<table>
<thead>
<tr>
<th>Title</th>
<th>Dynamical braided monoids and dynamical Yang-Baxter maps (Quantum groups and quantum topology)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Shibukawa, Youichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2010, 1714: 80-89</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/170281">http://hdl.handle.net/2433/170281</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Dynamical braided monoids
and
dynamical Yang-Baxter maps

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan

Abstract

By means of torsors (principal homogeneous spaces), we prove that dynamical braided monoids can produce dynamical Yang-Baxter maps.

1 Introduction

Finding solutions to the quantum Yang-Baxter equation [1, 21] is essential in the study of integrable systems [2, 8]. This quantum Yang-Baxter equation is exactly the braid relation in a suitable tensor category; for example, the usual quantum Yang-Baxter equation is the braid relation in the tensor category of vector spaces, and the quantum group [3, 7] is useful for the construction of solutions.

Lu, Yan, and Zhu [12] constructed Yang-Baxter maps [4, 20], solutions to the braid relation in the tensor category of sets, by means of braided groups [19]. Let $S$ and $B$ be groups whose unit elements are respectively denoted by $1_S$ and $1_B$, and let $\sigma$ be a map from $S \times B$ to $B \times S$.

Definition 1.1. A triple $(S, B, \sigma)$ is a matched pair of groups [18], iff the map $\sigma : S \times B \ni (s, b) \mapsto (s \leftarrow b, s \leftarrow b) \in B \times S$ satisfies:

\[ s \leftarrow (t \leftarrow b) = (st) \leftarrow b; \]  
\[ (st) \leftarrow b = (s \leftarrow (t \leftarrow b))(t \leftarrow b); \]  
\[ (s \leftarrow b) \leftarrow c = s \leftarrow (bc); \]  
\[ s \leftarrow (bc) = (s \leftarrow b)((s \leftarrow b) \leftarrow c); \]  
\[ 1_S \leftarrow b = b; \]  
\[ s \leftarrow 1_B = s \quad (\forall s, t \in S, \forall b, c \in B). \]

The Cartesian product $B \times S$ is a group with the multiplication

\[ (b, s)(c, t) = (b(s \leftarrow c), (s \leftarrow c)t) \quad ((b, s), (c, t) \in B \times S). \]
To be more precise, the unit element is $(1_B, 1_S)$, and the inverse of the element $(b, s) \in B \times S$ is $(s^{-1} \mapsto b^{-1}, s^{-1} \mapsto b^{-1})$.

**Definition 1.2.** A pair $(G, \sigma)$ of a group $G$ and a map $\sigma : G \times G \to G \times G$ is a braided group, iff:

1. $(G, G, \sigma)$ is a matched pair of groups;
2. if $(y', x') = \sigma(x, y)$, then $y'x' = xy$ ($x, y, x', y' \in G$).

In [12], Lu, Yan, and Zhu showed

**Theorem 1.3.** If $(G, \sigma)$ is a braided group, then $\sigma$ satisfies the braid relation.

$$(\sigma \times \text{id}_G) \circ (\text{id}_G \times \sigma) \circ (\sigma \times \text{id}_G) = (\text{id}_G \times \sigma) \circ (\sigma \times \text{id}_G) \circ (\text{id}_G \times \sigma).$$

We can rephrase the definition of the matched pair of groups by using category theory.

Let $I_{\text{Set}}$ denote the set $\{e\}$ of one element. We write $m_S$ and $m_B$ for the multiplications of the groups $S$ and $B$, respectively. We define the maps $\eta_S : I_{\text{Set}} \to S$ and $\eta_B : I_{\text{Set}} \to B$ by

$$\eta_S(e) = 1_S; \eta_B(e) = 1_B.$$

The above equations (1.1)-(1.6) are equivalent to:

$$\begin{align*}
(id_B \times m_S) \circ (\sigma \times \text{id}_S) \circ (\text{id}_S \times \sigma) &= \sigma \circ (m_S \times \text{id}_B); \\
(m_B \times \text{id}_S) \circ (\text{id}_B \times \sigma) \circ (\sigma \times \text{id}_B) &= \sigma \circ (\text{id}_S \times m_B); \\
(id_B \times m_S) \circ (\sigma \times \text{id}_S) \circ (\eta_S \times \text{id}_{B \times S}) &= l_{B \times S}; \\
(m_B \times \text{id}_S) \circ (\text{id}_B \times \sigma) \circ (\text{id}_{B \times S} \times \eta_B) &= r_{B \times S}.
\end{align*}$$

Here, the maps $l_{B \times S} : I_{\text{Set}} \times B \times S \to B \times S$ and $r_{B \times S} : B \times S \times I_{\text{Set}} \to B \times S$ are defined by

$$l_{B \times S}(e, b, s) = (b, s); r_{B \times S}(b, s, e) = (b, s) \quad (I_{\text{Set}} = \{e\}, b \in B, s \in S).$$

It is natural to try to solve the braid relation in another tensor category similarly.

The aim of this article is to make an analogy between the Yang-Baxter maps and dynamical Yang-Baxter maps (Definition 2.1) [14], solutions to the braid relation in a tensor category $\text{Set}_H$ [15] defined in the next section. We construct the dynamical Yang-Baxter maps by means of dynamical
braided monoids in Definition 4.2. Torsors [9, 11], also known as the principal homogeneous spaces, are important in this construction.

The organization of this article is as follows.

In Section 2, we briefly sketch a tensor category $\text{Set}_H$. Section 3 explains monoids in $\text{Set}_H$. After introducing dynamical braided monoids, our main results are stated and proved in Sections 4 and 5. The crucial fact is that the dynamical braided monoid satisfying (3.1) is exactly a torsor (See Proposition 5.6).

2 Tensor category $\text{Set}_H$ and dynamical Yang-Baxter maps

This section explains the tensor category $\text{Set}_H$ (cf. the tensor category $\mathcal{V}$ in [5, Section 3]), in which we will focus on the braid relation (For the tensor category, see [10, Chapter XI]).

Let $H$ be a nonempty set. $\text{Set}_H$ is a category whose object is a pair $(X, \cdot X)$ of a nonempty set $X$ and a map $\cdot X : H \times X \ni (\lambda, x) \mapsto \lambda \cdot X x \in H$ and whose morphism $f : (X, \cdot X) \to (Y, \cdot Y)$ is a map $f : H \to \text{Map}(X, Y)$ satisfying that

$$\lambda \cdot f(\lambda)(x) = \lambda \cdot X x \quad (\forall \lambda \in H, \forall x \in X). \quad (2.1)$$

To simplify notation, we will often use the symbol $\lambda x$ instead of $\lambda \cdot X x$.

The identity $\text{id}$ and the composition $\circ$ are defined as follows: for objects $X, Y, Z$ and morphisms $f : X \to Y, g : Y \to Z$,

$$\text{id}_X(\lambda)(x) = x \quad (\lambda \in H, x \in X); (g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) \quad (\lambda \in H).$$

This $\text{Set}_H$ is a tensor category: the tensor product $X \otimes Y$ of the objects $X = (X, \cdot X)$ and $Y = (Y, \cdot Y)$ is a pair $(X \times Y, \cdot)$ of the Cartesian product $X \times Y$ and the map $\cdot : H \times (X \times Y) \to H$ defined by

$$\lambda \cdot (x, y) = (\lambda \cdot X x) \cdot Y y \quad (\lambda \in H, (x, y) \in X \times Y); \quad (2.2)$$

the tensor product of the morphisms $f : X \to X'$ and $g : Y \to Y'$ is defined by $(f \otimes g)(\lambda)(x, y) = (f(\lambda)(x), g(\lambda x)(y)) \quad (\lambda \in H, (x, y) \in X \times Y)$.

The other ingredients of the tensor category $\text{Set}_H$ are: the associativity constraint $a_{XYZ}(\lambda)((x, y), z) = (x, (y, z))$; the unit $I = (\{e\}, \cdot I)$, a pair of the set $\{e\}$ of one element and the map $\cdot I$ defined by $\lambda \cdot I e = \lambda$; the left and the right unit constraints $l_X(\lambda)(e, x) = x = r_X(\lambda)(x, e)$.

In what follows, the associativity constraint will be omitted.
Definition 2.1. A morphism $\sigma : X \otimes X \to X \otimes X$ of $\text{Set}_H$ is a dynamical Yang-Baxter map \cite{14,15}, iff $\sigma$ satisfies the following braid relation in $\text{Set}_H$.

\[(\sigma \otimes \text{id}_X) \circ (\text{id}_X \otimes \sigma) \circ (\sigma \otimes \text{id}_X) = (\text{id}_X \otimes \sigma) \circ (\sigma \otimes \text{id}_X) \circ (\text{id}_X \otimes \sigma). \quad (2.3)\]

Remark 2.2. (1) If $H$ is a set of one element, the tensor category $\text{Set}_H$ is exactly the tensor category $\text{Set}$ consisting of nonempty sets, and the dynamical Yang-Baxter map is a Yang-Baxter map.

(2) The dynamical Yang-Baxter maps satisfying suitable conditions can produce bialgebroids, each of which gives birth to a tensor category of its dynamical representations \cite{16}. Note that the definition of the tensor product in \cite{16} is slightly different from that in this section.

3 Monoids in $\text{Set}_H$

In this section, we introduce the monoid in $\text{Set}_H$ (See \cite[VII.3]{13}).

Let $X$ be an object of the tensor category $\text{Set}_H$ and let $m_X : X \otimes X \to X$ and $\eta_X : I \to X$ be morphisms of $\text{Set}_H$.

Definition 3.1. The triple $(X, m_X, \eta_X)$ is a monoid, iff:

\[m_X \circ (m_X \otimes \text{id}_X) = m_X \circ (\text{id}_X \otimes m_X);\]

\[m_X \circ (\eta_X \otimes \text{id}_X) = l_X;\]

\[m_X \circ (\text{id} \otimes \eta_X) = r_X.\]

We explain a construction of the monoid in $\text{Set}_H$, which is due to Mitsuhito Takeuchi. Let $X$ be an object of $\text{Set}_H$. Suppose that

\[\forall \lambda, \lambda' \in H, \exists! x \in X \text{ such that } \lambda x = \lambda'. \quad (3.1)\]

We will denote by $\lambda \backslash \lambda'$ the unique element $x \in X$.

Proposition 3.2. $X$ satisfying (3.1) is a monoid, together with the morphisms $m_X$ and $\eta_X$:

\[m_X(\lambda)(x, y) = \lambda \backslash ((\lambda x)y); \eta_X(\lambda)(e) = \lambda \backslash \lambda \quad (\lambda \in H, x, y \in X, I = \{e\}).\]

Furthermore, this monoid structure is unique.

Proof. We give the proof only for the uniqueness of the morphism $m_X$. Suppose that $m_X : X \otimes X \to X$ is a morphism of $\text{Set}_H$. It follows from (2.1) and (2.2) that $\lambda m_X(\lambda)(x, y) = \lambda(x, y) = (\lambda x)y$ ($\lambda \in H, x, y \in X$). By taking (3.1) into account, $m_X(\lambda)(x, y)$ is uniquely determined. \hfill \Box

Example 3.3. The set $H$ with the map $\lambda \cdot_H \lambda' = \lambda'$ ($\lambda, \lambda' \in H$) is an object of $\text{Set}_H$, and obviously satisfies (3.1); hence, $H = (H, \cdot_H)$ is a monoid.
4 Dynamical braided monoids

After introducing dynamical braided monoids in $\text{Set}_H$, we show in this section that each dynamical braided monoid satisfying (3.1) gives birth to the dynamical Yang-Baxter map.

Let $(X, m_X, \eta_X)$ be a monoid in the tensor category $\text{Set}_H$. Suppose that a morphism $\sigma : X \otimes X \to X \otimes X$ of $\text{Set}_H$ satisfies:

\begin{align*}
(id_X \otimes m_X) \circ (\sigma \otimes id_X) \circ (id_X \otimes \sigma) &= \sigma \circ (m_X \otimes id_X); \\
(m_X \otimes id_X) \circ (id_X \otimes \sigma) \circ (\sigma \otimes id_X) &= \sigma \circ (id_X \otimes m_X); \\
(id_X \otimes m_X) \circ (\sigma \otimes id_X) \circ (\eta_X \otimes id_X) &= l_X \otimes x; \\
(m_X \otimes id_X) \circ (id_X \otimes \sigma) \circ (\eta_X \otimes id_X) &= r_X \otimes x.
\end{align*}

We define the morphisms $m_{X \otimes X} : (X \otimes X) \otimes (X \otimes X) \to X \otimes X$ and $\eta_{X \otimes X} : I \to X \otimes X$ by:

\begin{align*}
m_{X \otimes X} &= (m_X \otimes m_X) \circ (id_X \otimes \sigma \otimes id_X); \\
\eta_{X \otimes X} &= (\eta_X \otimes \eta_X) \circ l_I^{-1}.
\end{align*}

A straightforward computation shows

**Proposition 4.1.** $(X \otimes X, m_{X \otimes X}, \eta_{X \otimes X})$ is a monoid.

**Definition 4.2.** $(X, \sigma)$ is a dynamical braided monoid, iff the morphism $\sigma$ satisfies (4.1)-(4.4).

**Remark 4.3.** (1) By taking (1.7)-(1.10) into account, the conditions (4.1)-(4.4) correspond to (1) in Definition 1.2, while (2) in Definition 1.2 corresponds to (2.1) for the morphism $\sigma$. If the monoid $X$ satisfies (3.1), then $m_X(\lambda)(x, y) = \lambda \backslash ((\lambda x)y)$ ($\lambda \in H, x, y \in X$) because of Proposition 3.2, and (2.1) for the morphism $\sigma$ is equivalent to that $m_X \circ \sigma = m_X$, which is similar to (2) in Definition 1.2.

(2) Let $(X, m_X, \eta_X)$ and $(Y, m_Y, \eta_Y)$ be a monoid in the tensor category $\text{Set}_H$. Suppose that a morphism $\sigma : X \otimes Y \to Y \otimes X$ of $\text{Set}_H$ satisfies:

\begin{align*}
(id_Y \otimes m_X) \circ (\sigma \otimes id_X) \circ (id_X \otimes \sigma) &= \sigma \circ (m_X \otimes id_Y); \\
(m_Y \otimes id_X) \circ (id_Y \otimes \sigma) \circ (\sigma \otimes id_Y) &= \sigma \circ (id_X \otimes m_Y); \\
(id_Y \otimes m_X) \circ (\sigma \otimes id_X) \circ (\eta_X \otimes id_Y) &= l_Y \otimes x; \\
(m_Y \otimes id_X) \circ (id_Y \otimes \sigma) \circ (\eta_X \otimes id_Y) &= r_Y \otimes x.
\end{align*}

We define the morphisms $m_{Y \otimes X} : (Y \otimes X) \otimes (Y \otimes X) \to Y \otimes X$ and $\eta_{Y \otimes X} : I \to Y \otimes X$ by:

\begin{align*}
m_{Y \otimes X} &= (m_Y \otimes m_X) \circ (id_Y \otimes \sigma \otimes id_X); \\
\eta_{Y \otimes X} &= (\eta_Y \otimes \eta_X) \circ l_I^{-1}.
\end{align*}
Then \((Y \otimes X, m_{Y\otimes X}, \eta_{Y\otimes X})\) is a monoid, which is called a matched pair of monoids.

The following theorem is an analogue of Theorem 1.3.

**Theorem 4.4.** If a dynamical braided monoid \((X, \sigma)\) satisfies (3.1), then \(\sigma\) is a dynamical Yang-Baxter map (Definition 2.1).

We give a proof of this theorem in the next section.

5 Torsors (Principal homogeneous spaces)

This section is devoted to proving Theorem 4.4, in which the notion of a torsor [11, Section 4.2] plays an essential role.

**Definition 5.1.** A pair \((M, \mu)\) of a nonempty set \(M\) and a ternary operation \(\mu : M \times M \times M \to M\) is called a torsor, iff \(\mu\) satisfies:

\[
\begin{align*}
\mu(u, v, v) &= u = \mu(v, v, u); \\
\mu(\mu(u, v, w), x, y) &= \mu(u, v, \mu(w, x, y)) \quad (\forall u, v, w, x, y \in M).
\end{align*}
\]

**Remark 5.2.** (1) A Mal'cev operation is a ternary operation satisfying (5.1) [9, Section 1]; moreover, an associative Mal'cev operation is a ternary operation satisfying (5.1) and (5.2). The torsor is also called a herd, a Schar (in German), a flock, and a heap [17, Section 1].

(2) For a pair \((M, \mu)\), the following conditions are equivalent (cf. [6, Section 2.1]):

(a) (5.1) and (5.2);

(b) (5.1) and (5.3).

\[
\begin{align*}
\mu(\mu(u, v, w), x, y) &= \mu(u, \mu(x, w, v), y) = \mu(u, v, \mu(w, x, y)) \\
&= \mu(u, v, \mu(w, x, \mu(w, x, v))) \\
&= \mu(u, v, \mu(w, x, v), y) \\
&= \mu(u, \mu(x, w, v), y).
\end{align*}
\]

In fact, (5.1) and (5.2) induce (5.3), because

\[
\begin{align*}
\mu(u, v, \mu(w, x, y)) &= \mu(u, v, \mu(w, x, \mu(w, x, v))) \\
&= \mu(u, v, \mu(w, x, \mu(w, x, v), \mu(x, w, v), y)) \\
&= \mu(u, v, \mu(w, x, v), y) \\
&= \mu(u, \mu(x, w, v), y).
\end{align*}
\]

Thus, a pair \((M, \mu)\) satisfying (5.1) and (5.3) is exactly a torsor.
The torsor $(M, \mu)$ is a principal homogeneous space \[11, \text{Section 4.2}].

Let $\mu(a, b) (a, b \in M)$ denote the map from $M$ to itself defined by $\mu(a, b)(c) = \mu(a, b, c)$ ($c \in M$). The set $G = \{\mu(a, b); a, b \in M\}$ is a subgroup of $\text{Aut}(M)$, which makes $M$ a $G$-principal homogeneous space. Conversely, the principal homogeneous space gives birth to a torsor.

Each group $G$ produces a torsor. Define the ternary operation $\mu_G$ on $G$ by
\[\mu_G(a, b, c) = ab^{-1}c \quad (a, b, c \in G).\] (5.4)

The pair $(G, \mu)$ is a torsor.

**Remark 5.3.** Every torsor $(M, \mu)$ is isomorphic to (5.4) \[17, \text{Section 1.6}]. We first fix any element $e \in M$. The nonempty set $M$, together with the binary operation
\[M \times M \ni (a, b) \mapsto \mu(a, e, b) \in M,\]
is a group \[9, \text{Section 1}]; in fact, the unit element is $e$, and the inverse of the element $a$ is $\mu(e, a, e)$. This group $M$ gives birth to the torsor (5.4), which is isomorphic to $(M, \mu)$.

Let $H = (H, \cdot H)$ denote the object of the category $\text{Set}_H$ in Example 3.3. Here, $\lambda \cdot_H \lambda' = \lambda'$ ($\lambda, \lambda' \in H$). Suppose that an object $X$ of $\text{Set}_H$ satisfies (3.1). We define the map $i : H \rightarrow \text{Map}(H, X)$ by
\[i(\lambda)(u) = \lambda \backslash u \quad (\lambda, u \in H).\]

**Proposition 5.4.** The map $i$ is an isomorphism of $\text{Set}_H$ from $H$ to $X$.

In fact, its inverse is as follows.
\[i^{-1}(\lambda)(x) = \lambda x \quad (\lambda \in H, x \in X).\]

Let $\sigma : X \otimes X \rightarrow X \otimes X$ be a morphism of $\text{Set}_H$. By virtue of (2.1) for the morphism $i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i : H \otimes H \rightarrow H \otimes H$,

**Proposition 5.5.** The second component of $(i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i)(\lambda)(u, v)$ ($\lambda, u, v \in H$) is $v$.

We define the ternary operation $\mu$ on the set $H$ by the first component of $(i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i)(\lambda)(u, v)$; that is,
\[(i^{-1} \otimes i^{-1} \circ \sigma \circ i \otimes i)(\lambda)(u, v) = (\mu(\lambda, u, v), v) \quad (\lambda, u, v \in H).\]
Proposition 5.6. \((H, \mu)\) is a torsor, if and only if \((X, \sigma)\) is a dynamical braided monoid.

Proof. We first observe (4.1) is equivalent to that
\[
\mu(u, v, \mu(v, w, x)) = \mu(u, w, x) \quad (\forall u, v, w, x \in H). \tag{5.5}
\]
On account of Proposition 5.4, the morphism \(\sigma\) satisfies (4.1), if and only if
\[
(id_H \otimes (i^{-1} \circ m_X \circ i \otimes i)) \circ ((i^{-1} \otimes i^{-1} \circ i \otimes i) \otimes id_H)
\circ(id_H \otimes (i^{-1} \otimes i^{-1} \circ i \otimes i))
\circ((i^{-1} \circ m_X \circ i \otimes i) \otimes id_H).
\tag{5.6}
\]
Because \((i^{-1} \circ m_X \circ i \otimes i)(\lambda)(u, v) = v \quad (\lambda, u, v \in H), \tag{5.5}\) and \(\tag{5.6}\) are equivalent.

Similar argument implies to: (4.2) is equivalent to that
\[
\mu(\mu(u, v, w), w, x) = \mu(u, v, x) \quad (\forall u, v, w, x \in H); \tag{5.7}
\]
(4.3) is equivalent to that \(\mu(v, v, u) = u \quad (\forall u, v \in H);\) and (4.4) is equivalent to that \(\mu(u, v, v) = u \quad (\forall u, v \in H).\)

An easy computation shows that (5.2) is equivalent to (5.5) and (5.7), if \(\mu\) satisfies (5.1); in fact, (5.5) and (5.7) induce (5.2), because
\[
\mu(\mu(u, v, w), x, y) = \mu(\mu(u, v, w), w, \mu(w, x, y)) = \mu(u, v, \mu(w, x, y)).
\]

Hence, \((H, \mu)\) is a torsor, if and only if \((X, \sigma)\) is a dynamical braided monoid. \(\Box\)

Proof of Theorem 4.4. Let \((X, \sigma)\) be a dynamical braided monoid satisfying (3.1). From (3.1) and Proposition 5.6, \((H, \mu)\) is a torsor. If \((H, \mu)\) is a torsor, then the morphism \((i^{-1} \otimes i^{-1}) \circ \sigma \circ (i \otimes i) : H \otimes H \to H \otimes H\) satisfies the braid relation (2.3), and so does the morphism \(\sigma\). Thus, \(\sigma\) is a dynamical Yang-Baxter map (Definition 2.1). \(\Box\)

Acknowledgments

The author wishes to express his thanks to the organizers of the Conference on Quantum Groups and Quantum Topology for the invitation and hospitality.
References


http://homepages.vub.ac.be/~scaenepe/proceedingsnomap.htm


