On the universal $sl_2$ invariant of bottom tangles

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Abstract

A bottom tangle is a tangle in a cube consisting of arc components whose boundary points are placed on the bottom, and every link can be represented as the closure of a bottom tangle. The universal $sl_2$ invariant of $n$-component bottom tangles takes values in the $n$-fold completed tensor power of the quantized enveloping algebra $U_h(sl_2)$, and has a universality property over the colored Jones polynomials of $n$-component links via quantum traces in finite dimensional representations. In this note, we study the values of the universal $sl_2$ invariant of certain three types of bottom tangles which are called boundary, ribbon, and brunnian bottom tangles. For each types of bottom tangles, we give certain small subalgebras in which the universal $sl_2$ invariant of bottom tangles of the type takes values. As applications, it follows that each boundary, ribbon, and brunnian link has stronger divisibility by cyclotomic polynomials than algebraically split links for Habiro's reduced version of the colored Jones polynomials.

1 Introduction

First of all, we recall tangles and bottom tangles. Then we define the three types of bottom tangles, boundary, ribbon, and brunnian bottom tangles. After that, we will mention the background of my research.

1.1 Tangles and bottom tangles

A tangle is the image of an embedding

$$\prod_{i=0}^{m}[0,1] \prod_{j=0}^{n} S^1 \hookrightarrow S^3$$

for $m, n \geq 0$, whose boundary is on the two lines $[0,1] \times \{\frac{1}{2}\} \times \{0,1\}$ on the bottom and on the top of the cube. We equip the image of an embedding both orientation and framing. In this note, the image of $[0,1]$ (resp. $S^1$) is called an arc (resp. cycle) component, see Figure 1 for example, and a point in boundary of arc components is called endpoint.

A bottom tangle is a tangle satisfying

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(1) there are no cycle components,
(2) every endpoint is on the line \([0,1] \times \{\frac{1}{2}\} \times \{0\}\) on the bottom,
(3) two endpoints of each component are adjacent to each other, and
(4) each component runs from its right endpoint to its left endpoint.

For example, see Figure 2 (a). We draw a diagram of a bottom tangle in a rectangle, see Figure 2 (b). For each \(n \geq 0\), let \(BT_n\) denote the set of the ambient isotopy classes, relative to endpoints, of \(n\)-component bottom tangles. The closure link \(\text{cl}(T)\) of \(T\) is defined as the unique isotopy class of links obtained from \(T\) by closing, see Figure 2 (c). For every \(n\)-component link \(L\), there is an \(n\)-component bottom tangle whose closure is isotopic to \(L\). For a bottom tangle, we can define the linking matrix as that of the closure link.
1.2 Boundary, ribbon, and brunnian bottom tangles

A Seifert surface of a knot $K$ is a compact connected orientable surface $F$ in $S^3$ bounded by $K$. An $n$-component link $L = L_1 \cup \cdots \cup L_n$ is called a boundary link if it bounds a disjoint union of $n$ Seifert surfaces $F_1, \ldots, F_n$ in $S^3$ such that $L_i$ bounds $F_i$ for $i = 1, \ldots, n$. For a 1-component bottom tangle $T \in BT_1$, there is a knot $L_T = (T \cup \gamma) \subset [0,1]^3$ where $\gamma$ is the line segment on the bottom $[0,1]^2 \times \{0\}$ such that $\partial\gamma = \partial T$. A Seifert surface of a 1-component bottom tangle $T$ is a Seifert surface of the knot $L_T$ in $[0,1]^3$. A bottom tangle $T = T_1 \cup \cdots \cup T_n$ is called a boundary bottom tangle if its components have disjoint Seifert surfaces $F_1, \ldots, F_n$ in $[0,1]^3$ such that $L_{T_i}$ bounds $F_i$ for $i = 1, \ldots, n$. For every boundary link $L$, there is a boundary bottom tangle whose closure is $L$.

An $n$-component link $L$ is called a ribbon link (cf. [1]) if it bounds the image of an immersion

$$D \cup \cdots \cup D \to S^3$$

from a disjoint union of two dimensional disks into $S^3$ with only ribbon singularities. Here a ribbon singularity is a singularity whose preimage consists of two lines one of which is in the interior of the disks. A ribbon bottom tangle is defined as a bottom tangle whose closure is a ribbon link.

A link is called brunnian link if its every proper sublink is trivial. Similarly, a bottom tangle is called brunnian bottom tangle if every proper subtangle is trivial, where a bottom tangle is said to be trivial if it has the trivial diagram that is copies of $\cap$. For each brunnian link $L$, there is a brunnian bottom tangle whose closure is $L$. 

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1.3 Back ground

In the 80's, Jones constructed a polynomial invariant of links by using von Neumann algebras. Shortly after, Reshetikhin and Turaev [8] defined invariants of framed links colored by finite dimensional representations of a ribbon Hopf algebra, which we call colored link invariants. The quantized enveloping algebra associated to a simple Lie algebra has a complete ribbon Hopf algebra structure, and Jones polynomial can be defined as the colored link invariant associated to the universal enveloping algebra $U_h := U_h(sl_2)$ and its 2-dimensional irreducible representation attached to all components of links. By a colored Jones polynomial, we mean a colored link invariant associated to $U_h$.

For a ribbon Hopf algebra, Lawrence [5, 4] and Ohtsuki [7] defined an invariant of framed tangle, which is called the universal invariant. By the universal $sl_2$ invariant, we mean the universal invariant associated to $U_h$. In [2], Habiro studied the universal invariant of bottom tangles associated to an arbitrary ribbon Hopf algebra, and in [3], he studied the universal $sl_2$ invariant of bottom tangles in detail. The universal $sl_2$ invariant of an $n$-component bottom tangle takes values in the $n$-fold completed tensor power $U_h^\otimes n$ of $U_h$. The universal invariant of bottom tangles has a universality property such that the colored link invariants of a link $L$ is obtained from the universal invariant of a bottom tangle $T$ whose closure is isotopic to $L$, by taking the quantum trace in the representations attached to the components of the link $L$. In particular, one can obtain colored Jones polynomials of links from the universal $sl_2$ invariant of bottom tangles.

In this note, we study algebraic properties of the universal $sl_2$ invariant of boundary, ribbon, and of brunnian bottom tangles.

2 The quantized enveloping algebra $U_h$ and its sub-algebras

In this note, we use the following $q$-integer notations:

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q \{i - 1\}_q \cdots \{i - n + 1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$

$$[i]_q = \{i\}_q / \{1\}_q, \quad [n]_q! = \{n\}_q \{n - 1\}_q \cdots \{1\}_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n} / \{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

We denote by $U_h$ the $h$-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by the elements $H, E,$ and $F$, satisfying the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.$$

We equip $U_h$ with a topological $\mathbb{Z}$-graded algebra structure with $\deg E = 1$, $\deg F = -1$, and $\deg H = 0$. For a homogeneous element $x$ of $U_h$, the degree of $x$ is denoted by $|x|$. 

There is a unique complete ribbon Hopf algebra structure on $U_h$ as follows. The comultiplication $\Delta: U_h \rightarrow U_h \hat{\otimes} U_h$, the counit $\varepsilon: U_h \rightarrow \mathbb{Q}[[h]]$, and the antipode $S: U_h \rightarrow U_h$ are given by

$$\begin{align*}
\Delta(H) &= H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H, \\
\Delta(E) &= E \otimes 1 + K \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -K^{-1}E, \\
\Delta(F) &= F \otimes K^{-1} + 1 \otimes F, \quad \varepsilon(F) = 0, \quad S(F) = -FK.
\end{align*}$$

The universal $R$-matrix $R \in U_h \hat{\otimes} U_h$ and its inverse are given by

\begin{align*}
R &= D \sum_{n \geq 0} q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n, \\
R^{-1} &= D^{-1} \sum_{n \geq 0} (-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n,
\end{align*}

where we set

$$\begin{align*}
D &= v^{\frac{1}{2}H \otimes H} = \exp \left( \frac{h}{4} H \otimes H \right) \in U_h \otimes^\wedge 2, \\
e &= (q^{1/2} - q^{-1/2}) E, \quad \tilde{F}^{(n)} = F^n K^n / [n]_q!,
\end{align*}$$

for $n \geq 0$.

The ribbon element $r \in U_h$ and its inverse are given by

$$\begin{align*}
r &= \sum R' K^{-1} \bar{R}' = \sum R'' K \bar{R}', \quad r^{-1} = \sum R' K R'' = \sum R'' K^{-1} R',
\end{align*}$$

where we set $R = \sum R' \otimes R''$, and $R^{-1} = (S \otimes 1) R = \sum \bar{R}' \otimes \bar{R}''$.

### 2.1 Subalgebras of $U_h$ and their completions

Let $U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_\mathbb{Z}$ generated by $K, K^{-1}, \tilde{E}^{(n)} = (v^{-1}E)^n / [n]_q!$, and $\tilde{F}^{(n)}$ for $n \geq 1$, and $U_{\mathbb{Z},q}^{ev}$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by the elements $K^2, K^{-2}, \tilde{E}^{(n)}$ and $\tilde{F}^{(n)}$ for $n \geq 1$.

**Remark 2.1.** Let $U_\mathbb{Z}$ denote Lusztig's integral form of $U_h$ (cf. [6]), which is defined to be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_h$ generated by $K, K^{-1}, E^{(n)} = E^n / [n]!$, and $F^{(n)} = F^n / [n]!$ for $n \geq 1$, where $[i] = \frac{q^{i/2} - q^{-i/2}}{q^{1/2} - q^{-1/2}}$ for $i \in \mathbb{Z}$ and $[n]! = [n] \cdots [1]$ for $n \geq 0$. We have

$$U_\mathbb{Z} = U_{\mathbb{Z},q} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Z}[v, v^{-1}].$$

Let $\bar{U}_q$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by the elements $K, K^{-1}, e$ and $f = (q - 1)FK$, and $\bar{U}_q^{ev}$ the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\bar{U}_q$ generated by the elements $K^2, K^{-2}, e$ and $f$.

Let $U_q^{ev}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}^{ev}$ generated by the elements $K^2, K^{-2}, e$ and $\tilde{F}^{(n)}$ for $n \geq 1$. 

We recall from [3] a filtration and a completion of $\mathcal{U}_q^{ev}$. For $p \geq 0$, let $\mathcal{F}_p(\mathcal{U}_q^{ev})$ be the two-sided ideal in $\mathcal{U}_q^{ev}$ generated by $e^p$. We define $\tilde{\mathcal{U}}_q^{ev}$ as the completion in $U_h$ of $\mathcal{U}_q^{ev}$ with respect to the decreasing filtration $\{\mathcal{F}_p(\mathcal{U}_q^{ev})\}_{p \geq 0}$, i.e., $\tilde{\mathcal{U}}_q^{ev}$ is the image of the homomorphism

$$
\lim_{p \geq 0} \left( \frac{\mathcal{U}_q^{ev}}{\mathcal{F}_p(\mathcal{U}_q^{ev})} \right) \rightarrow U_h
$$

induced by $\mathcal{U}_q^{ev} \subset U_h$. Then $\tilde{\mathcal{U}}_q^{ev}$ is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_h$.

For $n \geq 1$, let $((\tilde{\mathcal{U}}_q^{ev})^\otimes n)$ be the completion of the $n$-fold tensor product $(\mathcal{U}_q^{ev})^\otimes n$ of $\mathcal{U}_q^{ev}$ with respect to the decreasing filtration $\{\mathcal{F}_p((\mathcal{U}_q^{ev})^\otimes n)\}_{p \geq 0}$ such that

\begin{equation}
\mathcal{F}_p((\mathcal{U}_q^{ev})^\otimes n) = \sum_{i=1}^{n} (\mathcal{U}_q^{ev})^\otimes(i-1) \otimes \mathcal{F}_p(\mathcal{U}_q^{ev}) \otimes (\mathcal{U}_q^{ev})^\otimes(n-i).
\end{equation}

It is natural to set

\begin{equation}
\mathcal{F}_p((\mathcal{U}_q^{ev})^\otimes 0) = \mathcal{F}_p(\mathbb{Z}[q, q^{-1}]) = \begin{cases} 
\mathbb{Z}[q, q^{-1}] & \text{if } p = 0, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Thus we have

\begin{equation}
((\tilde{\mathcal{U}}_q^{ev})^\otimes 0) = \mathbb{Z}[q, q^{-1}].
\end{equation}

For a $\mathbb{Z}[q, q^{-1}]$-subalgebra $A$ of $(\mathcal{U}_q^{ev})^\otimes n$, we define the closure $(A)^\sim$ of $A$ in $((\tilde{\mathcal{U}}_q^{ev})^\otimes n)$ as the completion of $A$ with respect to the decreasing filtration $\{\mathcal{F}_p((\mathcal{U}_q^{ev})^\otimes n) \cap A\}_{p \geq 0}$. Especially, we denote by $(\tilde{\mathcal{U}}_q^{ev})^\otimes n$ the closure of $(\mathcal{U}_q^{ev})^\otimes n$ in $((\tilde{\mathcal{U}}_q^{ev})^\otimes n)$.

## 3 The universal $sl_2$ invariant of bottom tangles

In this section, we define the universal $sl_2$ invariant of bottom tangles (cf. [2]).

### 3.1 The universal $sl_2$ invariant of bottom tangles

In what follows, we write the $R$-matrix and its inverse as $R^{\pm 1} = \sum_{i \geq 0} R_i^{\pm}$, where we set

\begin{equation}
R_i = D(\alpha_i^+ \otimes \beta_i^+), \quad R_i^{-} = D^{-1}(\alpha_i^- \otimes \beta_i^-),
\end{equation}

\begin{equation}
\alpha_i^+ \otimes \beta_i^+ = q^{\frac{1}{2}(i-1)} \tilde{F}^{(i)} K^{-i} \otimes e^i, \quad \alpha_i^- \otimes \beta_i^- = (-1)^i \tilde{F}^{(i)} \otimes K^{-i} e^i.
\end{equation}

(We cannot define $\alpha_i^+, \beta_i^+, \alpha_i^-, \text{ or } \beta_i^-$, independently.)

**Remark 3.1.** In [9], we used different notations $R_i^+ = q^{\frac{1}{2}(i-1)} \tilde{F}^{(i)} K^{-i} \otimes e^i$ and $R_i^- = (-1)^i \tilde{F}^{(i)} \otimes K^{-i} e^i$. 

Figure 3: Fundamental tangles. The orientations of the strands are arbitrary.

\[(S' \otimes S') (R^+_{S(c)})\]

Figure 4: How to attach elements on the fundamental tangles.

We use diagrams of tangles obtained from copies of the fundamental tangles, as depicted in Figure 3, by pasting horizontally and vertically. For a bottom tangle \(T = T_1 \cup \cdots \cup T_n\), we define the universal \(sl_2\) invariant \(J_T \in U^\otimes n_h\) of \(T\) as follows. We choose a diagram \(P\) of \(T\). We denote by \(C(P)\) the set of the crossings of the diagram. We call a map

\[s: C(P) \rightarrow \{0, 1, 2, \ldots\}\]
a state. We denote by \(S(P)\) the set of states of the diagram \(P\).

For each fundamental tangle in the diagram, we attach elements of \(U_h\) or of \(U^\otimes 2_h\) associated to a state \(s \in S(P)\) following the rule described in Figure 4, where "\(S_m\) should be replaced with \(id\) if the string is oriented downward, and with \(S\) otherwise, see Figure 5. We define an element \(J_{P,s} \in U^\otimes n_h\) as follows. The \(i\)th component of \(J_{P,s}\) is defined to be the product of the elements put on the component corresponding to \(T_i\), where the elements are read off along each component reversing the orientation of \(P\), and written from left to right. Here we read an element \(y = \sum y_{[1]} \otimes y_{[2]} \in U^\otimes 2_h\) on arrowed dashed line by assuming that the first tensorand is attached to the startpoint of the arrow and the second tensorand to the endpoint of the arrow, see Figure 6. (The result does not depend on how one expresses the element on each dashed line as a sum of tensors.)

Set

\[J_T = \sum_{s \in S(P)} J_{P,s} \cdot\]

As is well known [7], \(J_T\) does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

For example, let us compute the universal \(sl_2\) invariant \(J_C\) of a bottom tangle \(C\) with a diagram \(P\) as depicted in Figure 7 (a), where \(c_1\) (resp. \(c_2\)) denotes the upper (resp. lower) crossing of \(P\). The diagram attached the elements for a state \(s \in S(P)\) is depicted in Figure 7 (b), where we set \(m = s(c_1), n = s(c_2)\). We have
$S'(x) = x = S(x)$

Figure 5: The definition of $S'$.

$y = \sum^{y_{[1]}} y[2]$  

Figure 6: How we read an element $y = \sum y_{[1]} \otimes y_{[2]} \in U_h^\otimes 2$.

$J_C = \sum_{s \in S(P)} J_{P,s} = \sum_{m,n \geq 0} \sum S(D_1^r \alpha^+_m) S(D_2^r \beta^+_n) \otimes D_2'' \alpha^+_n D'' \beta^+_m$

$= \sum_{m,n \geq 0} (-1)^{m+n} q^{-n+2mn} D^{-2}(\tilde{F}^{(m)} K^{-2n} e^n \otimes \tilde{F}^{(n)} K^{-2m} e^m)$.

where we set $D = \sum D_1^r \otimes D_1'^f = \sum D_2^r \otimes D_2''$.

The following propositions is fundamental.

**Proposition 3.2 ([9]).** Let $T$ be an $n$-component bottom tangle with 0-framing, and $P$ a diagram of $T$. We have

$J_{P,s} \in (\mathcal{U}_q^{ev})^\otimes n$.

Later, we use the following lemma.

**Lemma 3.3.** Let $T$ be an $n$-component bottom tangle with 0-framing, and $P$ a diagram of $T$. Set $|s| = \max\{s(c) \mid c \in C(P)\}$. We have

$J_{P,s} \in F_{|s|}((\mathcal{U}_q^{ev})^\otimes n)$.

(5)

### 3.2 Colored Jones polynomials

If $V$ is a finite dimensional representation of $U_h$, then the quantum trace $tr^V_q(x)$ in $V$ of an element $x \in U_h$ is defined by

$tr^V_q(x) = tr^V(\rho_V(K^{-1}x)) \in \mathbb{Q}[[\hbar]]$,
where $\rho_V: U_h \rightarrow \text{End}(V)$ denotes the left action of $U_h$ on $V$, and $\text{tr}^V: \text{End}(V) \rightarrow \mathbb{Q}[[h]]$ denotes the trace in $V$. For every element $y = \sum_n a_n V_n \in R$, $a_n \in \mathbb{Q}(v)$, we set

$$\text{tr}^y_q(x) = \sum_n a_n \text{tr}^V_q(x) \in \mathbb{Q}((h))$$

for $x \in U_h$. Here $\mathbb{Q}((v))$ denote the quotient field of $\mathbb{Q}[[h]]$.

The universal $sl_2$ invariant of bottom tangles has a universality property to the colored Jones polynomials of links as the following.

**Proposition 3.4** (Habiro [3]). Let $L = L_1 \cup \cdots \cup L_n$ be an $n$-component, ordered, oriented, framed link in $S^3$. Choose an $n$-component bottom tangle $T$ whose closure is isotopic to $L$. For $y_1, \ldots, y_n \in R$, the colored Jones polynomial $J_{L;y_1,\ldots,y_n}$ of $L$ can be obtained from $J_T$ by

$$J_{L;y_1,\ldots,y_n} = (\text{tr}^y_1 \otimes \cdots \otimes \text{tr}^y_n)(J_T).$$

## 4 Main results

In this section, we give the main results. The results for boundary bottom tangles, which was conjectured by Habiro [3], and for ribbon bottom tangles are similar to each other as follows.

**Theorem 4.1.** Let $T$ be an $n$-component boundary bottom tangle with 0-framing. Then we have $J_T \in (\overline{U}^\text{ev}_q)^{\wedge \otimes n}$.

**Theorem 4.2** ([9]). Let $T$ be an $n$-component ribbon bottom tangle with 0-framing. Then we have $J_T \in (\overline{U}^\text{ev}_q)^{\wedge \otimes n}$.

In fact, We have a refinement of each Theorem 4.1 and 4.2 with a smaller subalgebra $(\overline{U}^\text{ev}_q)^{\otimes n} \subset (\overline{U}^\text{ev}_q)^{\wedge \otimes n}$ in place of $(\overline{U}^\text{ev}_q)^{\wedge \otimes n}$, see Section 8.2 for the definition of $(\overline{U}^\text{ev}_q)^{\wedge \otimes n}$. Here, we do not know whether the inclusion is proper or not, but the definition of $(\overline{U}^\text{ev}_q)^{\wedge \otimes n}$ is more natural than that of $(\overline{U}^\text{ev}_q)^{\wedge \otimes n}$ in our setting.

The result for brunnian bottom tangles is as follows.
Theorem 4.3. Let $T$ be an $n$-component brunnian bottom tangle. Then we have

\[ J_T \in \bigcap_{i=0}^{n} \{ (\overline{U}_q^{ev})^\otimes i-1 \otimes U_{Z,q}^{ev} \otimes (\overline{U}_q^{ev})^\otimes n-i \}^\wedge. \]

For a bottom tangle $T = T_1 \cup \cdots \cup T_n$, let denote by $\overline{T}_{i_1, \ldots, i_m}$ the subtangle obtained from $T$ by removing its components $T_{i_1}, \ldots, T_{i_m}$. In fact, Theorem 4.3 is a corollary of the following result.

Theorem 4.4. Let $T$ be an $n$-component bottom tangle with 0-framing whose subtangle $\overline{T}_{i_1, \ldots, i_m}$ is trivial. Then we have

\[ J_T \in (A_1 \otimes A_2 \otimes \cdots \otimes A_n). \]

where

\[
A_i = \begin{cases} 
U_{Z,q}^{ev} & i = i_1, \ldots, i_m \\
\overline{U}_q^{ev} & \text{other.}
\end{cases}
\]

5 Applications

Here, we give an application of each Theorem 4.1, 4.2, and 4.4. For $m \geq 1$, let $V_m$ denote the $m$-dimensional irreducible representation of $U_h$. Let $\mathcal{R}$ denote the representation ring of $U_h$ over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

\[ \mathcal{R} = \text{Span}_{\mathbb{Q}(q^{\frac{1}{2}})} \{ V_m \mid m \geq 1 \} \]

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

Habiro [3] studied the following elements in $\mathcal{R}$

\[ \tilde{P}_l' = \frac{q^l}{(l)_q} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \]

for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. He proved the following.

Theorem 5.1 (Habiro [3]). Let $L$ be an $n$-component, algebraically-split link with 0-framing. We have

\[ J_{L;P_{l_1}', \ldots, P_{l_n}'} \in \frac{\{2l_j + 1\}_{q,l_j+1}Z[q,q^{-1}]}{\{1\}_q}, \]

for $l_1, \ldots, l_n \geq 0$, where $j$ is an integer such that $l_j = \max\{l_i\}_{1 \leq i \leq n}$.

Habiro [3] proved that Theorem 4.1 implies the following result.
**Theorem 5.2.** Let \( L \) be an \( n \)-component boundary link with 0-framing. We have

\[
J_{L; \tilde{P}, \ldots, \tilde{P}_{l_{n}}'} \in \left\{ \frac{2l_{j} + 1}{q, l_{j} + 1} \right\}_{q} \prod_{1 \leq i \leq n, i \neq j} I_{l_{i}},
\]

for \( l_{1}, \ldots, l_{n} \geq 0 \), where \( j \) is an integer such that \( l_{j} = \max\{l_{i}\}_{1 \leq i \leq n} \). Here, for \( l \geq 0 \), \( I_{l} \) is the ideal in \( \mathbb{Z}[q, q^{-1}] \) generated by the elements \( \{l-k\}_{q}!\{k\}_{q}! \) for \( k = 0, \ldots, l \).

**Remark 5.3.** For \( m \geq 1 \), let \( \Phi_{m}(q) \in \mathbb{Z}[q] \) denote the \( m \)th cyclotomic polynomial. It is not difficult to prove that \( I_{l}, l \geq 0 \), is contained in the principle ideal generated by \( \prod_{m} \Phi_{m}(q)^{f(l, m)} \), where \( f(l, m) = \max\{0, \left\lfloor \frac{l+1}{m} \right\rfloor - 1\} \). Here for \( r \in \mathbb{Q} \), we denote by \( \lfloor r \rfloor \) the largest integer smaller than or equal to \( r \).

Similarly, we have the following.

**Theorem 5.4.** Let \( L \) be an \( n \)-component ribbon link with 0-framing. We have

\[
J_{L; \tilde{P}, \ldots, \tilde{P}_{l_{n}}'} \in \left\{ \frac{2l_{j} + 1}{q, l_{j} + 1} \right\}_{q} \prod_{1 \leq i \leq n, i \neq j} I_{l_{i}},
\]

for \( l_{1}, \ldots, l_{n} \geq 0 \), where \( j \) is an integer such that \( l_{j} = \max\{l_{i}\}_{1 \leq i \leq n} \).

For a link \( L = L_{1} \cup \ldots \cup L_{n} \), we denote by \( \check{L}_{i_{1}, \ldots, i_{m}} \) the sublink obtained from \( L \) by removing its components \( L_{i_{1}}, \ldots, L_{i_{m}} \). In a similar way in which Habiro proved Theorem 5.2 by assuming Theorem 4.1, we can prove the following.

**Theorem 5.5.** Let \( L = L_{1} \cup \ldots \cup L_{n} \) be a link with 0-framing whose sublink \( \check{L}_{i_{1}, \ldots, i_{m}} \) is trivial. We have

\[
J_{L; \check{P}, \ldots, \check{P}_{l_{n}}'} \in \left\{ \frac{2l_{j} + 1}{q, l_{j} + 1} \right\}_{q} \prod_{1 \leq i \leq n, i \neq j, i_{1}, \ldots, i_{m}} I_{l_{i}},
\]

for \( l_{1}, \ldots, l_{n} \geq 0 \), where \( j \) is an integer such that \( l_{j} = \max\{l_{i} \mid 1 \leq i \leq n, i \neq i_{1}, \ldots, i_{m}\} \).

**Corollary 5.6.** Let \( L \) be an \( n \)-component brunnian link with 0-framing. We have

\[
J_{L; \tilde{P}, \ldots, \tilde{P}_{l_{n}}'} \in \left\{ \frac{2l_{j} + 1}{q, l_{j} + 1} \right\}_{q} \prod_{1 \leq i \leq n, i \neq j, k} I_{l_{k}},
\]

for \( l_{1}, \ldots, l_{n} \geq 0 \), where \( j \) is an integer such that \( l_{j} = \max\{l_{i} \mid 1 \leq i \leq n\} \) and \( k \) is an integer such that \( l_{k} = \min\{l_{i} \mid 1 \leq i \leq n\} \).

### 6 The universal \( sl_{2} \) invariant of boundary, ribbon, and of brunnian bottom tangles

In this section, we study the universal \( sl_{2} \) invariant of boundary, ribbon, and of brunnian bottom tangles. We recall Habiro's formulas for the universal invariant of boundary bottom tangles and of ribbon bottom tangles, which we used in a proof of Theorem 4.1 and 4.2. (We do not write the proofs in this note.) For brunnian bottom tangles, we prove Theorem 4.4.
Let $Y: U_h \otimes U_h \to U_h$ be the $U_h$-module homomorphism defined by

$$Y(x \otimes y) = \sum x_{(1)} \beta_k S((\alpha_k \triangleright y)_{(1)}) S(x_{(2)})(\alpha_k \triangleright y)_{(2)}$$

for $x, y \in U_h$.

**Remark 6.1.** The morphism $Y$ is equal to $Y_H$ for $H = U_h$ in [2, Section 9.3].

For $T \in BT_{i+j+2}$, $i,j \geq 0$, let $(Y_b)_{i,j}(T) \in BT_{i+j+1}$ and $(\mu_b)_{(i,j)}(T) \in BT_{i+j+1}$ denote the bottom tangles as depicted in Figure 8.

In what follows, we use a notation

$$f_{i,j} = \text{id} \otimes f \otimes \text{id} : U_h^{\otimes^i} \to U_h^{\otimes^i}$$

for $f : U_h^{\otimes^k} \to U_h^{\otimes^l}$.

**Lemma 6.2** (Habiro [2]). For a bottom tangle $T \in BT_{i+j+2}$, $i,j \geq 0$, we have

$$J_{(Y_b)_{i,j}}(T) = Y_{i,j}(J_T),$$
$$J_{(\mu_b)_{i,j}}(T) = \mu_{i,j}(J_T).$$

**where $\mu : U_h \hat{\otimes} U_h \to U_h$ is the multiplication of $U_h$.**

Let $T = T_1 \cup \cdots \cup T_n$ be a boundary bottom tangle and $F_1, \ldots, F_n$ a disjoint compact, oriented surfaces such that $\partial F_i = T_i$ for $i = 1, \ldots, n$. We can arrange the surfaces $F_1, \ldots, F_n$ as depicted in Figure 9, where $\text{Double}(T')$ is the tangle obtained from a bottom tangle $T'$ by duplicating and then reversing the orientation of the inner component of each duplicated component. This implies the following proposition, which is implicit in [2, Theorem 9.9].
Figure 9: An arranged Seifert surfaces of the bottom tangle $T$.

Figure 10: A bottom tangle $T \in BT_{2g}$ and the bottom tangle $Y_{b}^{\otimes g}(T) \in BT_{g}$.

**Proposition 6.3.** For an $n$-component bottom tangle $T$, the following conditions are equivalent.

1. $T$ is a boundary bottom tangle.
2. There is a bottom tangle $T' \in BT_{2g}, g \geq 0$, and there are integers $g_{1}, \ldots, g_{n} \geq 0$ satisfying $g_{1} + \cdots + g_{n} = g$, such that
   
   $$T = \mu_{b}^{[g_{1}, \ldots, g_{n}]} Y_{b}^{\otimes g}(T'),$$

   where

   $$Y_{b}^{\otimes g}: BT_{2g} \rightarrow BT_{g}$$

   is as depicted in Figure 10, and

   $$\mu_{b}^{[g_{1}, \ldots, g_{n}]}: BT_{g_{1} + \cdots + g_{n}} \rightarrow BT_{n}$$

   is as depicted in Figure 11.

   If (8) holds, then we call $(T'; g_{1}, \ldots, g_{n})$ a boundary data for $T$.

   For $n \geq 1$, let

   $$\mu^{[n]}: U_{h}^{\otimes n} \rightarrow U_{h}, \ x_{1} \otimes \cdots \otimes x_{n} \mapsto x_{1}x_{2} \cdots x_{n}$$
Given integers $g_1, \ldots, g_n \geq 0$, set $g_1 + \cdots + g_n = g$, set
\[ \mu^{[g_1, \ldots, g_n]} = \mu^{[g_1]} \otimes \cdots \otimes \mu^{[g_n]} : U^\otimes k \to U^\otimes n. \]

Lemma 6.2 and Proposition 6.3 imply the following.

**Proposition 6.4 (Habiro [2]).** Let $T$ be an $n$-component boundary bottom tangle and $(T' \in BT_{2g}; g_1, \ldots, g_n)$ a boundary data for $T$. Then we have
\[ J_T = \mu^{[g_1, \ldots, g_n]} Y^\otimes g(J_{T'}). \]

### 6.2 The universal $sl_2$ invariant of ribbon bottom tangles

Habiro [3] studied the universal $sl_2$ invariant of 1-component ribbon bottom tangles. We generalize those to $n$-component ribbon bottom tangles for $n \geq 1$. We use the left adjoint action $\text{ad}: U_h \otimes U_h \to U_h$ defined by
\[ \text{ad}(a \otimes b) = \sum a'bS(a''), \]
for $a, b \in U_h$, where we set $\Delta(a) = \sum a' \otimes a''$. We also use the notation $a \triangleright b = \text{ad}(a \otimes b)$.

For $T \in BT_{i+j+2}$, $i, j \geq 0$, let $(\text{ad}_b)_{i,j}(T) \in BT_{i+j+1}$ denote the bottom tangle as depicted in Figure 12. We use the following lemma.

**Lemma 6.5 (Habiro [2]).** For a bottom tangle $T \in BT_{i+j+2}$, $i, j \geq 0$, we have
\[ J_{(\text{ad}_b)_{i,j}(T)} = \text{ad}_{i,j}(J_T). \]
Figure 13: A bottom tangle $T \in BT_{2k}$ and the bottom tangle $\text{ad}_b^{\otimes k}(T) \in BT_k$.

For a $2k$-component bottom tangle $W = W_1 \cup \cdots \cup W_{2k} \in BT_{2k}$, $k \geq 0$, set

$$W^{ev} = \bigcup_{i=1}^{k} W_{2i} \in BT_k, \quad \text{and} \quad W^{odd} = \bigcup_{i=1}^{k} W_{2i-1} \in BT_k.$$ 

For a diagram $P$ of $W$, let $P^{ev}$ (resp. $P^{odd}$) denote the part of the diagram $P$ corresponding to $W^{ev}$ (resp. $W^{odd}$). We say a bottom tangle $W \in BT_{2k}$ is even-trivial if $W^{ev}$ is a trivial bottom tangle. For example, see Figure 14. We also say a diagram $P$ of $W$ is even-trivial if and only if $P^{ev}$ has no self crossings. Note that a bottom tangle $W$ has an even-trivial diagram if and only if $W$ is even-trivial.

The following Proposition is almost the same as [2, Theorem 11.5].

**Proposition 6.6.** For an $n$-component bottom tangle $T$, the following conditions are equivalent.

1. $T$ is a ribbon bottom tangle.
2. There is an even-trivial bottom tangle $W \in BT_{2k}$, $k \geq 0$, and there are integers $N_1, \ldots, N_n \geq 0$ satisfying $N_1 + \cdots + N_n = k$, such that

$$T = \mu^{[N_1, \ldots, N_n]} \text{ad}_b^{\otimes k}(W),$$

where

$$\text{ad}_b^{\otimes k} : BT_{2k} \to BT_k$$

is as depicted in Figure 13.

If (9) holds, then we call $(W; N_1, \ldots, N_n)$ a ribbon data for $T$. For example, the ribbon bottom tangle $\mu^{[1,2,0]}(\text{ad}_b)^{\otimes 3}(W) \in BT_3$ with the ribbon data $(W \in BT_3; 1, 2, 0)$, where $W$ is the bottom tangle in Figure 14, is as depicted in Figure 15.

Lemma 6.5 and Proposition 6.6 imply the following.

**Proposition 6.7.** Let $T$ be an $n$-component ribbon bottom tangle and $(W \in BT_{2k}; N_1, \ldots, N_n)$ a ribbon data for $T$. Then we have

$$J_T = \mu^{[N_1, \ldots, N_n]} \text{ad}_b^{\otimes k}(J_W),$$

where $\text{ad}_b^{\otimes k} : U_h^{\otimes 2k} \to U_h^{\otimes k}$ is the $k$-fold tensor power of the adjoint action.
Figure 14: An even-trivial bottom tangle $W \in BT_6$. Here $W^{ev}$ is depicted with thick lines.

Figure 15: The ribbon bottom tangle $\mu^{[1,2,0]}(ad_b)^{\otimes 3}(W) \in BT_3$ for the even-trivial bottom tangle $W \in BT_3$ in Figure 14.
6.3 The universal $sl_2$ invariant of brunnian bottom tangles

We prove Theorem 4.4. We only have to prove the following claim.

Claim: There is a diagram $P$ of $T$, such that every state $s \in S(P)$, we have

$$J_{P,s} \in A_1 \otimes \cdots A_n. \quad (10)$$

By Lemma 3.3 and (10), we will have

$$J_{P,s} \in (A_1 \otimes \cdots A_n) \cap F_{|s|}((\mathcal{U}_q^{ev})^\otimes n).$$

It will imply that

$$J_T = \sum_{p \geq 0} \sum_{s \in S(P), |s| = p} J_{P,s} \in (A_1 \otimes \cdots A_n).$$

We prove (10). By definition, the subtangle $T_{i_1, \ldots, i_m}$ has the trivial diagram, hence $T$ has a diagram $P = P_1 \cup \cdots \cup P_n$ whose subdiagram $P_{i_1, \ldots, i_m}$ corresponding to $T_{i_1, \ldots, i_m}$ is the trivial diagram. Figure 16 is an example with the Borromean tangle that is a 3-component brunnian bottom tangle, whose closure is Borromean rings. Note that $P$ has two kinds of crossings:

- Crossings between $P_{i_1, \ldots, i_m}$ and $P_j, j \neq i_1, \ldots, i_m$
- Self crossings of $P_{i_1, \ldots, i_m}$

Let calculate $J_{P,s}$ for a state $s \in S(P)$. We modify the elements attached to crossings as follows. Let $c$ be a crossing of the diagram with strands oriented downward, and set $m = s(c)$. As depicted in Figure 17, we replace the two dots labeled by $R_m^\pm$ with two black dots labeled by $D_m^\pm$ and two white dots labeled by $\alpha_m^\pm \otimes \beta_m^\pm$. Similarly, we modify the dots on the other crossings. We have completed the modification. We have

$$R = D \sum_{n \geq 0} q^{\frac{1}{4}n(n-1)} F^{(n)} K^{-n} \otimes e^n$$

$$= D \sum_{n \geq 0} q^{n(n-1)} f^n K^{-n} \otimes E^{(n)},$$

$$R^{-1} = D^{-1} \sum_{n \geq 0} (-1)^n F^{(n)} K^{-n} e^n$$

$$= D^{-1} \sum_{n \geq 0} (-1)^n q^{\frac{1}{4}n(n-1)} f^n K^{-n} E^{(n)}.$$
Figure 17: The modification process of elements on positive and negative crossings.

Figure 18: How we treat $\alpha_m \otimes \beta_m$.

Hence we have

$$\alpha_m^\pm \otimes \beta_m^\pm \in (U_{Z,q} \otimes \bar{U}_q) \cap (\bar{U}_q \otimes U_{Z,q}) \subset U_{Z,q} \otimes U_{Z,q}.$$ 

Hence for a crossings between $P_{i_1 \ldots i_m}$ and $P_j$, $j \neq i_1, \ldots, i_m$, we can assume that the element on the white dot on $P_{i_1 \ldots i_m}$ is in $U_{Z,q}$ and that on $P_j$ is in $\bar{U}_q$, and for a self crossing of $P_{i_1 \ldots i_m}$, we can assume the element on the white dot is in $\bar{U}_q$, see Figure 18. We slide the elements $D^\pm$ on the black dots to the heads of tensorands of $J_{P,s}$ by using the formula

$$(1 \otimes x)D = D(K^{|-x|} \otimes x)$$

where $x$ is a homogeneous element of $U_h$, see Figure 19. Since $T$ is with 0-framing, those $D^\pm$s are cancelled. Hence, $i_1, \ldots, i_m$th tensorands of $J_{P,s}$ are contained in $U_{Z,q}$ and others in $\bar{U}_q$. In the view of Proposition 3.2, $J_{P,s}$ is contained in even part of the subalgebra, hence we have the assertion.

7 Examples

The Borromean tangle $B \in BT_3$ is the bottom tangle depicted in Figure 16, which we can depict as in Figure 20 as well. Note that $B$ is a 3-component, algebraically-split,
0-framed bottom tangle, and the closure of $B$ is the Borromean rings $L_B$. It is well
known that $L_B$ is not a ribbon link. In [3], the formulas of the universal $sl_2$ invariant
of $B$ is observed:

$$J_B = \sum_{m_1,m_2,m_3,n_1,n_2,n_3 \geq 0} q^{m_3+n_3} (-1)^{n_1+n_2+n_3} q^{\sum_{i=1}^{3} (-\frac{1}{2}m_i(m_i+1)-n_i+m_i+1-2m_i n_i-1)}$$

$$\tilde{F}^{(n_3)} e^{m_1} \tilde{F}^{(m_3)} e^{n_1} K^{-2m_2} \otimes \tilde{F}^{(n_1)} e^{m_2} \tilde{F}^{(m_1)} e^{n_2} K^{-2m_3} \otimes \tilde{F}^{(n_2)} e^{m_3} \tilde{F}^{(m_2)} e^{n_3} K^{-2m_1}$$

$$\not\in (\overline{U}_{q}^{ev})^{\wedge \otimes 3},$$

(12)

where the index $i$ should be considered modulo 3. The following is also observed in [3];

$$J_{L_{B;\tilde{P}_{1}',\ldots,\tilde{P}_{1}'}} = \begin{cases} (-1)^{i} q^{-i(3i-1) \{2i+1\}_{q,i+1} / \{1\}_{q}} & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

(13)

Since $\frac{\{2i+1\}_{q,i+1}}{\{1\}_{q}} \not\in \mathbb{Z}[q,q^{-1}]$ for $i \geq 1$, each of (12) and (13) implies that the
Borromean rings $L_B$ is not a boundary or a ribbon link.

For $n \geq 3$, Milnor's link $L_{M,n}$ is the $n$-component brunnian link as depicted in Figure
21. Note that $L_{M,3}$ is the Borromean rings $L_B$. For $m \geq 1$, recall that $\Phi_m(q)$ is the
$m$th cyclotomic polynomial in $q$. We have

$$J_{L_{M,n;\tilde{P}_{1}',\ldots,\tilde{P}_{1}'}} = (-1)^{n-2} q^{-2n+4}\Phi_4(q)^{n-3}\Phi_3(q)\Phi_2(q)^{n-2}\Phi_1(q)^{n-2} \not\in \mathbb{Z}[q,q^{-1}]\Phi_1(q)^{n}.$$ 

Hence, for all $n \geq 3$, $L_{M,n}$ is not a boundary or a ribbon link.
8 Completion for $\overline{U}^{ev}_{q}$

8.1 Filtrations of $\overline{U}^{ev}_{q}$

In this subsection, we define two filtrations $\{A_p\}_{p\geq 0}$ and $\{C_p\}_{p\geq 0}$ of $\overline{U}^{ev}_{q}$, which are cofinal with each other. We give four equivalent definitions for $\{A_p\}_{p\geq 0}$, and two for $\{C_p\}_{p\geq 0}$.

For a subset $X \subset \overline{U}^{ev}_{q}$, let $(X)_{\text{ideal}}$ denote the two-sided ideal of $\overline{U}^{ev}_{q}$ generated by $X$. For $p \geq 0$, set

$$A_p = \langle U_{Z,q} \triangleright e^p \rangle_{\text{ideal}}, \quad A'_p = \langle U_{Z,q} \triangleright f^p \rangle_{\text{ideal}},$$

$$B_p = \langle K^p(U_{Z,q} \triangleright K^{-p}e^p) \rangle_{\text{ideal}}, \quad B'_p = \langle K^p(U_{Z,q} \triangleright f^pK^{-p}) \rangle_{\text{ideal}},$$

$$C_p = \left( \sum_{p' \geq p} (U_{Z,q} \tilde{E}^{(p')} \triangleright \overline{U}^{ev}_{q}) \right)_{\text{ideal}}, \quad C'_p = \left( \sum_{p' \geq p} (U_{Z,q} \tilde{F}^{(p')} \triangleright \overline{U}^{ev}_{q}) \right)_{\text{ideal}}.$$

Proposition 8.1 ([9]).

(i) $\{A_p\}_{p\geq 0}$ is a decreasing filtration.

(ii) For $p \geq 0$, we have

$$A_p = A'_p = B_p = B'_p.$$

Proposition 8.2 ([9]).

(i) For $p \geq 0$, we have $C_p = C'_p$.

(ii) For $p \geq 0$, we have $C_{2p} \subset A_p$.

(iii) If $p \geq 0$ is even, then we have $C_{2p} = A_p$.

Corollary 8.3. For $p \geq 0$, we have

$$C_{2p} \subset h^pU_h.$$

Proof. Since $e^p \in h^pU_h$, we have $A_p \subset h^pU_h$. Then the assertion follows from Proposition 8.2 (iii).

8.2 The completion $(\overline{U}^{ev}_{q})^{\wedge n}$ of $(\overline{U}^{ev}_{q})^{\otimes n}$

In this subsection we define the completion $(\overline{U}^{ev}_{q})^{\wedge n}$ of $(\overline{U}^{ev}_{q})^{\otimes n}$. Let $(\overline{U}^{ev}_{q})^{\wedge}$ denote the completion in $U_h$ of $\overline{U}^{ev}_{q}$ with respect to the decreasing filtration $\{C_p\}_{p\geq 0}$, i.e., $(\overline{U}^{ev}_{q})^{\wedge}$ is
the image of the homomorphism
\[
\lim_{p} (\overline{U}_{q}^{ev}/C_{p}) \to U_{h}.
\]
induced by the inclusion \(\overline{U}_{q}^{ev} \subset U_{h}\), which is well defined since \(C_{2p} \subset h^{p}U_{h}\) for \(p \geq 0\).

For \(n \geq 1\), we define a filtration \(\{C_{p}^{(n)}\}_{p \geq 0}\) for \((\overline{U}_{q}^{ev})^{\otimes n}\) by
\[
C_{p}^{(n)} = \sum_{j=1}^{n} \overline{U}_{q}^{ev} \otimes \cdots \otimes \overline{U}_{q}^{ev} \otimes C_{p} \otimes \overline{U}_{q}^{ev} \otimes \cdots \otimes \overline{U}_{q}^{ev},
\]
where \(C_{p}\) is at the \(j\)th position. Define the completion \((\overline{U}_{q}^{ev})^{\wedge \otimes n}\) of \((\overline{U}_{q}^{ev})^{\otimes n}\) as the image of the homomorphism
\[
\lim_{p} ((\overline{U}_{q}^{ev})^{\otimes n}/C_{p}^{(n)}) \to U_{h}^{\otimes n}.
\]
For \(n = 0\), it is natural to set
\[
C_{p}^{(0)} = \begin{cases} 
\mathbb{Z}[q, q^{-1}] & \text{if } p = 0, \\
0 & \text{otherwise.}
\end{cases}
\]
Thus, we have
\[
(\overline{U}_{q}^{ev})^{\wedge \otimes 0} = \mathbb{Z}[q, q^{-1}].
\]

References


