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Unstable periodic orbits and complex behavior of two interacting Keynes-Wicksell-Goodwin economies – A characterization of high dimensional macroeconomic model phenomena –

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1 Introduction

Some interesting macroeconomic models described by nonlinear equations were proposed about half a century ago (e.g., Kaldor 1940, Hicks 1950, Goodwin 1967). Again, since the nonlinearity in time series of main economic indices was pointed out by many empirical researches (e.g., Brock and Sayers 1988), nonlinear models have been attracting much attention and generalizations of those models have been studied energetically. Asada et al. (2003, Ch.10) have exemplified that interaction between two large open economies can generate chaotic business fluctuations by a high-dimensional nonlinear model. We have observed large increases in trade and capital flows for several decades. Recent interactions between large countries are worth focusing on. Therefore, such a two-country model is considered to be more important than simple ones in this context.

In this paper, taking the two-country KWG model in Asada et al. (2003), we try to reveal essential properties of complicated business fluctuations when there are strong linkages between the countries. As pointed out in Ishiyama and Saiki (2005), unstable periodic orbits chosen from an appropriate viewpoint are useful in order to capture statistical and dynamical characteristics of chaotic behavior. We study the high-dimensional nonlinear model phenomena by way of a periodic orbit analysis.

In the next section, we review the two-country KWG model. In Section 3 it is confirmed that the model exhibits chaotic behavior for a setting of parameters. In Section 4, by using unstable periodic orbits, we extract the essential properties of the chaotic phenomenon as typical behavior strong linkages between countries give rise to. Final section concludes our results.

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2 The model

In this section we review a variant of the two-country KWG model given in Asada et al. (2003, Ch.10). Let us begin with the definitions of important macroeconomic magnitudes. The symbol ω denotes the real wage, which is measured by the unit of domestic output. The real wage is defined as follows:

$$\omega = \frac{w}{p}, \tag{1}$$

where w and p are the money wage and the price level, respectively. The actual rate of profit ρ is defined by

$$\rho = \frac{Y - \delta K - \omega L^d}{K}, \tag{2}$$

where Y denotes the real output, K the capital stock, L^d the employed labor force, and δ the capital depreciation rate. Hereafter, we employ the following abbreviations:

$$y = \frac{Y}{K}, \quad l^d = \frac{L^d}{K}. \tag{3}$$

Thus, equation (2) can be rewritten as

$$\rho = y - \delta - \omega l^d. \tag{4}$$

We assume δ is constant. In addition, assuming a Leontief-type production function, y and l^d are also constant.[1] Economic agents in the two-country KWG model are workers, asset holders, firms, and governments. Reviewing assumptions with respect to their expenditure, we consider how excess demand in the goods market is determined.

Householders consist of workers and asset holders. Workers spend all their money to consume domestic goods solely. A part of the expenditure of asset holders is directed to imports. The amount in the case of the home country[2] is

$$(1 - \gamma_c(\eta))(1 - s_c)Y_c^D, \tag{5}$$

where variable η means the real exchange rate, $\gamma_c(\eta)$ is a negative function of η, s_c is the average saving rate of asset holders[3], and Y_c^D denotes their disposable income. We express the disposable income per unit of capital as

$$\rho - t_c, \tag{6}$$

where $t_c K$ means the difference between all taxes they pay and all their interest income. For simplicity, t_c is assumed to be fixed. This simplification removes the effect of the accumulation of the domestic and the foreign bonds from the model. Thus, for domestic goods,

$$c_1 = \omega l^d + \gamma_c(\eta)(1 - s_c)(\rho - t_c) \tag{7}$$

[1] In other words, we assume that the capital output ratio and the labor productivity are fixed.

[2] In the case of the foreign country, it is expressed as $(1 - \gamma_c^*(\eta))(1 - s_c^*)Y_c^{D*}$, where $\gamma_c^*(\eta)$ is a positive function of η.

[3] The saving rate is assumed to be smaller than unity and greater than zero, i.e., $0 < s_c < 1$.

expresses the households' demand per unit of capital.[4] On the other hand, firms determine their investment level according to the following equation:

$$I = i(\rho - (r - \pi))K + nK, \tag{8}$$

where variables r and π are the nominal interest rate and the expected inflation rate, respectively; i is a positive constant, and n is the growth rate on a balanced growth path.[5] Note that the capital accumulation rate (and the growth rate of GDP) in the model is represented by

$$\hat{K} = \frac{I}{K} = i(\rho - (r - \pi)) + n, \tag{9}$$

where a hat over a variable denotes the change rate of the variable. The government expenditure is proportional to the capital stock, that is,

$$G = gK. \tag{10}$$

By assuming initial levels of labor supply in both two countries are equivalent, the export per unit of capital of the home country can be expressed as

$$c_1^* = \frac{l}{\eta l^*}(1 - \gamma_c(\eta))(1 - s_c^*)(\rho^* - t_c^*), \tag{11}$$

where l means the labor supply capital ratio in the home country, and superscript $*$ indicates a foreign country variable. From equations (7), (8), (10) and (11), excess demand on the goods market in the home country is expressed as

$$X^p = c_1 + c_1^* + i(\rho - (r - \pi)) + n + \delta + g - y. \tag{12}$$

Similarly that in the foreign country is

$$X^{p*} = c_2^* + c_2 + i^*(\rho^* - (r^* - \pi^*)) + n^* + \delta^* + g^* - y^*, \tag{13}$$

where $c_2^* K^*$ means the foreign householders' demand level for the foreign goods, and $c_2 K^*$ is the amount of imports from the foreign country to the home country.

Next, let us consider the labor market. Excess demand on the labor market in the home country is indicated by

$$X^w = \frac{l^d}{l} - \bar{V}, \tag{14}$$

where $\bar{V}$ is NAIRU-type normal utilization rate concept of labor. Each growth rate of p and w is assumed to be influenced by both X^p and X^w, and wage deflation is excluded from the model.[6] Namely, $\hat{p}$ and $\hat{w}$ are determined by the following simultaneous equations:

$$\begin{cases} \hat{p} = \beta_p X^p + k_p \hat{w} + (1 - k_p)\pi \\ \hat{w} = \max[\beta_w X^w + k_w \hat{p} + (1 - k_w)\pi, 0]. \end{cases} \tag{15}$$

[4]The tax rate on wage income is assumed to be zero.

[5]The trajectory on which the capital stock, the labor supply and the national income grow at the same rate is called a balanced growth path. The balanced growth is realized at the long run equilibrium point.

[6]Such a wage rigidity contributes to bound fluctuations of state variables in an economically meaningful region (Chiarella et al. 2003).

From equation (1), the change rate of the real wage is determined as

$$\hat{\omega} = \hat{w} - \hat{p}. \tag{16}$$

Labor force is assumed to grow at the natural rate n. Hence,

$$\hat{l} = n - \hat{K} = -i(\rho - (r - \pi)). \tag{17}$$

Expectation formation for the domestic price is expressed as

$$\dot{\pi} = \beta_\pi(\alpha_\pi(\hat{p} - \pi) + (1 - \alpha_\pi)(\hat{p}_o - \pi)), \tag{18}$$

where $\hat{p}_o$ indicates the long run inflation rate. Equation (18) means that with an adjustment speed β_π, the expectation for inflation is determined by a weighted average of backward looking and forward looking adjustments.

Then, we consider the money market. We assume asset market clearing. The stock demand for real money balances is assumed to depend on output, capital[7] and the nominal interest as follows:

$$\frac{M^d}{p} = h_1 Y + h_2 K(r_o - r), \tag{19}$$

where r_o is the long run nominal interest rate, and parameters h_1 and h_2 are positive constants. The only rule of monetary policy by the central bank is to keep the domestic money supply M growing a constant rate μ. The nominal interest rate is determined so that the following equation holds.

$$\frac{M}{p} = h_1 Y + h_2 K(r_o - r), \tag{20}$$

That is,

$$r = r_o + \frac{h_1 y - m}{h_2}, \tag{21}$$

where variable m denotes real money balances per unit of capital. The change rate of m is

$$\hat{m} = \mu - \hat{p} - i(\rho - (r - \pi)) - n. \tag{22}$$

Finally, we consider the dynamics of the foreign exchange market. By using the nominal exchange rate e, the definition of η is written as

$$\eta = \frac{p}{ep^*}. \tag{23}$$

Hence,

$$\hat{\eta} = \hat{p} - \hat{e} - \hat{p}^*. \tag{24}$$

The way and the extent of international capital flows per unit of capital depend on the interest differential under imperfect capital mobility. The nominal exchange rate is assumed to be adjusted according to the capital flows and net exports as follows:

$$\hat{e} = \beta_e(\beta(r^* + \varepsilon - r) - nx) + \hat{e}_o, \tag{25}$$

[7] For simplicity, we think K real wealth.

where nx denotes net export (per unit of capital) of the home country and $\hat{e}_o$ is the growth rate of the nominal exchange at the long run equilibrium[8], and ε is the expected rate of exchange depreciation. The expectation formation for the nominal exchange rate is similar to the case of inflation.[9] That is,

$$\dot{\varepsilon} = \beta_\varepsilon(\alpha_\varepsilon(\hat{e} - \varepsilon) + (1 - \alpha_\varepsilon)(\hat{e}_o - \varepsilon)). \tag{26}$$

The dynamics of the foreign county is modeled analogously. Hence, if the magnitudes of ω, l, m, π, η, ε, ω^*, l^*, m^*, π^* are known, their developments can be derived from the following equations.

$$\hat{\omega} = \hat{w} - \hat{p}, \tag{27}$$

$$\hat{l} = -i(\rho - (r - \pi)), \tag{28}$$

$$\hat{m} = \mu - \hat{p} - i(\rho - (r - \pi)) - n, \tag{29}$$

$$\dot{\pi} = \beta_\pi(\alpha_\pi(\hat{p} - \pi) + (1 - \alpha_\pi)(\hat{p}_o - \pi)), \tag{30}$$

$$\hat{\eta} = \hat{p} - \hat{e} - \hat{p}^*, \tag{31}$$

$$\dot{\varepsilon} = \beta_\varepsilon(\alpha_\varepsilon(\hat{e} - \varepsilon) + (1 - \alpha_\varepsilon)(\hat{e}_o - \varepsilon)), \tag{32}$$

$$\hat{\omega}^* = \hat{w}^* - \hat{p}^*, \tag{33}$$

$$\hat{l}^* = -i^*(\rho^* - (r^* - \pi^*)), \tag{34}$$

$$\hat{m}^* = \mu^* - \hat{p}^* - i^*(\rho^* - (r^* - \pi^*)) - n^*, \tag{35}$$

$$\dot{\pi}^* = \beta_\pi^*(\alpha_\pi^*(\hat{p}^* - \pi^*) + (1 - \alpha_\pi^*)(\hat{p}_o^* - \pi^*)). \tag{36}$$

This is the 10 dimensional differential equation system to be analyzed in this paper.

An economy modeled by the above equations moves in response to gaps between actual and expected values if it does not exist at the long run equilibrium. Now we turn our attention to the long run equilibrium. The time derivatives of 10 main variables are equal to zero at the equilibrium, where clearing all the markets is realized and GDP of each country grows at the natural rate. For simplicity, we assume that there is no difference between the natural growth rates of two countries, namely, $n = n^*$. Concerning each variable fixed at the long run equilibrium, the equilibrium value is expressed using subscript o.

From equation (14), the equilibrium level of l is determined as

$$l_o = \frac{l^d}{\overline{V}}. \tag{37}$$

[8] In the long run, the prices and the nominal exchange rate grow depending only on the natural growth rates and increases in money supply in both countries.

[9] See equation (18).

Substituting $r = r_o$ into equation (21), we obtain

$$m_o = h_1 y. \tag{38}$$

From $\hat{l} = 0$,

$$r_o = \rho_o + \pi_o. \tag{39}$$

From $\hat{l} = 0$, $\hat{m} = 0$ and $\hat{p}_o = \pi_o$,

$$\pi_o = \mu - n. \tag{40}$$

At the long run equilibrium, net export nx is zero, therefore, we can derive the equilibrium values of ρ and ω from

$$\begin{cases} \rho_o = y - \delta - \omega_o l^d \\ \omega_o l^d + (1 - s_c)(\rho_o - t_c) + n + \delta + g - y = 0. \end{cases} \tag{41}$$

They are solved as

$$\rho_o = t_c + \frac{n + g - t_c}{s_c}, \tag{42}$$

$$\omega_o = \frac{y - \delta - \rho_o}{l^d}. \tag{43}$$

The real exchange rate at the long run equilibrium is

$$\eta_o = \frac{l_o(1 - \gamma_c^*)(1 - s_c^*)(\rho_o^* - t_c^*)}{l_o^*(1 - \gamma_c)(1 - s_c)(\rho_o - t_c)}, \tag{44}$$

where both $\gamma_c = \gamma_c(\eta_o)$ and $\gamma_c^* = \gamma_c^*(\eta_o)$ are assumed to be greater than zero and smaller than unity. From $\pi_o = \mu - n$, $\pi_o^* = \mu^* - n^*$, $n = n^*$ and $\dot{\varepsilon} = 0$,

$$\varepsilon_o = \mu - \mu^*. \tag{45}$$

It is obvious that the long run equilibrium is independent of any adjustment speed included in the model. Moreover, it has been also obvious from the analytical examination in Asada et al. (2003, Ch.10) that the equilibrium can be locally unstable if the adjustment speeds are sufficiently large.

3 Chaotic behavior of the model

To see the model phenomena in detail, hereafter, we restrict our examination to a setting of parameters, for which two countries are linked through international trade and capital flows. Parameters are set as follows:
$s_c = s_c^* = 0.8$, $\delta = \delta^* = 0.1$, $t_c = t_c^* = 0.35$, $g = g^* = 0.35$, $n = n^* = 0.02$, $h_1 = h_1^* = 0.1$, $h_2 = h_2^* = 0.2$, $y = y^* = 1$, $l^d = l^{d*} = 0.5$, $k_w = k_w^* = 0.5$, $k_p = k_p^* = 0.5$, $i = i^* = 0.5$, $\bar{V} = \bar{V}^* = 0.8$, $\alpha_\varepsilon = 0.5$, $\alpha_\pi = 0.5$, $\mu = 0.025$, $\mu^* = 0.022$; $\beta_w = \beta_w^* = 2$, $\beta_p = \beta_p^* = 3$, $\beta_k = \beta_k^* = 1$, $\beta_\pi = 3.9$, $\beta_\pi^* = 3.8$, $\beta = 1$, $\beta_e = 2$, $\beta_\varepsilon = 9.5$,
where the unit of time is considered to be one year. For example, $\mu = 0.025$ means

money supply in the home country grows 2.5% in a year. Note that the two countries are different with respect to monetary policy rules as well as adjustment speeds of inflation expectation. In the two-country KWG model, the level of an individual adjustment speed expresses characteristics of the corresponding market. Note that there are differences among the speeds set above. For the setting, the long run equilibrium

$$(\omega_o, l_o, m_o, \pi_o, \eta_o, \varepsilon_o, \omega_o^*, l_o^*, m_o^*, \pi_o^*) = (1.05, 0.625, 0.1, 0.005, 1, 0.003, 1.05, 0.625, 0.1, 0.002)$$

is locally unstable.

Now the equations which determine the dynamics of main variables can be rewritten as follows.

$$\frac{d\omega}{dt} = (\hat{w} - \hat{p})\omega, \tag{46}$$

$$\frac{dl}{dt} = \hat{l}l, \tag{47}$$

$$\frac{dm}{dt} = (\mu - \hat{p} + \hat{l} - 0.02)m, \tag{48}$$

$$\frac{d\pi}{dt} = 1.95(\hat{p} - 2\pi + \mu - 0.02), \tag{49}$$

$$\frac{d\eta}{dt} = (\hat{p} - \hat{p}^* - \hat{e})\eta, \tag{50}$$

$$\frac{d\varepsilon}{dt} = 4.75(\hat{e} - 2\varepsilon + \mu - \mu^*), \tag{51}$$

$$\frac{d\omega^*}{dt} = (\hat{w}^* - \hat{p}^*)\omega^*, \tag{52}$$

$$\frac{dl^*}{dt} = \hat{l}^* l^*, \tag{53}$$

$$\frac{dm^*}{dt} = (m^* - \hat{p}^* + \hat{l}^* - 0.02)m^*, \tag{54}$$

$$\frac{d\pi^*}{dt} = 1.9(\hat{p}^* - 2\pi^* + \mu^* - 0.02). \tag{55}$$

Other variables are determined by the following equations.

$$\hat{w} = \max[2X_p + \frac{8}{3}X_w + \pi, 0], \tag{56}$$

$$\hat{w}^* = \max[2X_p^* + \frac{8}{3}X_w^* + \pi^*, 0], \tag{57}$$

$$\hat{p} = \begin{cases} 3X_p + 0.5\pi & (\hat{w} = 0) \\ 4X_p + \frac{4}{3}X_w + \pi & (\hat{w} > 0) \end{cases}, \tag{58}$$

$$\hat{p}^* = \begin{cases} 3X_p^* + 0.5\pi^* & (\hat{w}^* = 0) \\ 4X_p^* + \frac{4}{3}X_w^* + \pi^* & (\hat{w}^* > 0) \end{cases}, \tag{59}$$

$$\hat{l} = 0.25\omega + 0.5\mu - 2.5m - 0.5\pi + 0.9X_p - 0.0225, \tag{60}$$

$$\hat{l}^* = 0.25\omega^* + 0.5\mu^* - 2.5m^* - 0.5\pi^* + 0.9X_p^* - 0.0225, \tag{61}$$

$$X_p = c_1 + c_1^* - 0.25\omega - 0.5\mu + 2.5m + 0.5\pi - 0.5075, \tag{62}$$

$$X_p^* = c_2^* + c_2 - 0.25\omega^* - 0.5\mu^* + 2.5m^* + 0.5\pi^* - 0.5075, \tag{63}$$

$$X_w = \frac{1}{2l} - 0.8, \tag{64}$$

$$X_w^* = \frac{1}{2l^*} - 0.8, \tag{65}$$

$$c_1 = 0.5\omega + \gamma(\eta)(0.11 - 0.1\omega), \tag{66}$$

$$c_1^* = \left(\frac{l}{l^*}\right)(1 - \gamma^*(\eta))(0.11 - 0.1\omega^*)/\eta, \tag{67}$$

$$c_2 = \left(\frac{l^*}{l}\right)(1 - \gamma(\eta))(0.11 - 0.1\omega)\eta, \tag{68}$$

$$c_2^* = 0.5\omega^* + \gamma^*(\eta)(0.11 - 0.1\omega), \tag{69}$$

$$\hat{e} = 2\left(\varepsilon - c_1^* + c_2\left(\frac{l}{l^*}\right)/\eta\right) + 10(m - m^*) + (\mu^* - \mu), \tag{70}$$

$$\gamma(\eta) = -\max[-1, -\max[-0.2\eta + 0.7, 0]], \tag{71}$$

$$\gamma^*(\eta) = -\max[-1, -\max[0.2\eta + 0.3, 0]], \tag{72}$$

where functions $\gamma_c(\eta)$ and $\gamma_c^*(\eta)$ are specialized as piecewise linear functions.

Figure 1 shows the attractor of the 10 dimensional dynamical system numerically integrated by using fourth order Runge-Kutta method. The maximum Lyapunov exponent evaluated along the attractor is about 0.13. This value means the system exhibits behavior that depends sensitively on the initial conditions. In this case, it may be difficult to predict trade conditions in a few years. On the other hand, kinks of Phillips curves and the functions of real exchange rate seem to contribute to the viability of economically meaningful fluctuations, and the oscillation in a bounded region generates recurrent time series. Therefore, recurrence and sensitivity on the initial values are characteristics of the dynamics of the system. In other words, the dynamics is chaotic.

Is generation of such a chaotic attractor associated with weak linkages between countries? It is true that we assume that workers consume only domestic goods. We may downgrade the role of international trade.[10] On the other hand, we stress the international mobility of capital. Let us examine the amount of capital flows from the home country in proportion to the domestic capital stock. A time series of the amount on the attractor is depicted in figure 2 (left). This figure exemplifies that, for our parameter setting, two countres are strongly connected through capital flows.

In general, it is not easy to see the dynamics of 10 main variables at the same time. As our concern is typical patterns of growth cycles of two countries the dynamical system depicts, let us focus on the deviations of GDP growth rates of two countries from their natural growth rates. Thus we introduce new variables: $x = \hat{Y} - n$ and $x^* = \hat{Y}^* - n^*$.[11] The time series of these variables on the attractor are illustrated in figure 2 (right). We cannot see any distinct relationships between x and x^* in this figure. However, it should be noted that the length of the chaotic trajectory plotted on the figure is too short to capture the typical patterns of business cycles of the countries.

Figure 3 shows the attractor projected onto x-x^* plane. Few points are plotted on the left lower area. It implies that the worldwide recession seldom occurs for the setting of parameters. The next section discusses the characteristics of the attractor by using unstable periodic orbits.

4 Unstable periodic orbits

It is known that there are an infinite number of unstable periodic orbits embedded in a chaotic attractor. When similar patterns are observed subsequently in a long time development in a chaotic attractor, it can be considered that the trajectory is going along an unstable periodic orbit embedded in the attractor. Figure 4 shows a chaotic trajectory which is a part of the complicated orbit illustrated in figure 3.

For an initial guess $\boldsymbol{X}_{(i)} \in \mathbf{R}^{10}$ chosen from the chaotic attractor and $T_{(i)} \in \mathbf{R}_+$, we iterate the following algorithm:
Step 1: Solve

$$\begin{pmatrix} \Phi_{T_{(i)}}(\boldsymbol{X}_{(i)}) - I & F(\phi_{T_{(i)}}(\boldsymbol{X}_{(i)})) \\ F(\boldsymbol{X}_{(i)})^t & 0 \end{pmatrix} \begin{pmatrix} \Delta \boldsymbol{X}_{(i)} \\ \Delta T_{(i)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_{(i)} - \phi_{T_{(i)}}(\boldsymbol{X}_{(i)}) \\ 0 \end{pmatrix} \tag{73}$$

about $\Delta \boldsymbol{X}_{(i)}$ and $\Delta T_{(i)}$, where $F(\boldsymbol{x})$ is the time derivative of $\boldsymbol{x}$, $\{\phi_t(\boldsymbol{x})\}_{t \in \mathbf{R}}$ denotes the orbit passing through $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbf{R}^{10}$) at $t = 0$, I is the 10×10 unit matrix, and 10×10 matrix $\Phi_t(\boldsymbol{x})$ means the variation of $\phi_t(\boldsymbol{x})$ about $\boldsymbol{x}$.
Step 2: Modify the guess as

$$(\boldsymbol{X}_{(i+1)}, T_{(i+1)}) = (\boldsymbol{X}_{(i)} + 2^{\lambda} \Delta \mathbf{X}_{(i)}, T_{(i)} + 2^{\lambda} \Delta T_{(i)}), \tag{74}$$

and go back to *step 1* after replacing $i+1$ with i. This is a damped-Newton method, and parameter λ is a damping exponent.[12] When both $|\Delta \boldsymbol{X}_{(i)}|$ and $|\Delta T_{(i)}|$ are sufficiently

[10] In addition, we ignore migrant workers.
[11] Note that their equilibrium values are zero.
[12] In order to find an unstable periodic orbit, we have to give not only initial guess but also λ appropriately.

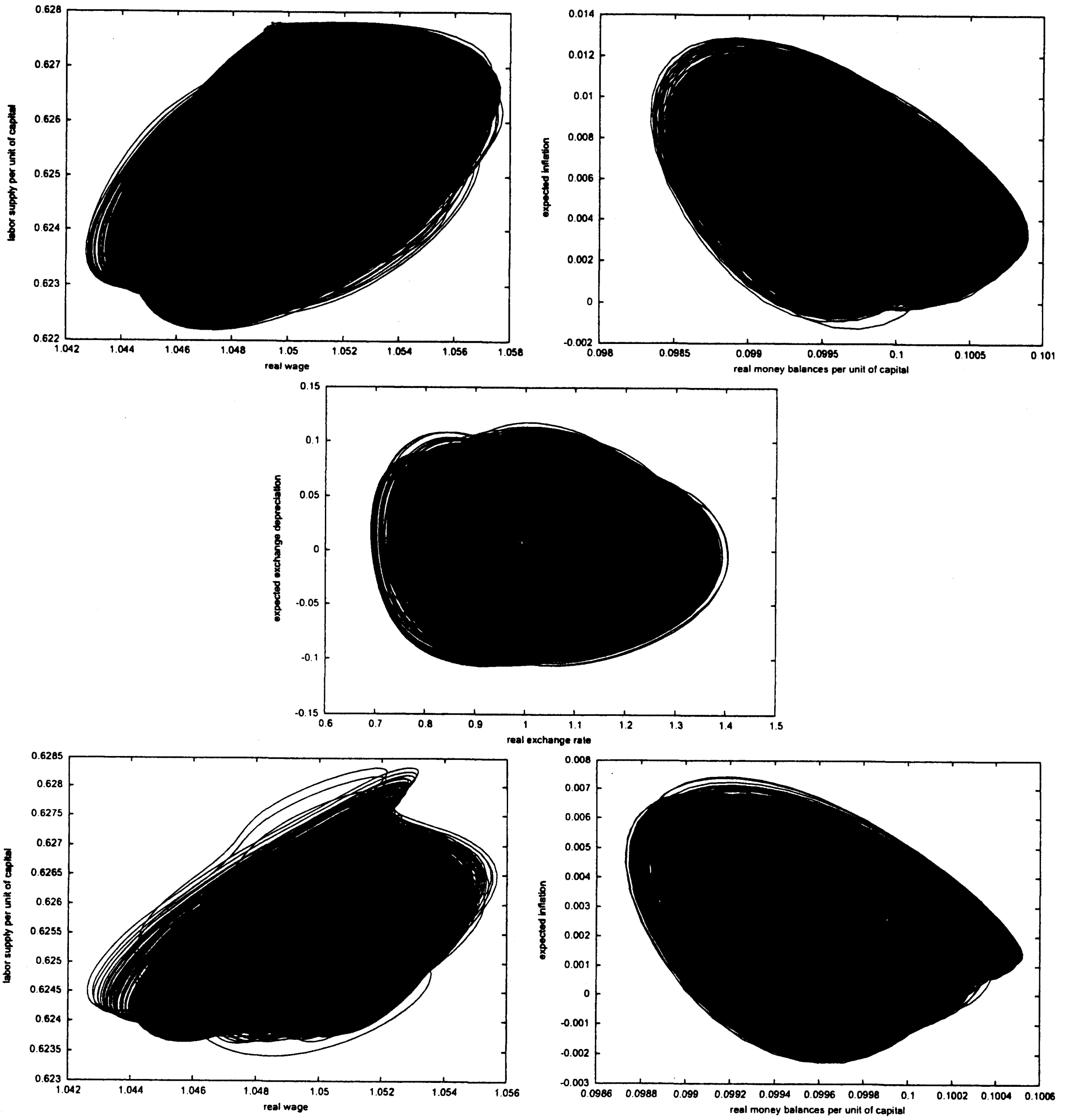

Figure 1: Chaotic attractor. The attractor is projected onto five planes (ω-l (top left), m-π (top right), η-ε (middle), ω^*-l^* (bottom left), m^*-π^* (bottom right)). The maximum Lyapunov exponent evaluated on the attractor is about 0.13.

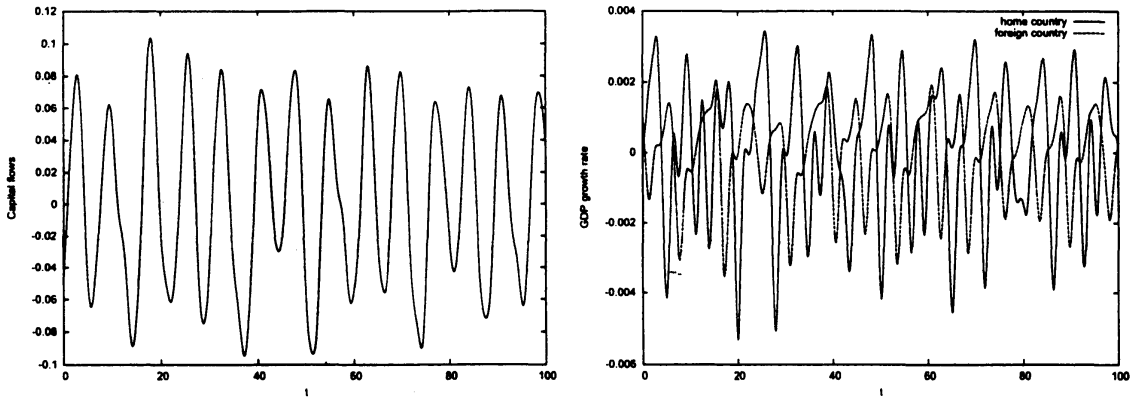

Figure 2: Chaotic movements of capital flows and GDP growth rates. The left figure shows the variation of $\beta(r^* + \varepsilon - r)$, which means a relative amount of capital flows from the home country. Time series of $x(t) = \hat{Y}(t) - n$ (solid line) and that of $x^*(t) = \hat{Y}^*(t) - n^*$ (dashed line) are illustrated in the right figure. The levels of x, x^*, r, r^* are calculated by using time series data of main variables on a chaotic trajectory.

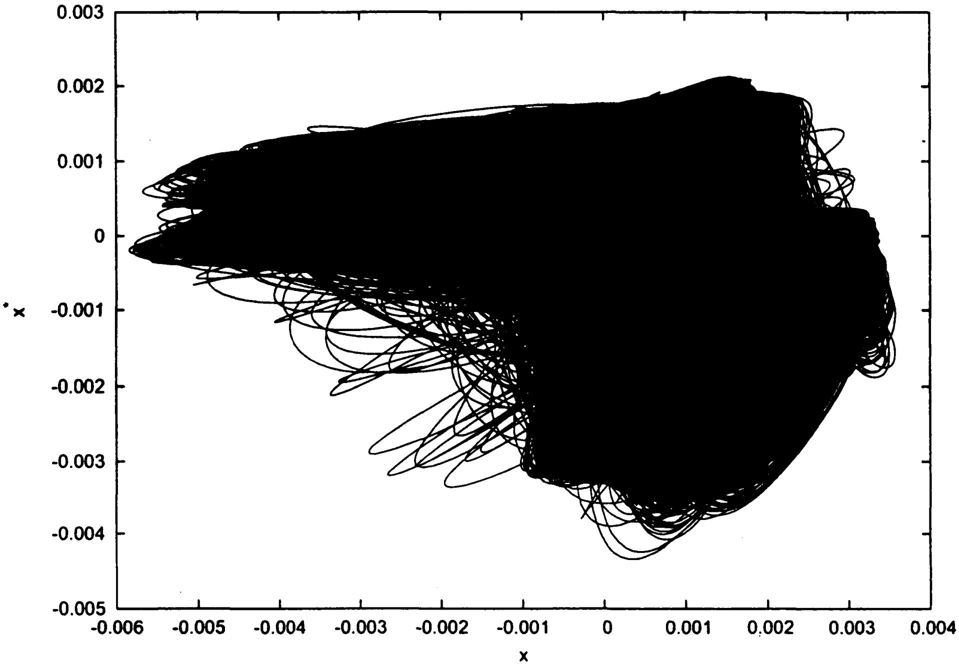

Figure 3: Chaotic attractor projected onto x-x^* plane. The trajectory of $(x(t), x^*(t))$ is plotted for $0 \leq t \leq 30000$.

small, we think that $\boldsymbol{X}_{(i)}$ is a point of a periodic orbit, and that $T_{(i)}$ is the period of the orbit. By using this method, we find an unstable periodic orbit which exists sufficiently close to the chaotic orbit. It is also depicted in figure 4.

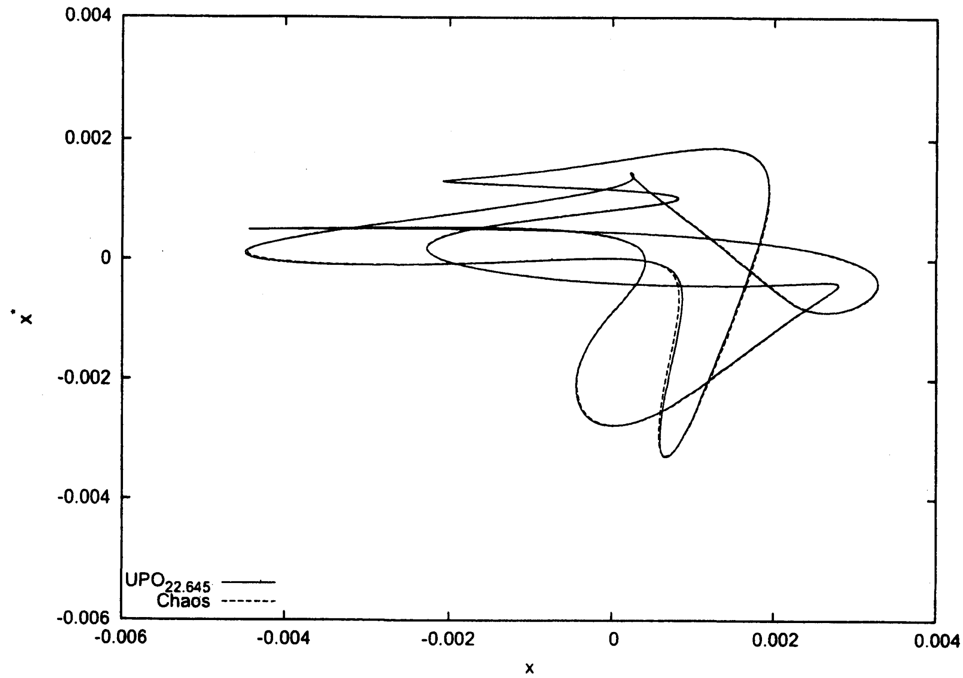

Figure 4: Unstable periodic orbit and chaotic attractor projected onto x-x^* plane. Plotted are an unstable periodic orbit (solid line) whose period is about 22.645 and a chaotic trajectory $\{(x(t), x^*(t)) | 0 \leq t \leq 22.645\}$ (dashed line). The chaotic trajectory starts from a point near the most stable point of the unstable periodic orbit.

As long as a chaotic orbit is going along an unstable periodic orbit, the phase difference between business cycles of the countries seems to be fixed in a sense. It depends on the unstable periodic orbit how the phase difference can be fixed. In this context, the variety of unstable periodic orbit embedded in the attractor is worth investigating. Hence, we try to find all periodic orbits with short period. As a result we obtain ten unstable periodic orbits in total.

When we attempt to know frequently observed phenomena by way of unstable periodic orbits, it is important to examine the instability of the orbits in advance. Here we consider how far a point near an unstable periodic orbit goes in a direction orthogonal to the orbit as time goes by. This local instability is evaluated by the maximum modulus of eigenvalues of a 9×9 square matrix as a linear map from a point near $\boldsymbol{x}(t)$ to a point near $\boldsymbol{x}(t + \Delta t)$, where $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t + \Delta t)$ are points on the unstable periodic orbit and Δt is sufficiently small. Even if a periodic orbit is unstable, the orbit is able to have some points of local stability.[13] It is possible that such a point existing near the chaotic attractor is the starting point when a chaotic trajectory goes along an unstable periodic orbit. Thus, we calculate the distance between a chaotic orbit and the most stable point of each unstable periodic orbit found numerically. The result is summarized in figure 5. According to this figure, some unstable periodic orbits seem to be useful.

[13] In fact, most periodic orbits we found have not a few points with local stability, at which the maximum modulus of eigenvalues of the Jacobian matrix is smaller than 1.000 as shown later.

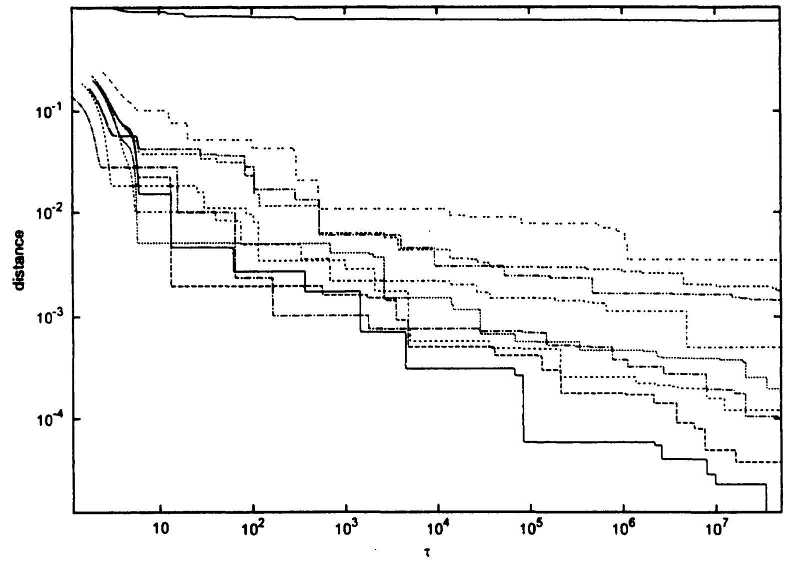

Figure 5: Distance between a point of the most stable point of each unstable periodic orbit and a chaotic trajectory $\{X(t)|0 \leq t \leq \tau\}$. The distance is measured as $\inf_{t \leq \tau} \|X(t) - A\|$, where A is the most stable point of an unstable periodic orbit.

Picking up five periodic orbits, we project each one onto x-x^* plane with a chaotic orbit $\{(x(t), x^*(t))|0 \leq t \leq T\}$, where $(x(0), x^*(0))$ is the closest point to the most stable point of the unstable periodic orbit in the 10 dimensional phase space, while T is equal to the period of the unstable periodic orbit. They are shown in figure 6. This figure suggests an important fact: the closer to an unstable periodic orbit the starting point of a chaotic trajectory is, the more patterns like the unstable periodic orbit the chaotic trajectory depicts. Therefore, it is essential that an unstable periodic orbit is embedded in the chaotic attractor. The unstable periodic orbits in figure 6 are expected to be embedded in the attractor.

We have defined the local stability of a periodic orbit in the beginning of this section. Figure 7 shows locally stable points of the unstable periodic orbits embedded in the chaotic attractor. Since a chaotic trajectory is considered to approach such stable points, it would be significant how many points of stability an unstable periodic orbit has, and where they exist. In figure 7, we can see that the region which attracts chaotic trajectories is not narrow on every unstable periodic orbit, and that such a region is divided into some subsets among which the correlative relationship between x and x^* is different. Figure 7 visually captures the source of the complexity of business cycles the two-country KWG model generates. On the other hand, figure 8 shows the time developments of the local instability and GDP growth rates along each unstable periodic orbit. It is exemplified in this figure that temporal comovements of business cycles caused by international trade and capital flows can be observed not for a long term but rahtr for a short term though the linkage is significantly strong. This figure also demonstrates the complexity and diversity of the model phenomena. Finally, it should be noted that similar results have been obtained for many different settings of parameters.

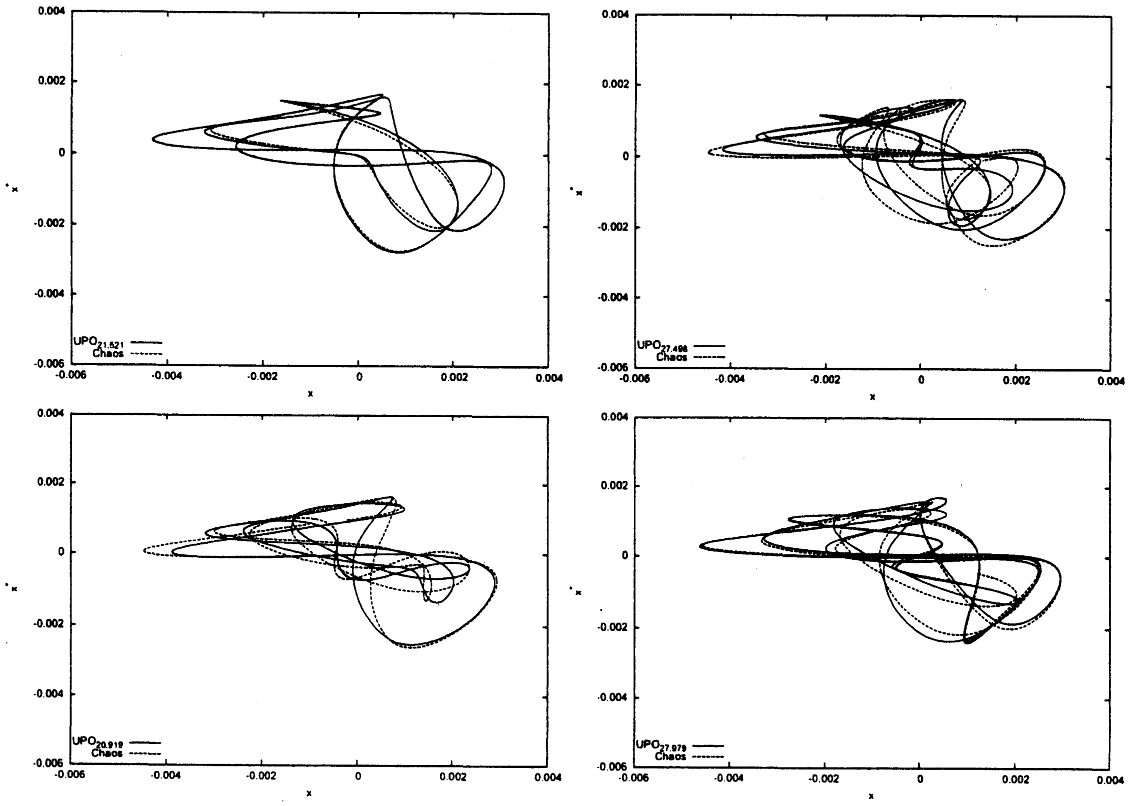

Figure 6: Unstable periodic orbit (solod line) and chaotic orbit (dashed line). Each unstable periodic orbit is named UPO_T, where T means the period of the orbit.

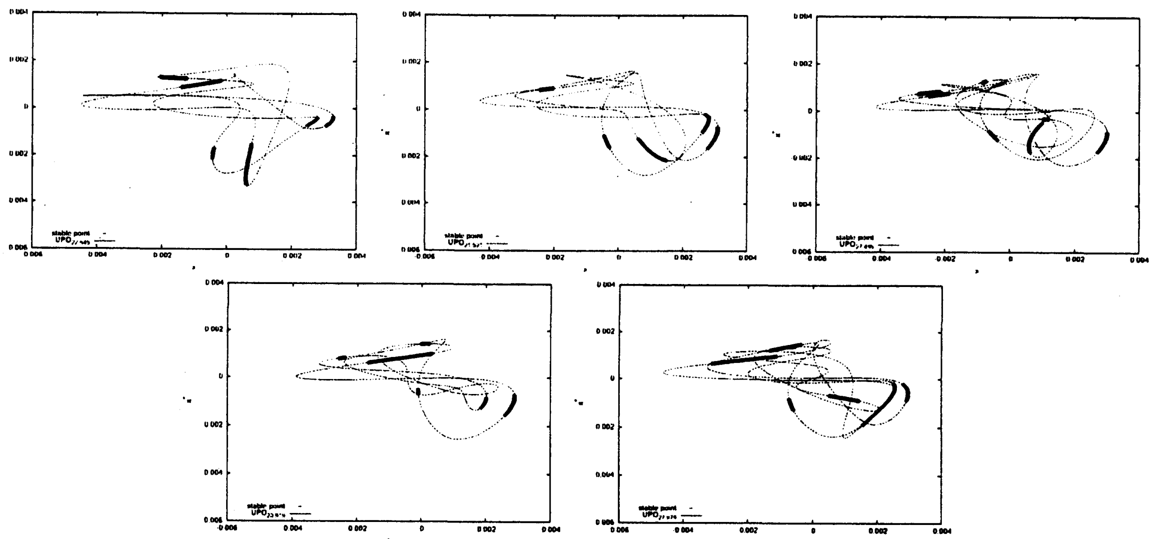

Figure 7: Unstable periodic orbit in the attractor and locally stable point. Each unstable periodic orbit is named UPO_T, where T denotes the period of the orbit. The locally stable points (cross) are superimposed on the corresponding unstable periodic orbit (dashed line).

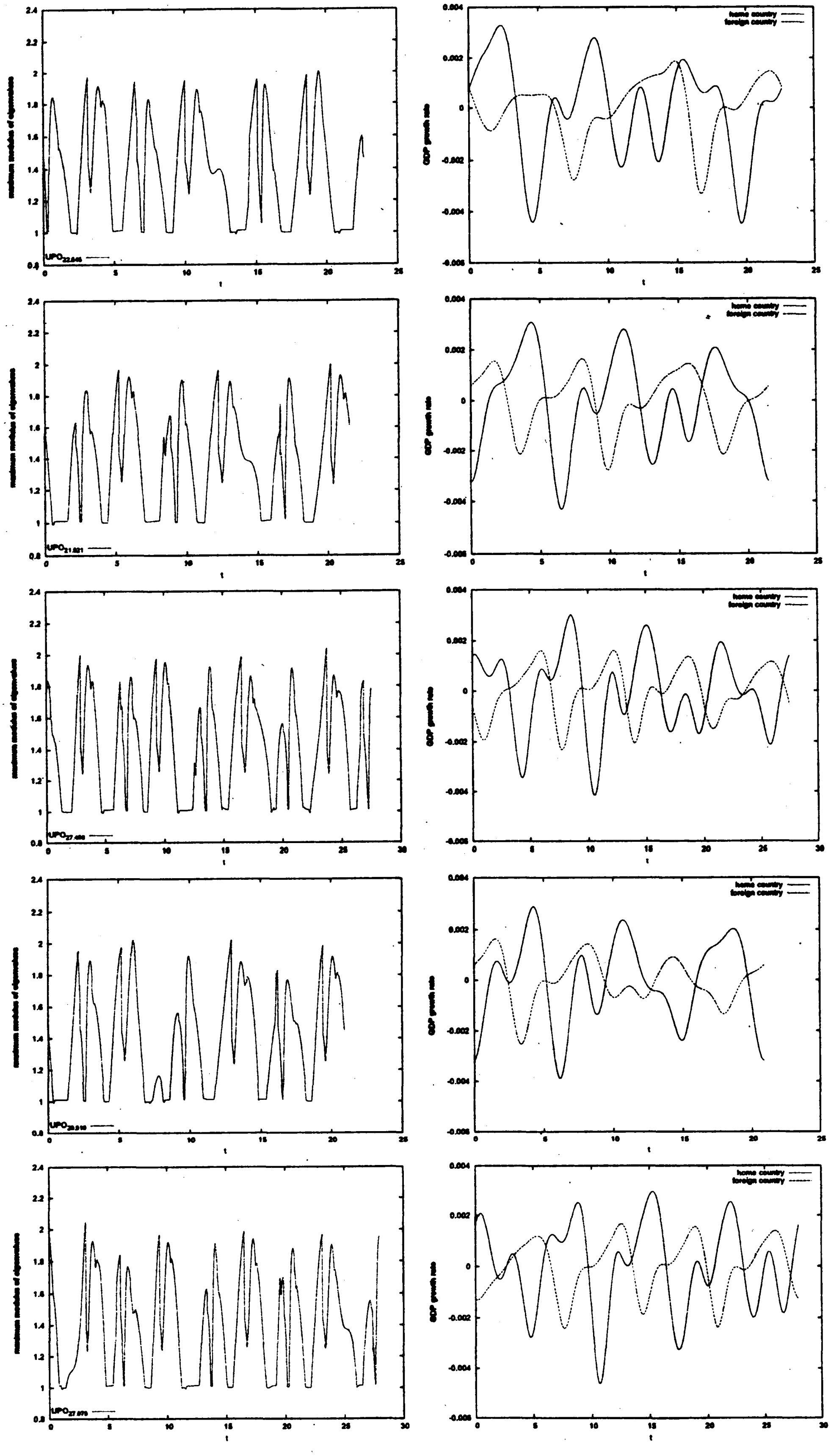

Figure 8: Fluctuation of local instability along an unstable periodic orbit and oscillations of GDP growth rates in two countries. Two figures in the same row show time series of the multiplier of a deviation to the most unstable direction orthgonal to an unstable periodic orbit evaluated at each point of the orbit (left), and of the GDP growth rates of the two countries along the unstable periodic orbit (right), respectively.

5 Conclusion

We numerically investigate the two-country KWG model with a setting of strong intensity of linkage between the countries mainly through capital flows. For our setting, the countries are slightly different with regard to monetary policy and the adjustment speed for inflation. Under the circumstances, an economy starting from almost every meaningful point reaches an attracting set after going through a transient period. We confirm the attractor is chaotic in the sense that it is bounded and the maximum Lyapunov exponent evaluated on the attractor is positive. In order to obtain more detailed information about the dynamics of the system, we detect unstable periodic orbits by using a damped-Newton method and then choose some orbits to be useful for our purpose among them. We demonstrate that a chaotic trajectory can draw the growth pattern like that of an unstable periodic orbit over and over. In addition, locally stable regions of the unstable periodic orbits embedded in the chaotic attractor are revealed. As a result, we can see that the region which attracts chaotic trajectories is not narrow on every unstable periodic orbit, and that such a region is divided into some subsets among which the correlative relationship between the GDP growth rates of the countries is different. This is one of the interesting phenomena the two-country KWG model can represent, and in this paper we capture it by unstable periodic orbits appropriately chosen. Moreover, we exemplify that temporal comovements of the business cycles trhough strong linkage between countries can be observed not for a long term but rahtr for a short term.

Finally, we refer to future issues. It is likely that the size and the number of locally stable regions on an unstable periodic orbit embedded in the attractor can vary through increases and decreases in a parameter as the shape of the orbit and the distance from the attractor can change. Complex dynamics we exemplify in this paper may be caused by the setting of asymmetric parameters in terms of two interacting economies. In order to study effects of the asymmetry, effects of a coordinated monetary policy and so on, we have to follow branches of unstable periodic orbits we have found through varying a parameter. Furthermore, if we give an appropriate definition of phase in terms of the business cycle, we will be able to discuss about chaotic time series of the phase difference between countries in the two-country model.

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