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Arithmetical rank of squarefree monomial ideals of height two whose quotient rings are Cohen–Macaulay

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1. INTRODUCTION

Let $S$ be a polynomial ring over a field $K$ and $I$ a squarefree monomial ideal of $S$. We denote by $G(I)$ the minimal set of monomial generators of $I$. The arithmetical rank of $I$ is defined by the minimum number of elements of $S$ such that those generate $I$ up to radical:

$$\text{ara } I := \min \{ r : \text{there exist } g_1, \ldots, g_r \in S \text{ such that } \sqrt{(g_1, \ldots, g_r)} = \sqrt{I} \}.$$

By the result due to Lyubeznik [12], inequalities

$$\text{height } I \leq \text{pd}_S S/I \leq \text{ara } I$$

hold, where height $I$ is the height of $I$ and pd$_S S/I$ is the projective dimension of $S/I$ over $S$. We sometimes write pd$_S S/I$ as pd $S/I$ if there is no fear of confusion. It is natural to ask when ara $I = \text{pd}_S S/I$ holds. Examples of squarefree monomial ideals those satisfy this equality are found in e.g., [1, 2, 3, 4, 5, 7, 9, 10, 11, 13, 14]. We say that $I$ is a set-theoretic complete intersection if ara $I = \text{height } I$ holds. When this is the case, the equality ara $I = \text{pd}_S S/I$ holds. On the other hand, we say that $S/I$ is Cohen–Macaulay if pd$_S S/I = \text{height } I$ holds. By definition, if $I$ is a set-theoretic complete intersection, then $S/I$ is Cohen–Macaulay. However for the converse, there are counterexamples (see [15, 10]), though these depend on the characteristic of $K$. Our main result on this report is the following theorem.

**Theorem 1.1.** Let $I$ be a squarefree monomial ideal of $S$. Suppose that $S/I$ is Cohen–Macaulay. If height $I = 2$, then $I$ is a set-theoretic complete intersection. That is,

$$\text{ara } I = \text{pd}_S S/I = \text{height } I = 2.$$
2. Preliminaries

In this section, we recall the notion of the Stanley–Reisner ring and the Alexander duality.

Let $X = \{x_1, \ldots, x_n\}$ be a set of vertices. A collection $\Gamma$ of subsets of $X$ is called a simplicial complex on the vertex set $X$ if (i) $\{x_i\} \in \Gamma$ for all $i = 1, \ldots, n$; (ii) if $F \in \Gamma$, then $G \in \Gamma$ for all $G \subset F$. If $\Gamma$ consists of all subsets of its vertex set, then $\Gamma$ is called a simplex. An element $F \in \Gamma$ is called a face and a maximal face of $\Gamma$ is called a facet. A simplicial complex is determined by its facets. When the set of facets of $\Gamma$ is $\{G_1, \ldots, G_s\}$, we write $\Gamma = \langle G_1, \ldots, G_s \rangle$. The dimension of $\Gamma$ is defined by $\dim \Gamma := \max\{|F| - 1 : F \in \Gamma\}$. Throughout this report, we assume $\dim \Gamma < |X| - 2$. The Alexander dual complex of $\Gamma$ is defined by

$$\Gamma^* := \{F \subset X : X \setminus F \notin \Gamma\}.$$ 

This is also a simplicial complex on $X$. Note that $(\Gamma^*)^* = \Gamma$.

We identify the vertex set $X = \{x_1, \ldots, x_n\}$ with the set of variables of $S = K[X] = k[x_1, \ldots, x_n]$. The Stanley–Reisner ideal of $\Gamma$ is defined by

$$I_{\Gamma} := (m_F : F \subset X, F \notin \Gamma), \quad \text{where} \quad m_F = \prod_{x_i \in F} x_i.$$ 

The quotient ring $K[\Gamma] := K[X]/I_{\Gamma}$ is called the Stanley–Reisner ring of $\Gamma$. The prime decomposition of $I_{\Gamma}$ is

$$I_{\Gamma} = \bigcap_{G \in \Gamma : \text{a facet}} P_G, \quad \text{where} \quad P_G = \langle x_i \in X : x_i \notin G \rangle.$$ 

On the other hand, the Stanley–Reisner ideal $I_{\Gamma^*}$ is minimally generated by

$$G(I_{\Gamma^*}) = \{m_{X \setminus G} : G \in \Gamma \text{ is a facet}\}.$$ 

In above, we construct a squarefree monomial ideal of $K[X]$ from a given simplicial complex $\Gamma$ on $X$ with $\dim \Gamma < |X| - 2$. On the contrary, we can construct a simplicial complex on $X$ when a squarefree monomial ideal $I$ of $K[X]$ with $\indeg I := \min\{\deg m : m \in G(I)\} \geq 2$. When $I = I_{\Gamma}$, then the ideal $I^* := I_{\Gamma^*}$ is called the Alexander dual ideal of $I$.

**Example 2.1.** Let $\Gamma$ be the simplicial complex on $X = \{x_1, \ldots, x_6\}$ whose facets are

$$\{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_3, x_5, x_6\}$$

(see Figure 1). Then

$$I = I_{\Gamma} = (x_4, x_5, x_6) \cap (x_1, x_2, x_6) \cap (x_1, x_2, x_4),$$

$$I^* = I_{\Gamma^*} = (x_4x_5x_6, x_1x_2x_6, x_1x_2x_4).$$

**Remark 2.2.** By the Alexander duality, it is clear that $\indeg I^* = \height I$. Moreover, Eagon–Reiner [6] proved that $I^*$ has a linear resolution if and only if the quotient ring is Cohen–Macaulay.
Figure 1. $\Gamma = \langle \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_3, x_5, x_6\} \rangle$

3. Generalized Tree

The generalized tree, which was introduced by Barile and Terai [4], is the notion on simplicial complexes. The definition is recursive: (i) a simplex is a generalized tree; (ii) if $\Gamma$ is a generalized tree on $X$, then for an arbitrary face $F \in \Gamma$ and an arbitrary new vertex $x_0$, the union $\Gamma' := \Gamma \cup \text{co}_{x_0} F$ is a generalized tree on $X' := X \cup \{x_0\}$, where $\text{co}_{x_0} F$ is the simplex on $F \cup \{x_0\}$.

Example 3.1. As noted below, the simplicial complex $\Gamma$ in Example 2.1 is a generalized tree (see also Figure 2).

First, $\Gamma_1 := \langle \{x_1, x_2, x_3\} \rangle$ is a simplex, thus it is a generalized tree. Second, set $F_1 = \{x_3\} \in \Gamma_1$. Then $\Gamma_2 := \Gamma_1 \cup \text{co}_{x_3} F_1 = \langle \{x_1, x_2, x_3\}, \{x_3, x_4\} \rangle$ is a generalized tree. Third, set $F_2 = \{x_3, x_4\} \in \Gamma_2$. Then $\Gamma_3 := \Gamma_2 \cup \text{co}_{x_5} F_2 = \langle \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\} \rangle$ is a generalized tree. Last, set $F_3 = \{x_3, x_5\} \in \Gamma_3$. Then $\Gamma_4 := \Gamma_3 \cup \text{co}_{x_6} F_3 = \langle \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_3, x_5, x_6\} \rangle = \Gamma$ is a generalized tree.

Figure 2. Generalized trees

The following lemma can be obtained by [4, Lemma 2] using the Alexander duality (see Remark 2.2).

Lemma 3.2. Let $\Gamma$ be a simplicial complex on $X$ with $\dim \Gamma < |X| - 2$. Then $\Gamma$ is a generalized tree if and only if $\text{height } I_{\Gamma^*} = 2$ and $S/I_{\Gamma^*}$ is Cohen–Macaulay.

Barile and Terai [4] used the original form of this lemma to prove that if $I$ has a 2-linear resolution, then $\text{ara } I = \text{pd}_S S/I$, which was first proved by Morales [13]. Thanks to the inductive definition of generalized tree, the proof due to Barile and Terai was proceeded by induction on $|X|$, and done by comparing the projective dimensions of $K[\Delta]$ and $K[\Delta']$, the arithmetical ranks of $I_\Delta$ and $I_{\Delta'}$, which is needed to guarantee the inductive step. In fact, our motivation is to consider the Alexander dual of these results.
4. KEY RESULT

In this section, we state the outline of the proof of Theorem 1.1. Let $\Delta$ be a simplicial complex on $X$ with $\dim \Delta < |X| - 2$. Set $\Gamma = \Delta^*$. Let $F$ be a face of $\Gamma$ and $x_0$ a new vertex. Set $\Gamma' = \Gamma \cup \text{co}_{x_0} F$, $X' = X \cup \{x_0\}$, and $\Delta' = (\Gamma')^*$.

First we compare the projective dimensions of $K[\Delta]$ and $K[\Delta']$.

**Lemma 4.1.** Using above notations, we have

$$\text{pd } K[\Delta'] = \text{pd } K[\Delta].$$

For the proof of this lemma, please see [8].

Second we compare the arithmetical ranks of $I_{\Delta}$ and $I_{\Delta'}$.

**Proposition 4.2.** We use the notations as above. If $\text{ara } I_{\Delta} = 2$, then $\text{ara } I_{\Delta'} \leq 2$. In particular, if $\text{ara } I_{\Delta} = \text{pd } K[\Delta] = 2$, then the same equalities hold for $\Delta'$.

The proof of Theorem 1.1 is done by induction on $|X|$ using Lemma 3.2. Proposition 4.2 guarantees the inductive step on the proof.

**Proof of Proposition 4.2.** Set

$$G(I_{\Delta}) = \{m_1, \ldots, m_\mu\}.$$

Then it is easy to see that

$$I_{\Delta'} = (m_0, x_0m_1, \ldots, x_0m_\mu),$$

where $m_0 = m_{X\setminus F}$. Let $G$ be a facet of $\Gamma$ which contains $F$. We may assume $m_1 = m_{X\setminus G}$. Then $m_1$ divides $m_0$.

Let $g_1, g_2 \in I_{\Delta}$ be elements which generate $I_{\Delta}$ up to radical. Since $m_1 \in I_{\Delta} = \sqrt{(g_1, g_2)}$, there exists an integer $\ell$ such that $m_1^\ell \in (g_1, g_2)$. Therefore we can write as

$$m_1^\ell = a_1g_1 + a_2g_2, \quad a_1, a_2 \in K[X].$$

Set

$$g'_1 = x_0g_1 - a_2m_0, \quad g'_2 = x_0g_2 + a_1m_0.$$

We claim that $g'_1, g'_2$ generate $I_{\Delta'}$ up to radical. Set $J = (g'_1, g'_2)$. Since $g'_1, g'_2 \in I_{\Delta'}$, it is clear $\sqrt{J} \subset I_{\Delta'}$. We prove the opposite inclusion.

Since

$$a_1g'_1 + a_2g'_2 = x_0(a_1g_1 + a_2g_2) = x_0m_1^\ell,$$

we have $x_0m_1^\ell \in J$, thus $x_0m_1 \in \sqrt{J}$. Since $m_1$ divides $m_0$, we have $x_0m_0 \in \sqrt{J}$. Then we have $x_0g_1, x_0g_2 \in \sqrt{J}$ because $x_0g'_1, x_0g'_2 \in J$. This leads that $x_0m_1, \ldots, x_0m_\mu \in \sqrt{J}$. On the other hand, we also have $a_2m_0, a_1m_0 \in \sqrt{J}$. Since

$$g_1(a_1m_0) + g_2(a_2m_0) = m_0(a_1g_1 + a_2g_2) = m_0m_1^\ell,$$

we have $m_0m_1 \in \sqrt{J}$. Again since $m_1$ divides $m_0$, we have $m_0 \in \sqrt{J}$, as required. \qed
Example 4.3. Let $\Gamma$ be the simplicial complex as in Example 2.1. Set $F = \{x_4\}$ and $\Gamma' = \Gamma \cup \text{co}_{x_0} F$. The vertex set of $\Gamma'$ is $X' = X \cup \{x_0\}$. Then facets of $\Gamma'$ are $\{x_0, x_4\}$ together with facets of $\Gamma$. Therefore

$$I_{\Gamma'} = (x_0, x_4, x_5, x_6) \cap (x_0, x_1, x_2, x_6) \cap (x_1, x_2, x_3, x_5, x_6),$$

$$I_{\Delta'} = (x_0x_4x_5x_6, x_0x_1x_2x_6, x_0x_1x_2x_4, x_1x_2x_3x_5x_6).$$

In this case, $m_0 = x_1x_2x_3x_5x_6$. Note that

$$I_{\Gamma} = (x_4, x_5, x_6) \cap (x_1, x_2, x_6),$$

$$I_{\Delta} = (x_4x_5x_6, x_1x_2x_6, x_1x_2x_4).$$

The facet of $\Gamma$ which contains $F$ is $\{x_3, x_4, x_5\}$. It corresponds to $m_1 := x_1x_2x_6 \in G(I_{\Delta})$ and this divides $m_0$.

Set

$$\begin{aligned}
g_1 &= x_1x_2x_6, \\
g_2 &= x_4x_5x_6 + x_1x_2x_4,
\end{aligned}$$

and

$$\begin{aligned}
h_1 &= x_1x_2x_4, \\
h_2 &= x_4x_5x_6 + x_1x_2x_6.
\end{aligned}$$

Then $\sqrt{(g_1, g_2)} = \sqrt{(h_1, h_2)} = I_{\Delta}$ (this fact can be easily seen by [14, Lemma, p. 249]). Since $g_1 = m_1$, we can easily construct two elements $g'_1, g'_2$ which generate $I_{\Delta'}$ up to radical from $g_1, g_2$ as in the proof of Proposition 4.2:

$$\begin{aligned}
g'_1 &= x_0x_1x_2x_6, \\
g'_2 &= x_0x_4x_5x_6 + x_0x_1x_2x_4 + x_1x_2x_3x_5x_6.
\end{aligned}$$

(In this case, $a_1 = 1, a_2 = 0$. We can also prove that this $g'_1, g'_2$ generate $I_{\Delta'}$ up to radical by [14, Lemma, p. 249].) On the other hand, for $h_1, h_2$, the construction of two elements $h'_1, h'_2$ which generate $I_{\Delta'}$ up to radical is rather complicated. In this case,

$$m_1^2 = -x_5x_6^2 h_1 + x_1x_2x_6 h_2.$$

Thus $a_1 = -x_5x_6^2, a_2 = x_1x_2x_6$. Therefore

$$\begin{aligned}
h'_1 &= x_0x_1x_2x_4 - x_1x_2x_6 \cdot x_1x_2x_3x_5x_6, \\
h'_2 &= x_0x_4x_5x_6 + x_0x_1x_2x_6 - x_5x_6^2 \cdot x_1x_2x_3x_5x_6.
\end{aligned}$$

By (4.1), if $I_{\Delta}$ is generated by $g_1, \ldots, g_h$ up to radical, then $m_0, g_1, \ldots, g_h$ generate $I_{\Delta'}$ up to radical. Therefore the inequality $\text{ara} I_{\Delta'} \leq \text{ara} I_{\Delta} + 1$ always holds. Proposition 4.2 says that more precisely, the inequality $\text{ara} I_{\Delta'} \leq \text{ara} I_{\Delta}$.
holds when $\text{ara} I_\Delta = 2$. In general, does this inequality hold? By the similar technique to the proof of Proposition 4.2, we have the following corollary.

**Corollary 4.4.** We use the notations as above. If $\text{ara} I_\Delta$ is even, then

$$\text{ara} I_{\Delta'} \leq \text{ara} I_\Delta.$$

**Proof.** Set $\text{ara} I_\Delta = 2h$. We assume that $g_1, \ldots, g_{2h}$ generate $I_\Delta$ up to radical. Since $m_1 \in I_\Delta = \sqrt{(g_1, \ldots, g_{2h})}$, there is an integer $\ell$ such that $m_1^\ell \in (g_1, \ldots, g_{2h})$. Then we can write as

$$m_1^\ell = a_1 g_1 + \cdots + a_{2h} g_{2h}, \quad a_1, \ldots, a_{2h} \in K[X].$$

Set

$$g_{2i-1}' = x_0 g_{2i-1} - a_{2i} m_0, \quad g_{2i}' = x_0 g_{2i} + a_{2i-1} m_0, \quad i = 1, \ldots, h.$$ We claim that $g_1', \ldots, g_{2h}'$ generate $I_{\Delta'}$ up to radical.

Set $J = (g_1', \ldots, g_{2h}')$. First we note that

$$x_0(a_{2i-1} g_{2i-1} + a_{2i} g_{2i}) = a_{2i-1} g_{2i-1}' + a_{2i} g_{2i}' \in J.$$ Then

$$x_0 m_1^\ell = x_0(a_1 g_1 + \cdots + a_{2h} g_{2h}) = \sum_{i=1}^{h} x_0(a_{2i-1} g_{2i-1} + a_{2i} g_{2i}) \in J.$$ Thus $x_0 m_1 \in \sqrt{J}$. Since $m_1$ divides $m_0$, we have $x_0 m_0 \in \sqrt{J}$. Then $x_0 g_i' \in J$ implies $x_0 g_i \in \sqrt{J}$ for $i = 1, \ldots, 2h$. Therefore $x_0 m_1, \ldots, x_0 m_h \in \sqrt{J}$. On the other hand, we also have $a_i m_0 \in \sqrt{J}$ for $i = 1, \ldots, 2h$. Since

$$m_0 m_1^\ell = m_0 (a_1 g_1 + \cdots + a_{2h} g_{2h}) = g_1(a_1 m_0) + \cdots + g_{2h}(a_{2h} m_0) \in \sqrt{J},$$ we have $m_0 m_1 \in \sqrt{J}$. Again since $m_1$ divides $m_0$, we have $m_0 \in \sqrt{J}$. \[\square\]

Then the following question occurs.

**Question.** If $\text{ara} I_\Delta = 3$, then does the inequality $\text{ara} I_{\Delta'} \leq \text{ara} I_\Delta$ hold?

If this is true, then the same technique as in the proof of Corollary 4.4 would lead the inequality $\text{ara} I_{\Delta'} \leq \text{ara} I_\Delta$ with no condition on $\text{ara} I_\Delta$.

**REFERENCES**


