q}$.

Example 4 *Suppose that two group elements $g_1 = (\pi_1, \psi_0, \varphi(100))$ and $g_2 = (\pi_2, \psi_0, \varphi(001))$ in $G_{3,2}$ act[4] on $f_{110} \in \mathcal{P}_{3,2}$ in this order where $\psi_0 = \mathbf{1}$. That is*

$$\begin{aligned} g_1 \circ f_{110} &= x_1x_2x_3 + x_2 + x_3 = f_{230} \\ g_2 \circ (g_1 \circ f_{110}) &= g_2 \circ f_{230} = x_1x_2x_3 + x_1x_2 + x_1 + x_3 + 1 = f_{229}. \end{aligned}$$

[3]Note that this symbol φ is independent from our permutation φ.

[4]The symbol of group action $\circ$ is usually omitted like group operation.

On the other hand, by applying the rule of the semi-direct product (8), we see

$$\begin{aligned} g_2 g_1 &= (\pi_2, \psi_0, \varphi(001))(\pi_1, \psi_0, \varphi(100)) \\ &= (\pi_2\pi_1, \psi_0, \varphi(001)^{\pi_1}\varphi(100)) \\ &= (\pi_2\pi_1, \psi_0, \varphi(010)\varphi(100)) \\ &= (\pi_3, \psi_0, \varphi(110)) \\ &= g_3 \end{aligned}$$

But

$$g_3 \circ f_{110} = x_1x_2x_3 + x_1x_2 + x_1 + x_3 + 1 = f_{229}.$$

Lemma 3 *Any g-automorphic CA are equivalent (have the same global function) up to permutation.*

Proof 2 *It is obvious from Equation (7). Permute the local function f with the inverses of $\varphi(n)$ and ψ.*

Example 5 *[g-automorphism class of f_{110}] As a typical example of g-automorphism classification, we consider f_{110} again. Table 1 below lists up the (n,q)-permutations of f_{110} only for the case of $\psi_0 = \mathbf{1}$. The permutation $\psi_1 f^{\pi}\varphi$ where $\psi_1 = (12)$ is obtained by adding 1 to the polynomial of each entry. For example for $\psi_0 f^{\pi_2}\varphi(010) = f_{167} = x_1x_2x_3 + x_1x_3 + x_2x_3 + x_1 + 1$, we have $\psi_1 f^{\pi_2}\varphi(010) = x_1x_2x_3 + x_1x_3 + x_2x_3 + x_1 = f_{88}$.*

Table 1: g-automorphism class of f_{110}

$\psi, \varphi\backslash\pi$	π_0	π_1	π_2	π_3	π_4	π_5
$\psi_0 f\varphi(000)$	110	110	122	122	124	124
$\psi_0 f\varphi(100)$	230	230	218	218	188	188
$\psi_0 f\varphi(010)$	155	157	167	181	199	211
$\psi_0 f\varphi(001)$	157	155	181	167	211	199
$\psi_0 f\varphi(110)$	185	217	173	229	203	227
$\psi_0 f\varphi(101)$	217	185	229	173	203	203
$\psi_0 f\varphi(011)$	103	103	91	91	103	61
$\psi_0 f\varphi(111)$	118	118	94	94	62	62

For $f \neq f' \in \mathcal{P}_{3,2}$, it is seen that $\psi_1 f \neq f$ and $\psi_1 f \neq \psi_1 f'$. Since Table 1 contains 24 different functions among the $3!2^3 = 48$ entries, it is seen that the number of the functions that are g-automorphic with f_{110} is $24 \times 2 = 48$. Then by Lemma 3, we see

Lemma 4 *There are 48 local functions which are computation universal up to permutation.*

This is compared with 6 functions which are automorphic with f_{110} [4, 5].

5 Generalized automorphism classification of CA

g-automorphism $\cong_g$ is an equivalence relation in $\mathcal{P}_{n,q}$ and naturally induces a generalized classification of CA called *g-automorphism classification*. Every local function in a class has the same global property *up to permutation* by Lemma 3.

5.1 g-automorphism classification of ELF

The classification of 256 ELFs into 11 g-automorphism classes (denoted GN class) is shown in Table 2, where every g-automomophism classes a union of NW classes. The NW classification will be found in [6, 4]. 6 functions in GN6** are reversible and 32 functions in GN9*, GN10* and GN11* are surjective but not injective. The rests are not surjective nor injective. GN8 consists of 48 universal functions.

Table 2: g-automorphism classification of 2-state 3-neighbor CA

GN class	size	NW classes
$GN1$	2	$NW1$
$GN2$	44	$NW2, NW6, NW10, NW22, NW38, NW43$
$GN3$	22	$NW3, NW7, NW11, NW29, NW34$
$GN4$	24	$NW4, NW9, NW37$
$GN5$	24	$NW5, NW8, NW20, NW35$
$GN6**$	6	$NW12**, NW44**(reversible)$
$GN7$	54	$NW13, NW14, NW15, NW18, NW21, NW23, NW26,$ $NW33, NW36, NW39, NW45, NW46$
$GN8$	48	$NW16, NW17, NW24, NW28, NW32, NW41$ $(universal)$
$GN9*$	24	$NW19*, NW25*, NW31*, NW42*$
$GN10*$	6	$NW27*$
$GN11*$	2	$NW30*, NW40*$
total	256	46 NW classes

6 Concluding remarks

We have generalized the automorphism (classification) of CA by considering two kinds of permutations of the local structures; n-permutation of the neighborhood and q-permutation of the cell states. For explaining the idea, we inserted several examples using rule f_{110} and gave the table of g-automorphisms of f_{110}. As a byproduct we see that 48 local rules are universal up to permutation. We also gave the g-automorphism classification of 256 ELF into 11 g-automorphism classes. The counting problem of the number of the g-automorphism classes has been left for future research.

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