数理解析研究所講究録
第 1713 巻 2010 年 111-123

Unstable periodic orbits embedded in a continuous time dynamical system -time averaged properties-
連続カオス力学系の不安定周期軌道解析 -軌道平均値について-

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ABSTRACT

It is recently found in some dynamical systems in fluid dynamics that only a few unstable periodic orbits (UPOs) with low periods can give good approximations to mean properties of turbulent (chaotic) solutions. By employing the Kuramoto-Sivashinsky equation we compared time averaged properties of a set of UPOs embedded in a chaotic attractor and those of a set of segments of chaotic orbits, and reported that the distribution of a time average of a dynamical variable along UPOs with lower and higher periods are similar to each other and the variance of the distribution is small, in contrast with that along chaotic segments. The result is similar to those for low dimensional ordinary differential equations (Lorenz system, Rössler system and Economic system) reported in Saiki and Yamada, 2009, Physical Review E, 79(1) R015201.

1 Introduction

Chaos in dynamical systems has been discussed in relation to UPOs (unstable periodic orbits) embedded in a chaotic attractor, as a chaotic orbit is considered to be approximatable by an ensemble of UPOs which are densely distributed in the chaotic attractor [3]. Recently, in some turbulence systems in fluid dynamics, it has been shown that even only a few UPOs with relatively low periods can capture mean properties of chaotic motions [6]. For the turbulent Couette flow of rather low Reynolds number in the full

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Navier-Stokes system, Kawahara and Kida obtained a remarkable agreement of an averaged velocity profile along a single UPO with that along a chaotic orbit in phase space of a turbulent Couette flow. Later van Veen *et al.* [16] performed a numerical study of an isotropic Navier-Stokes turbulence with high symmetry, and found that among several UPOs there is an UPO with relatively low period where the energy dissipation rate appears to converge to a nonzero value as assumed in the Kolmogorov similarity theory in the limit of large Reynolds number. This suggests that the UPO corresponds to the isotropic turbulence of fluid motion, although the Reynolds number is not large enough to discuss the detailed properties of the fully developed turbulence because of computational difficulties. As for the universal statistical properties of fluid turbulence at high Reynolds numbers, employing the GOY shell model, Kato and Yamada [7] found a single UPO which gives a fairly good approximation to the scaling exponents of structure functions of velocity, which suggests that the intermittency in the model turbulence can be interpreted as a property of a single UPO, rather than a statistical contribution of complex orbits.

In the above studies, it seems that only a few UPOs with relatively low periods are enough to capture some mean properties of a chaotic solution. However, on the other hand, the chaotic attractor includes an infinite number of UPOs, and it appears that an UPO with longer period gives a better approximation to the statistical properties of chaotic solutions, as a set of long UPOs and a set of chaotic orbits are intuitively taken to have similar statistical properties. So we may have a question why in the above systems even a small number of UPOs with rather low periods can give a remarkably good approximation to the chaotic mean values. Some studies have been concerned with this problem [8, 5, 13, 14]. Kawasaki and Sasa studied a simple model of chaotic dynamical systems with a large degree of freedom, and found that there is an ensemble of UPOs with the special property that the expectation values of macroscopic quantities can be calculated using one UPO sampled from the ensemble. Hunt and Ott studied an optimal periodic orbit which yields the optimal (extreme) value of a time average of a given smooth performance function of dynamical variables. They obtained an implication that the optimal periodic orbit is typically a periodic orbit of low period, although they do not consider the relation of averaged statistical properties along UPOs and chaotic orbits. On the other hand, Yang *et al.* reported that the optimal UPO can be a periodic orbit of high period when the system is near crisis. In a study on UPOs of low dimensional map systems by Saiki and Yamada [13], it is reported that UPOs with low periods are not effective to approximate the time averaged properties of chaotic orbits.

Recently Saiki and Yamada [14] employed chaotic systems described by low dimensional ODEs and investigate the relation between the average of a dynamical quantity along an UPO and that along a chaotic orbit, especially with an attention focused on the dependence of the variance of averaged values on the periods of the UPOs. At a first glance, it may appear that if we take all the UPOs with the period around T, for example, and take the averages of a dynamical quantity along these UPOs, the variance of the averages would decrease as T increases, because an extremely long orbit would cover most part of the chaotic attractor, capturing possible dynamical states on the attractor. The aim was to see whether this intuitive discussion holds for chaotic systems simple enough to obtain a large number of UPOs by available numerical computation with double accuracy. For this purpose we take three chaotic systems; Lorenz system, Rössler model and a business cycle model. A set of UPOs in each model were obtained numerically, and found that for every chaotic system the distributions of a time average of a dynamical variable along UPOs with lower and higher periods are similar to each other and the variance of the distribution is small, in contrast with that along chaotic segments. Here, in this paper, we study Kuramoto-Sivashinsky equation as an example of a partial differential equation system and examine time averaged properties along UPOs and segments of chaotic orbits with the corresponding lengths.

2 Time averaged properties

UPOs in the Kuramoto-Sivashinsky equation are already studied in some ways [2, 17, 4, 9, 10, 11]. Christensen et al. reported that cycle expansion theory works in the system with a periodic boundary condition in some set of parameter values. Zoldi and Greenside investigated UPOs of the Kuramoto-Sivashinsky equation with a rigid boundary condition, which generates spatio temporal chaotic behaviors. In this paper, we study Kuramoto-Sivashinsky equation with a periodic boundary condition with the same setting as that studied in the previous studies [2, 11]. That is, the original system

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx} \tag{1}$$

is written in the Fourier space as

$$\dot{b_k} = (k^2 - \nu k^4) b_k + ik \sum_{m=-\infty}^{\infty} b_m b_{k-m} \tag{2}$$

by

$$u(x,t) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}, \tag{3}$$

where the coefficients b_k are in general complex variables. However, we simplify the system by assuming that b_k are pure imaginary, $b_k = ia_k$, where a_k are real and obtain evolution equations [2]

$$\dot{a_k} = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m}. \tag{4}$$

We reduce this system to 16 dimensional ODEs and fix ν as 0.02991. The system generates two chaotic attractors which are symmetric to each other.

Here we focus our attention to the distribution of time averaged values of a dynamical variable along UPOs of the Kuramoto-Sivashinsky system. In order to detect UPOs we employ in this paper the Newton-Raphson-Mees method in which the period of the UPO is regarded as a variable to be found in the numerical calculation [12]. We found more than 650 UPOs of the periods from 0.87072 through 12.30608, corresponding respectively from 1 through 14 Poincaré map periods (PERIODs). Detected UPOs are classified into three types. UPOs which are embedded in a chaotic attractor are classified into the first type. In Fig.1 two examples of the (a_1, a_2) projections of UPOs ((b)$T = 0.870729$, (c)$T = 6.172071$) are described in contrast with that of a chaotic attractor (a). The second type is a UPO which is outside a chaotic attractor but mediates an attractor merging crisis at the different parameter value. The stable manifold of the UPO forms the basin boundary of two chaotic attractors before the merging crisis and the orbit becomes embedded in a big attractor after the merging crisis [10]. Other existing UPOs which are outside a chaotic attractor are classified into the third type. Here in this paper we focus our attention to UPOs of the first type which are embedded in a chaotic attractor.

It should be remarked that the Poincaré map is defined by the Poincaré section $a_1 = 0$ with $da_1/dt > 0$. In our numerical calculation, we identified most UPOs with PERIOD less or equal to 12.

One of the most important indices representing the complexity of a dynamical system is the topological entropy [1], which is estimated by the exponential growth rate of the number of periodic orbits; $h_{top} = \limsup_{N\to\infty} \log(\#\{\text{PERIOD-}N \text{ UPOs}\})/N$, and the topological entropy h_{top} of the Poincaré map in this case is estimated to be log(1.6) from Fig. 2. We should remark a clear linear dependence of $\log\{\#\text{UPO}\}$ on N which suggests that the number of UPOs with PERIOD N detected in our computation is sufficient to study statistical properties of UPOs. We now calculate the time average of a_2 ($\langle a_2\rangle \equiv \int_{t=0}^{T} a_2/T dt$) along each UPO with period T. $\langle a_2\rangle$s along UPOs take similar but different values around the average value of $\langle a_2\rangle$s along chaotic segments (-0.06477) (Fig. 3). Fig.4 shows the

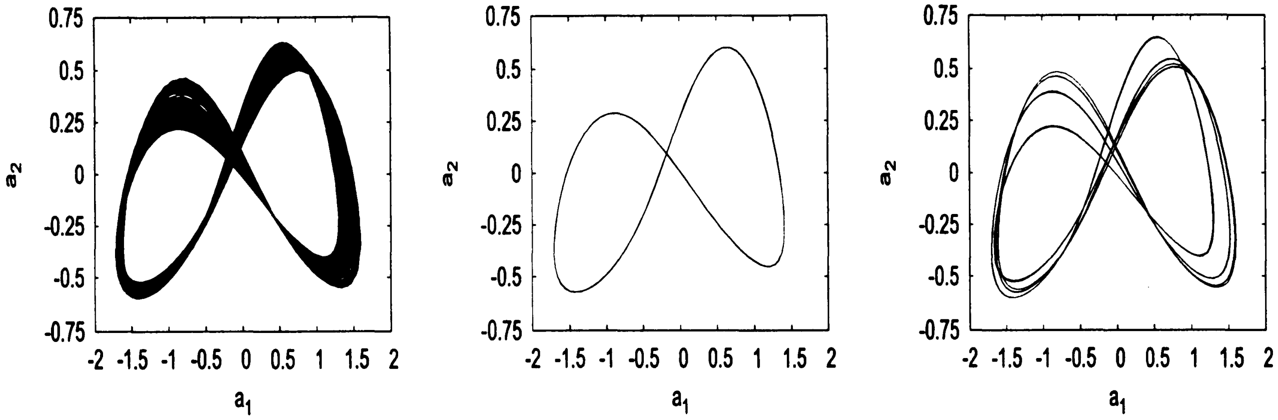

Fig. 1: Projections of a chaotic attractor (a) and UPOs ((b)T =0.870729, (c)T =6.172071) onto a_1-a_2 plane

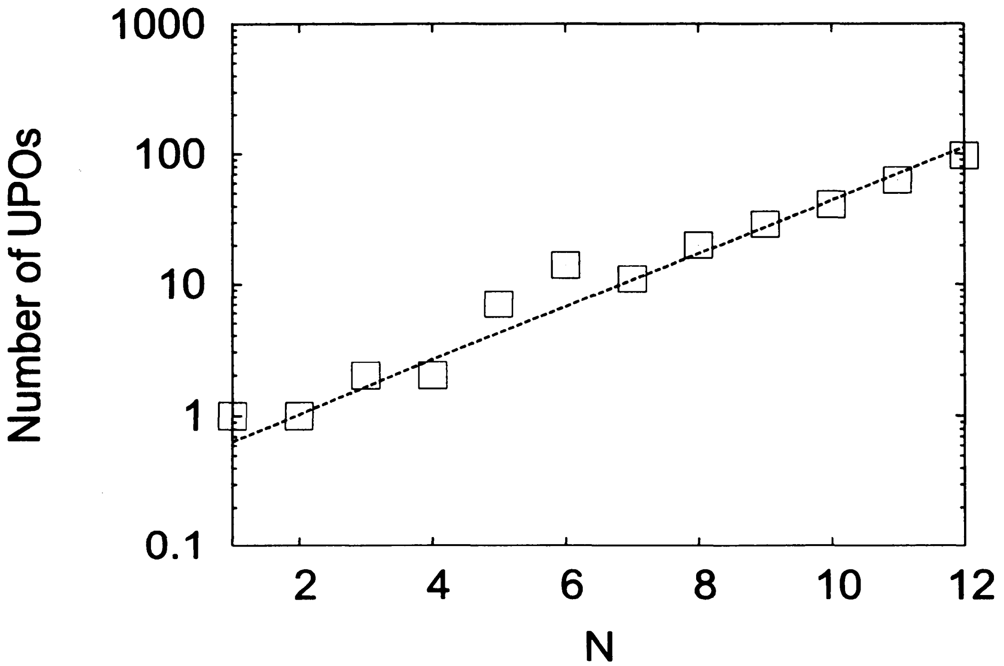

Fig. 2: Number of detected UPOs with PERIOD N of the Kuramoto-Sivashinsky system which are embedded in a chaotic attractor in comparison with $0.4 \cdot 1.6^N$.

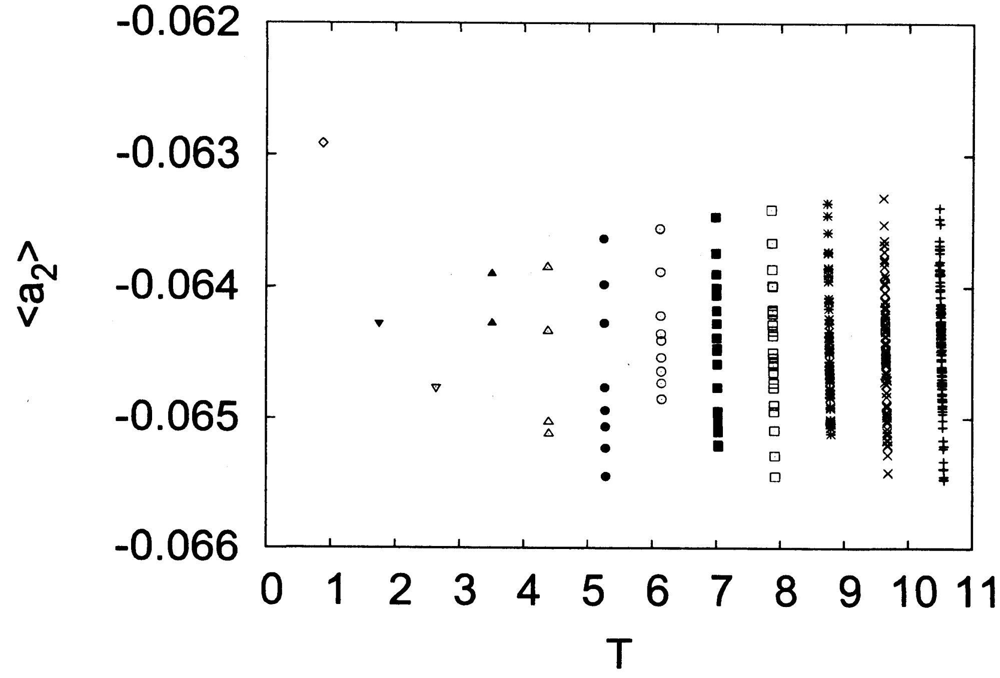

Fig. 3: Time averages $\langle a_2 \rangle$s ($\langle a_2 \rangle \equiv \int_{t=0}^{T} a_2/T dt$) along UPOs with period T.

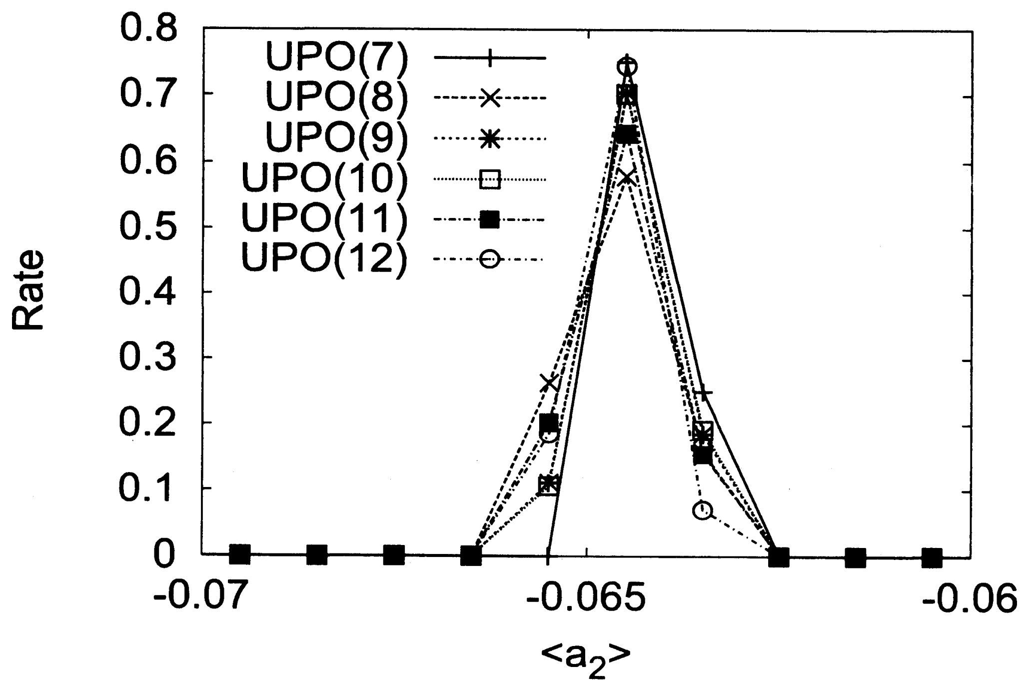

Fig. 4: Density distribution of time averages $\langle a_2 \rangle$s ($\langle a_2 \rangle \equiv \int_{t=0}^{T} a_2/T dt$) along UPOs with PERIOD $N(= 7, \cdots, 12)$.

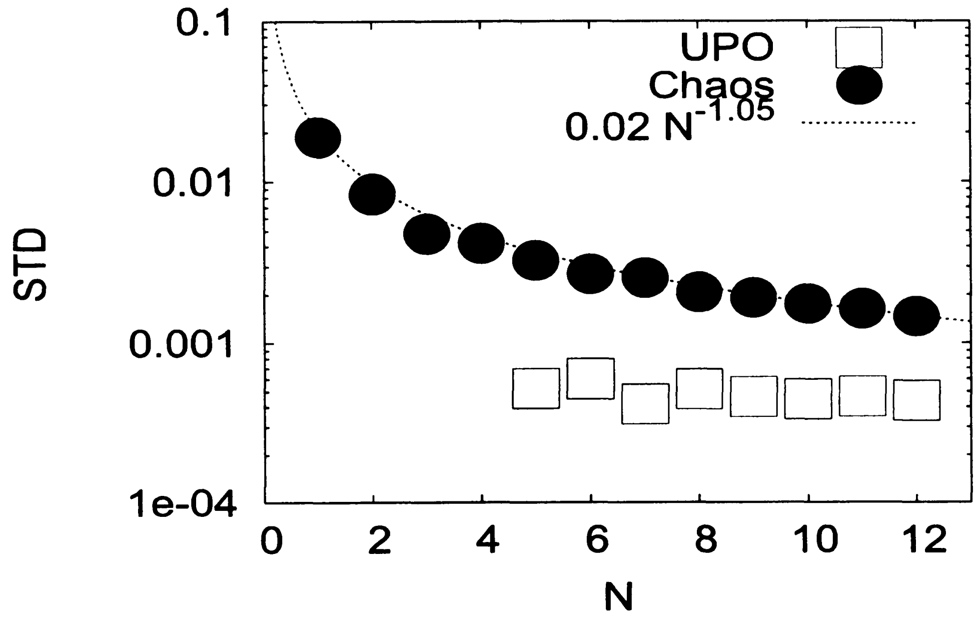

Fig. 5: Standard deviation of density distribution of $\langle a_2 \rangle$s along UPOs with PERIOD $N(+)$ and that along 10^5 chaotic segments with the corresponding time lengths $T(= 0.8774 \cdot N)(\times)$ and $0.02N^{-1.05}$(line).

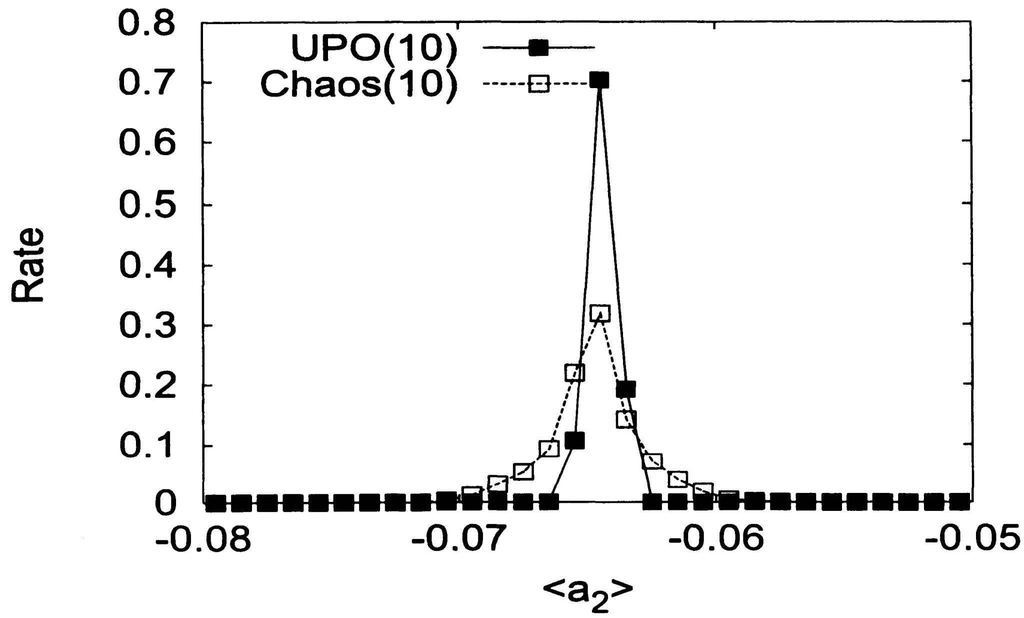

Fig. 6: Density distribution of $\langle a_2 \rangle$s along UPOs with PERIOD 10 ($\langle a_2 \rangle = -0.06442$) (average period=8.7625) in comparison with that along 10^5 chaotic segments with the corresponding time-length $T(= 0.8774 \cdot 10)$ ($\langle a_2 \rangle = -0.06477$).

density distribution of $\langle a_2 \rangle$s along UPOs for $N(=7, 8, \cdots, 12)$. We can see that the distribution stays similar shape though N varies, indicating that even longer UPO is not necessarily suitable for evaluation of a_2 averaged along a long chaotic orbit. This may be contrary to our expectation that an UPO with longer period would give better approximations to statistical properties of chaotic orbits. Actually in Fig. 5 the standard deviations of the density distribution of $\langle a_2 \rangle$s along UPOs with PERIOD N are seen to be nearly constant as N increases. The figure also shows that the standard deviations of $\langle z \rangle$s along segments of chaotic orbits with time length $T = N \cdot 0.8774$, where 0.8774 stands for the corresponding recurrent time to the Poincaré section. We can see that as N increases, the latter standard deviation decreases nearly as $N^{-1.05}$. The difference between the density distribution of time averages along UPOs is clearly observed in Fig. 6 in the case of the distribution of $\langle z \rangle$s along a set of UPOs of PERIOD 10 and chaotic segments with the corresponding lengths.

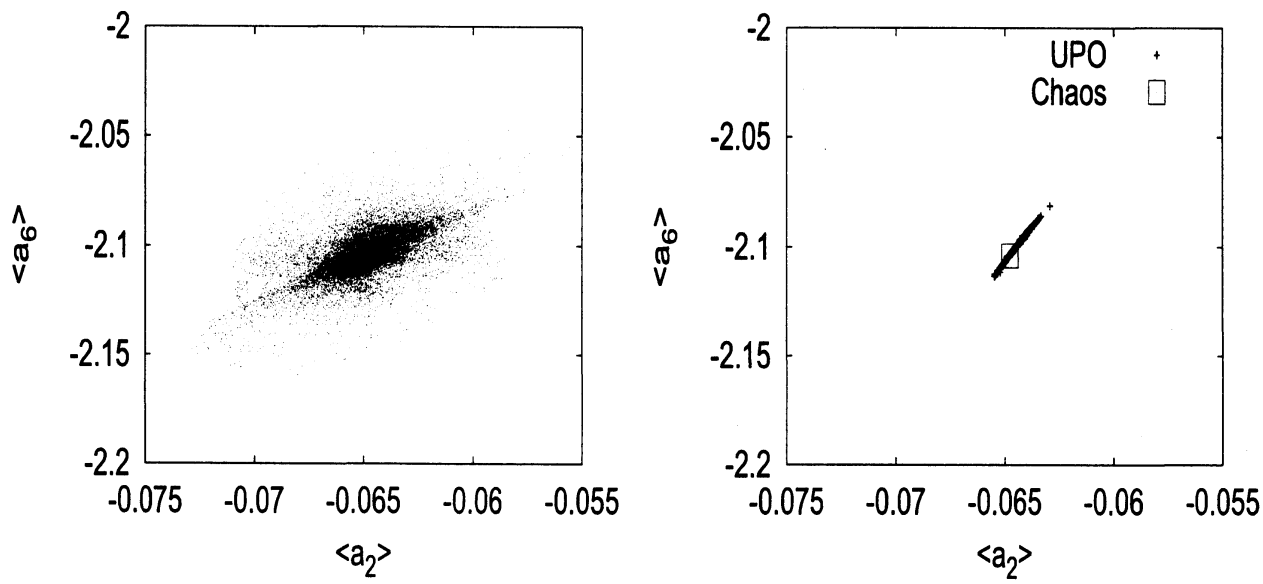

Fig. 7: Relations between time averaged values of a_2 ($\langle a_2 \rangle$)and a_6 ($\langle a_6 \rangle$) along chaotic segments (length $T = 8 \cdot 0.8774$) (left), UPOs(+) and chaotic mean value (□)(right)

In Fig.7 we investigate relations between time averaged values of a_2 ($\langle a_2 \rangle$) and a_6 ($\langle a_6 \rangle$) along chaotic segments (length $T = 8{\cdot}0.8774$) (left), and those along UPOs(+) and chaotic mean value ($\Box$)(right). Surprisingly there are linear correlations between two time averaged values along UPOs, whereas time averaged values along chaotic segments (length $T = 8 \cdot 0.8774$) are spreading on ($\langle a_2 \rangle$, $\langle a_6 \rangle$) plane. It can also be confirmed that chaotic mean value is on the constraint formed by a set of time averaged values along UPOs.

3 Summary

We have discussed time averages of dynamical variables along UPOs in the Kuramoto-Sivashinsky equation. We have calculated more than 650 UPOs, and found that time averaged properties along a set of UPOs and a set of chaotic orbits with finite lengths are totally different from each other. From our numerical result a longer UPO is not necessarily advantageous than shorter UPO to estimate mean properties of the chaotic state in the model. The result is similar to those obtained for the case of ODEs (the Lorenz system, the Rössler system and a 6-dimensional business cycle model). It is implied that we can employ a short UPO for the estimation of the mean properties of the chaotic state without significant reduction of plausibility. In some fluid dynamical systems, it has been found that only a few UPO with low periods give fairly good approximations to some statistical properties. Our result about the Kuramoto-Sivashinsky equation suggests that the estimation by using a short UPO is as reliable (or unreliable) as that by using a long UPO. It would be interesting to study chaotic macro-economic models from the point of view of unstable periodic orbits [15].

Acknowledgment

This work is partially supported by the Grant-in-Aids (No. 194048) and an incentive system for young researchers of the Academic Center for Computing and Media Studies, Kyoto University.

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