Fourier expansion of Arakawa lifting and central L-values

(Automorphic forms, automorphic representations and related topics)

Author(s)
Murase, Atsushi; Narita, Hiro-aki

Citation
数理解析研究所講究録 (2010), 1715: 143-158

Issue Date
2010-10

URL
http://hdl.handle.net/2433/170292

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
Fourier expansion of Arakawa lifting and central L-values

Atsushi Murase* and Hiro-aki Narita†

Abstract

This note is a write-up of our talk at the RIMS-conference on automorphic forms held at the University of Tokyo in January 2010. Our results deal with the Fourier coefficients of Arakawa lifts and central values of some automorphic L-functions. In our previous paper [M-N-2] we provide an explicit formula for the Fourier coefficients in terms of toral integrals of some automorphic forms with respect to Hecke characters of imaginary quadratic fields. After studying explicit relations between the toral integrals and the central L-values, we explicitly determine the constant of proportionality relating the square norm of a Fourier coefficient of an Arakawa lift with the central L-value. In some case we can relate such square norm with the central value of some degree eight L-function of convolution type attached to the lift and the character. We also discuss the existence of strictly positive central values of the L-functions in our concern.

1 Reviews on Arakawa lifting and its Fourier expansion.

1.1

In this section we review our results in [M-N-2]. Let $B$ be a definite quaternion algebra over $\mathbb{Q}$ and denote by $n$ its reduced norm. Let $G = GSp(1,1)$ be the $\mathbb{Q}$-algebraic group defined by

$$G_{\mathbb{Q}} := \{ g \in M_{2}(B) \mid ^{t}\overline{g}Qg = \nu(g)Q, \, \nu(g) \in \mathbb{Q}^\times \},$$

where $Q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By $Z_{G}$ we denote the center of $G$.

We put $G_{\infty}^{1} := \{ g \in M_{2}(\mathbb{H}) \mid ^{t}\overline{g}Qg = Q \},$ where $\mathbb{H} := B \otimes_{\mathbb{Q}} \mathbb{R}$ is the Hamilton quaternion algebra. Then

$$K_{\infty} := \{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a \pm b \in \mathbb{H}^{1} \}$$

forms a maximal compact subgroup of $G_{\infty}^{1}$, where $\mathbb{H}^{1} := \{ u \in \mathbb{H} \mid n(u) = 1 \}$. For a non-negative integer $\kappa$ we let $(\sigma_{\kappa}', V_{\kappa})$ be the $\kappa$-th symmetric tensor representation of $GL_{2}(\mathbb{C})$ and $\sigma_{\kappa}$ the pull-back of $\sigma_{\kappa}'$ to $\mathbb{H}^{\times}$ via the standard embedding $\mathbb{H} \subset M_{2}(\mathbb{C})$ (cf. [M-N-2, (1.4)]). This induces an irreducible representation $(\tau_{\kappa}, V_{\kappa})$ of $K$ by

$$\tau_{\kappa}(\begin{pmatrix} a & b \\ b & a \end{pmatrix}) := \sigma_{\kappa}(a + b), \quad (\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in K_{\infty}).$$

*Partially supported by Grand-in-Aid for Scientific Research C-20540031, JSPS
†Partially supported by Grand-in-Aid for Young Scientists (B) 21740025, JSPS
Let $H$ and $H'$ be $\mathbb{Q}$-algebraic groups defined by

$$H_{\mathbb{Q}} = GL_2(\mathbb{Q}), \quad H_{\mathbb{Q}}' := B^x$$

respectively.

Fix a maximal order $\mathfrak{O}$ of $B$. We denote by $d_B$ the discriminant of $B$ and fix a divisor $D$ of $d_B$. For $p|d_B$ let $\mathfrak{P}_p$ be the maximal ideal of the $p$-adic completion $\mathfrak{O}_p$ of $\mathfrak{O}$ and let

$$L_p := \begin{cases} t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \not| d_B \text{ or } p|D), \\ t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|D). \end{cases}$$

We put

$$K_p := \{k \in G_p | kL_p = L_p\}$$

for each finite prime $p$ and

$$K_f := \prod_{p<\infty} K_p.$$

It is known that, up to $G_p$-conjugate, the two groups $K_p$ for the two $L_p$'s exhaust maximal compact subgroups of $G_p$. For $\kappa > 4$ we then introduce the space $S_{\kappa}$ of $V_{\kappa}$-valued cusp forms $F$ on $G_{A_{\mathbb{Q}}}$ satisfying the following:

1. $F(z \gamma g k_f k_{\infty}) = \tau_\kappa(k_{\infty})^{-1} F(g)$ for $(z, \gamma, g, k_f, k_{\infty}) \in Z_{G,A_{\mathbb{Q}}} \times G_{\mathbb{Q}} \times G_{A_{\mathbb{Q}}} \times K_f \times K_{\infty}$,

2. for each fixed $g_f \in G_{A_f}$, the right translations of the coefficients of $F|_{G_{\infty}^1}(g_{f}^*)$ by $g_{\infty} \in G_{\infty}^1$ generate, as a $(g, K_{\infty})$-module, quaternionic discrete series with the minimal $K_{\infty}$-type $\tau_{\kappa}$ (cf. [Gr-W]). Here $g$ denotes the Lie algebra of $G_1$. For a positive integer $\kappa$ we let $S_{\kappa}(D)$ be the space of elliptic cusp forms of weight $\kappa$ with level $D$ (cf. [M-N-2, §3.1]) and $A_{\kappa}$ be the space of automorphic forms of weight $\sigma_{\kappa}$ with respect to $\prod_{v<\infty} \mathfrak{D}_v^{\times}$ (cf. [M-N-2, §3.2]), where $\mathfrak{D}_v^{\times}$ denotes the multiplicative group of $\mathfrak{O}_v$. Now we can review the definition of Arakawa lifting. By a metaplectic representation of $GSp(1,1)_{A_{\mathbb{Q}}} \times H_{A_{\mathbb{Q}}} \times H_{A_{\mathbb{Q}}}'$, we define the End($V_{\kappa}$)-valued theta function $\theta_{\kappa}(g, h, h')$ with some specified End($V_{\kappa}$)-valued Schwartz-Bruhat function on $B_{A_{\mathbb{Q}}}^{\oplus 2} \times A_{\mathbb{Q}}^{\times}$ (cf. [M-N-1, §3]). Then, for $\kappa > 4$, we define the Arakawa lifting by

$$S_{\kappa}(D) \times A_{\kappa} \ni (f, f') \mapsto \mathcal{L}(f, f')(g) \in S_{\kappa}$$

with

$$\mathcal{L}(f, f')(g) := \int_{B_{A_{\mathbb{Q}}}^\times \backslash (H \times H')_{A_{\mathbb{Q}}}} \overline{f(h)} \theta_{\kappa}(g, h, h') f'(h') dh dh'.$$

1.2

We now review the Fourier expansion of $\mathcal{L}(f, f')$ described in [M-N-2, §1.3]. We let $B^- := \{x \in B | tr(x) = 0\}$ and have

$$\mathcal{L}(f, f')(g) = \sum_{\xi \in B^- \backslash \{0\}} \mathcal{L}(f, f')_{\xi}(g),$$

where

$$\mathcal{L}(f, f')_{\xi}(g) := \int_{B^- \backslash B_{A_{\mathbb{Q}}}} \mathcal{L}(f, f')(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g) \psi(-tr(\xi x)) dx.$$
with the standard additive character $\psi$ on $\mathbb{Q}\backslash A$. Here we normalize the measure $dx$ so that the volume of $B\backslash B_{\mathbb{A}}$ is one. For $\xi \in B_{-}\backslash \{0\}$ we let $E_{\xi} := \mathbb{Q}(\xi)$, which is isomorphic to an imaginary quadratic field, and $X_{\xi}$ be the set of unitary characters on $A_{\mathbb{Q}}^{\times}E_{\xi}^{\times}\backslash A_{E_{\xi}}^{\times}$. The Fourier expansion is then refined as follows:

$$\mathcal{L}(f, f')(g) = \sum_{\xi \in B_{-}\backslash \{0\}} \sum_{\chi \in X_{\xi}} \mathcal{L}(f, f')_{\xi}(g),$$

with

$$\mathcal{L}(f, f')_{\xi}(g) := \text{vol}(\mathbb{R}_{+}^{\times}A_{\mathbb{Q}}^{\times}\backslash A_{E_{\xi}}^{\times} \mathbb{Q}_{\mathbb{A}}^{\times} \backslash A_{E_{\xi}}^{\times})^{-1} \int_{\mathbb{R}_{+}^{\times}A_{E_{\xi}}^{\times} \backslash A_{E_{\xi}}^{\times}} \mathcal{L}(f, f')(s1_{2}\cdot g)\chi(s)^{-1} ds.$$

1.3

To review our explicit formula for $\mathcal{L}(f, f')_{\xi}$, we let $(f, f') \in S_{\kappa}(D) \times A_{\kappa}$ and assume the following two:

(1) The two forms $f$ and $f'$ are Hecke eigenforms and have the same eigenvalue for the "Atkin-Lehner involution". More precisely, for each $p | D$, let $\epsilon_{p}$ (resp. $\epsilon'_{p}$) be the eigenvalue for the involutive action of $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ (resp. a prime element $\varpi_{B, p} \in B_{p}$) on $f$ (resp. $f'$). Then

$$\epsilon_{p} = \epsilon'_{p}.$$

Otherwise $\mathcal{L}(f, f') \equiv 0$ (cf. [M-N-1, Remark 5.2 (ii)]).

(2) We assume that $\xi \in B_{-}\backslash \{0\}$ is primitive. Namely, letting $a_{p} := \begin{cases} \mathcal{D}_{p} & (p \not| D) \\ \mathfrak{P}_{p} & (p | D) \end{cases}$ for each finite prime $p$,

$$\xi \in a_{p} \backslash pa_{p}.$$

For the second assumption we note that, in general, a Fourier coefficient $F_{\xi}$ of an automorphic form $F$ on $G_{\mathbb{A}}$ satisfies

$$F_{\xi}(g) = F_{\xi}(t \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} g) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^{\times}).$$

We then see that the problem determining $F_{\xi}$ is reduced to the case where $\xi$ is primitive.

1.4

For each $\xi \in B_{-}\backslash \{0\}$, $d_{\xi}$ denotes the discriminant of $E := E_{\xi}$. We put

$$a := \begin{cases} 2\sqrt{-n(\xi)}\sqrt{d_{\xi}} & (d_{\xi} \text{ is odd}) \\ \sqrt{n(\xi)}\sqrt{d_{\xi}} & (d_{\xi} \text{ is even}) \end{cases}, \quad b := \xi^{2} - \frac{a^{2}}{4}.$$

With these $a$ and $b$ we define $\iota_{\xi} : E^{\times} \rightarrow GL_{2}(\mathbb{Q})$ by

$$\iota_{\xi}(x + y\xi) = x \cdot 1_{2} + y \cdot \begin{pmatrix} \frac{a}{2} & b \\ 1 & -\frac{a}{2} \end{pmatrix} (x, y \in \mathbb{Q}).$$
Put $\theta := r^{-1}(\xi - \frac{a}{2})$ with $r = \frac{2\sqrt{n(\xi)}}{\sqrt{d_{\xi}}} \in \mathbb{Q}^\times$. Then $\{1, \theta\}$ forms a $\mathbb{Z}$-basis of the integer ring $\mathcal{O}_E$ of $E$. We can rewrite $\iota_\xi$ as

$$\iota_\xi(x + y\theta) = \begin{pmatrix} x & -r\text{N}\mathbb{E}/\mathbb{Q}(\theta)y \\ y & x + \text{Tr}\mathbb{E}/\mathbb{Q}(\theta)y \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

The completion $E_\infty$ of $E$ at $\infty$ is identified with $\mathbb{C}$ by

$$\delta_{\xi} : E_\infty \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} (x, y \in \mathbb{R}).$$

For a Hecke character $\chi = \prod_{v \leq \infty} \chi_v$ of $\mathbb{R}_+^\times \mathbb{E}^\times \backslash \mathbb{A}_E^\times$, we let $w_\infty(\chi) \in \mathbb{Z}$ be such that

$$\chi_\infty(u) = (\delta_{\xi}(u)/|\delta_{\xi}(u)|)^{w_\infty(\chi)} (u \in E_\infty).$$

Furthermore, for each prime $v = p < \infty$, we let $p^i_p(\chi)$ be the conductor of $\chi$ at $p$ and

$$\mu_p := \frac{\text{ord}_p(2\xi)^2 - \text{ord}_p(d_{\xi})}{2},$$

which coincides with $\text{ord}_p(r)$. We now state the following (cf. [M-N-2, Theorem 5.1.1]):

**Proposition 1.1.** $\mathcal{L}(f, f', \xi)^{\chi} \equiv 0$ unless $i_p(\chi) = 0$ for any $p|d_B$ and $w_\infty(\chi) = -\kappa$.

1.5

In what follows, we assume that $\chi$ satisfies the assumption in the proposition above. We need further notations to recall our formula for $\mathcal{L}(f, f', \xi)$. We define $\gamma_0 = (\gamma_{0,p})_{p \leq \infty} \in H_{\mathbb{A}_E}$ and $\gamma'_0 = (\gamma_{0,p}')_{p < \infty} \in H_{\mathbb{A}_f}'$ as follows:

$$\gamma_{0,p} := \begin{cases} 1 & (p \parallel d_B), \\ 2 & (p|d_B \text{ and } p \text{ is inert in } E), \\ 0 & (p|D \text{ and } p \text{ ramifies in } E), \\ n_H(a/2)d_H(N(\xi)^{1/4}) & (p = \infty), \end{cases}$$

$$\gamma_{0,p}' := \begin{cases} 1 & (p \mid d_B), \\ 2 \pi_{B,p}^{-1} & (p|d_B \text{ and } p \text{ is inert in } E), \\ (p+1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E). \end{cases}$$

Here recall that $\pi_{B,p}$ denotes a prime element of $B_p$ (cf. §1.3).

In addition, we introduce the following local constants:

$$C_p(f, \xi, \chi) := \begin{cases} p^{2\mu_p - i_p(\chi)}(1 - \delta(i_p(\chi) > 0)e_p(E)p^{-1}) & (p \not| d_B), \\ 1 & (p|d_B \text{ and } p \text{ is inert in } E), \\ 2\pi_p & (p|D \text{ and } p \text{ ramifies in } E), \\ (p+1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E). \end{cases}$$
where

$$e_p(E) = \begin{cases} 
-1 & (p \text{ is inert in } E), \\
0 & (p \text{ ramifies in } E), \\
1 & (p \text{ splits in } E). 
\end{cases}$$

For \((f, f') \in S_\kappa(D) \times A_\kappa\) we introduce their toral integrals with respect to \(\chi\) (cf. [Wa]).

$$P_\chi(f; h) := \int_{\mathbb{R}_+ \backslash A_\xi^\times} f(s) \chi(s)^{-1} ds,$$

$$P_\chi(f'; h') := \int_{\mathbb{R}_+ \backslash A_\xi^\times} f(s) \chi(s)^{-1} ds,$$

where \((h, h') \in GL_2(A_{\mathbb{Q}}) \times B_{A_{\mathbb{Q}}}^\times\).

As in [M-N-2, §2.4] we normalize the measure \(ds\) of \(A_\xi^\times\) so that

$$\text{vol}(\mathcal{O}_{E,p}^\times) = 1$$

for any \(p < \infty\),

$$\text{vol}(E_\infty^\times) = 1.$$

We denote by \(h(E)\) and \(w(E)\) the class number of \(E\) and the number of the roots of unity in \(E\) respectively. Then we are able to state our formula for \(\mathcal{L}(f, f')_{\xi}^\chi\) (cf. [M-N-2, Theorem 5.2.1]).

**Theorem 1.2.** Let \((f, f')\) be Hecke eigenforms and \(\xi \in B^- \backslash \{0\}\) be primitive. Suppose that \(\chi\) satisfies the assumption in Proposition 1.1 and that (1) and (2) in §1.3 hold. We then have the following formula:

$$\mathcal{L}(f, f')_{\xi}^\chi(g_0, d_G(\sqrt{\eta_\infty})) = 2^{\kappa-1}n(\xi)^{\kappa/4} \frac{w(E)}{h(E)} \cdot \frac{w(E)}{h(E)} \cdot \prod_{p < \infty} P_\chi(f; \gamma_0) P_\chi(f'; \gamma_0).$$

Here \(\eta_\infty \in \mathbb{R}_+^\kappa\) and \(g_0, f = (g_0, p)_{p < \infty} \in G_{\infty}\) is given by

$$g_0, f := \begin{cases} 
\text{diag}(p^{i_p(\chi) - \mu_p}, p^{2(i_p(\chi) - \mu_p)}, 1, p^{i_p(\chi) - \mu_p}) & (p \not| d_B), \\
1_2 & (p|d_B) 
\end{cases}$$

**Remark 1.3.** According to Sugano [Su, Theorem 2-1], the Fourier coefficient \(\mathcal{L}(f, f')_{\xi}^\chi\) is determined by the evaluation at \(g_0, f (\sqrt{\eta_\infty}, 0)\).

## 2 Relation with central \(L\)-values.

### 2.1

Let \((f, f')\) be Hecke eigenforms and assume that \(f\) is a primitive form (for the definition, see [Mi, §4.6]). Let \(\pi(f)\) (resp. \(\pi(f')\)) be the irreducible automorphic representation generated by \(f\) (resp. \(f')\), and let \(\pi(JL(f'))\) be the irreducible automorphic representation generated by the Jacquet-Langlands lift \(JL(f')\) of \(f'\). It is known that \(\pi(f)\) and \(\pi(JL(f'))\) (resp. \(\pi(f')\)) decompose into restricted tensor products over \(v \leq \infty\) of irreducible admissible representations of \(GL_2(\mathbb{Q}_v)\) (resp. \(B_v^\kappa\)). By \(\pi_v\), \(\pi'_v\) and \(\pi''_v\) we denote the \(v\)-component of \(\pi(f)\),
\(\pi(JL(f'))\) and \(\pi(f')\) respectively. According to such decomposition of \(\pi(f)\) and \(\pi(f')\), \(f\) and \(f'\) admit decompositions into pure tensor products

\[
\rho(f) = \prod_{v \leq \infty} f_v, \quad \rho'(f') = \prod_{v \leq \infty} f'_v,
\]

where we fix an isomorphism \(\rho\) (resp. \(\rho'\)) between \(\pi(f)\) and \(\otimes'_{v \leq \infty} \pi_v\) (resp. \(\pi(f')\) and \(\otimes'_{v \leq \infty} \pi'_v\)). We denote by \(\Pi\) (resp. \(\Pi'\)) the quadratic base change of \(\pi(f)\) (resp. \(\pi(JL(f'))\)) to \(GL_2(A_E)\). These \(\Pi\) and \(\Pi'\) also decompose into the restricted tensor products \(\otimes'_{v \leq \infty} \Pi_v\) and \(\otimes'_{v \leq \infty} \Pi'_v\), respectively, where each \(\Pi_v\) or \(\Pi'_v\) is a local base change lift of \(\pi_v\) or \(\pi'_v\) at every place \(v\) respectively.

### 2.2 Review on the adjoint L-functions and the L-functions of base change lifts for \(GL_2\).

Let \(L(\pi, s)\) be the standard \(L\)-function for an automorphic representation \(\pi\) of \(GL_2(A_{\mathbb{Q}})\) in the sense of Jacquet-Langlands [J-L]. We denote by \(L(\Pi, \chi^{-1}, s)\) (resp. \(L(\Pi', \chi^{-1}, s)\)) the \(L\)-function of \(\Pi\) (resp. \(\Pi'\)) with \(\chi^{-1}\)-twist, and let \(L(\pi(f), \text{Ad}, s)\) (resp. \(L(\pi(JL(f')), \text{Ad}, s)\)) be the adjoint \(L\)-function of \(\pi(f)\) (resp. \(\pi(JL(f'))\)).

We describe the local factors of \(L(\Pi, \chi^{-1}, s)\) and \(L(\Pi', \chi^{-1}, s)\) (resp. \(L(\pi(f), \text{Ad}, s)\) and \(L(\pi(JL(f')), \text{Ad}, s)\)), following Jacquet [J] (resp. Gelbart-Jacquet [G-J]). We note that \(\pi_p\) (resp. \(\pi'_p \simeq \pi''_p\)) is a unitary unramified principal series representation for each finite prime \(p\) (resp. \(p \mid d_B\)). This is due to the Ramanujan conjecture for holomorphic cusp forms on \(GL_2\). In addition, we remark that, for \(p \mid d_B\), \(\pi''_p\) is written as

\[
B_p^\times \ni b \mapsto \delta_p \cdot n(b) \in \{\pm 1\},
\]

with a character \(\delta_p\) of \(\mathbb{Q}_p^\times\) of order two. For \(p \mid d_B\), \(\pi'_p\) is thus the special representation of \(GL_2(Q_p)\) given by the irreducible subquotient of the induced representation \(\text{Ind}(\delta_p \cdot \cdot \cdot |^* \cdot \frac{1}{p} \cdot \delta \cdot \cdot \cdot |_{p}^{\frac{1}{2}})\) (cf. [J-L, Theorem 4.2 (iii)]), where \(| \cdot |_p\) is the \(p\)-adic absolute value. We furthermore note that, when \(p\) is inert or ramified in \(E\) and \(\chi_p\) is unramified, \(\chi_p\) can be written as

\[
\chi_p = \omega_p \cdot n_{E_p/Q_p}
\]

with a character \(\omega_p\) of \(\mathbb{Q}_p^\times\) of order two and the norm \(n_{E_p/Q_p}\) of \(E_p\). In fact, \(\omega_p \equiv 1\) when \(p\) is inert. In addition, at the Archimedean place, \(\pi_\infty\) and \(\pi''_\infty\) are the discrete series representations with weight \(\kappa\) and \(\kappa + 2\) respectively (see [Sh, §6]).

We let \(\pi_{ur}\) be a unitary unramified principal series representation of \(GL_2(Q_p)\) with Satake parameter \((\alpha_p, \alpha_p^{-1})\) and the trivial central character, and let \(\pi_{sp}\) be the special representation of \(GL_2(Q_p)\) corresponding to \(\pi''_p\). We denote by \(\Pi_{ur}\) (resp. \(\Pi_{sp}\)) the base change lift of \(\pi_{ur}\) (resp. \(\pi_{sp}\)) to \(GL_2(E_p)\).

We first deal with the local \(L\)-functions of \(\pi_{ur}\) and \(\Pi_{ur}\).

**Lemma 2.1.** (1)

\[
L_p(\pi_{ur}, s) = (1 - \alpha_p p^{-s})^{-1}(1 - \alpha_p^{-1} p^{-1})^{-1}.
\]
\begin{align}
L_p(\pi_{ur}, \text{Ad}, s) &= (1 - p^{-s})^{-1}(1 - \alpha_p^2 p^{-s})^{-1}(1 - \alpha_p^{-2} p^{-s})^{-1}, \\
L_p(\Pi_{ur}, \chi_p^{-1}, s) &= \begin{cases}
(1 - \alpha_p \chi_p(\varpi_p)\iota)^{-1} p^{-s})^{-1} (1 - \alpha_p^{-1} \chi_p(\varpi_p)^{-1} p^{-s})^{-1} & (p:\text{split}, \ i_p(\chi) = 0), \\
(1 - \alpha_p p^{-2s})^{-1} (1 - \alpha_p^{-2} p^{-2s})^{-1} & (p:\text{inert}, \ i_p(\chi) = 0), \\
(1 - \alpha_p \chi_p(\varpi_p)^{-1} p^{-s})^{-1} (1 - \alpha_p^{-1} \chi_p(\varpi_p)^{-1} p^{-s})^{-1} & (p:\text{ramified}, \ i_p(\chi) = 0), \\
1 & (i_p(\chi) > 0),
\end{cases}
\end{align}

where \(\varpi_{p,i} \in E_p\) with \(i = 1, 2\) (resp. \(\varpi_p \in E_p\)) denote two distinct prime elements dividing \(p\) (a prime element dividing \(p\)) when \(p\) is split (resp. \(p\) is ramified).

We next deal with the case of \(\pi_{sp}\) and \(\Pi_{sp}\).

**Lemma 2.2.** We have
\begin{align}
L_p(\pi_{sp}, s) &= (1 - \delta_p(p)p^{-(s+\frac{1}{2})})^{-1}, \\
L_p(\pi_{sp}, \text{Ad}, s) &= (1 - p^{-(s+1)})^{-1}, \\
L_p(\Pi_{sp}, \chi_p^{-1}, s) &= \begin{cases}
(1 - p^{-(2s+1)})^{-1} & (p:\text{inert}), \\
(1 - \delta_p(p)\omega_p(p)p^{-(s+\frac{1}{2})})^{-1} & (p:\text{ramified}).
\end{cases}
\end{align}

For a positive integer \(k \geq 2\) we let \(\pi_k\) be the discrete series representation with weight \(k\) and \(\Pi_k\) denote its base change. We give their Archimedean local \(L\)-functions as follows.

**Lemma 2.3.** We have
\begin{align}
L_\infty(\pi_k, s) &= \Gamma_\mathbb{C}(s + \frac{k-1}{2}), \\
L_\infty(\pi_k, \text{Ad}, s) &= \Gamma_R(s + 1)\Gamma_\mathbb{C}(s + k - 1), \\
L_\infty(\Pi_k, \chi_l, s) &= \begin{cases}
\Gamma_c(s + \frac{k-1}{2} + |l|)\Gamma_c(s - \frac{k-1}{2} + |l|) & (|l| \geq \frac{k-1}{2}) \\
\Gamma_c(s + \frac{k-1}{2} + |l|)\Gamma_c(s + \frac{k-1}{2} - |l|) & (|l| \leq \frac{k-1}{2})
\end{cases},
\end{align}

where \(l \in \frac{1}{2}\mathbb{Z}\).

### 2.3 Relation between \(P_\chi(f; \gamma_0)\) and \(L(\Pi, \chi^{-1}, \frac{1}{2})\).

By \(\eta\) we denote the quadratic character attached to the quadratic extension \(E/\mathbb{Q}\). We let \(L(\eta, s)\) be the \(L\)-function defined by \(\eta\) and \(L_v(\eta_0, s)\) the local factor of \(L(\eta, s)\) at a place \(v\).

Let \(T_p^\pm f = \lambda_p^\pm f\) for \(p|D\) (for the definition of \(T_p^\pm\), see [Mu, 2.4]) and let
\begin{align}
S_1 &= \{p < \infty \mid p|D, \ p \text{ is inert in } E\}, \\
S_2^\pm &= \{p < \infty \mid p|D, \ p \text{ ramifies in } E, \ \chi_p(\varpi_p)\lambda_p^\pm = \pm 1\},
\end{align}
where $\varpi_{p}$ denotes a prime element of $E_{p}$ (cf. Lemma 2.1). Note that $S_{1} \cup S_{2}^{+} \cup S_{2}^{-}$ coincides with $\delta(D) := \{ p < \infty \mid p \mid D \}$. We furthermore put $A(\chi) := \prod_{p<\infty}p^{i_{p}(\chi)}$.

Let us introduce the Petersson norm

$$\langle f, f \rangle := \int_{Z(\mathbb{A}_{\mathbb{Q}})GL_{2}(\mathbb{Q}) \backslash GL_{2}(\mathbb{A}_{\mathbb{Q}})} |f(g)|^{2} dg$$

of $f$, where $Z$ denotes the center of $GL_{2}$. Keeping the assumption on $\chi$ in Proposition 1.1 we can quote [Mu, Theorem 1.1] with a modification.

**Proposition 2.4.** We have

$$\frac{|P_{X}(f; \gamma_{0})|^{2}}{\langle f, f \rangle} = C(f, \chi)L(\Pi, \chi^{-1}, \frac{1}{2}),$$

with

$$C(f, \chi) := \begin{cases} \frac{2^{k(D)} |d_{\xi}| \prod_{p \mid A(\chi)} L_{p}(\eta_{p}, 1)^{2}}{4D^{2} A(\chi) L(\pi(f), Ad, 1)} (S_{1} = S_{2}^{+} = \emptyset), \\ 0 \quad \text{(otherwise)} \end{cases}$$

for $\pi(\Pi, \chi^{-1}, \frac{1}{2})$.

### 2.4 Relation between $P_{X}(f'; \gamma_{0}')$ and $L(\Pi', \chi^{-1}, \frac{1}{2})$

Next we provide an explicit relation between $|P_{X}(f'; \gamma_{0}')|^{2}$ and the central $L$-value $L(\Pi', \chi^{-1}, \frac{1}{2})$, which we prove in §3. To write down the relation we need several notations. We denote by $r_{p}$ the ramification index of $p$ for the quadratic extension $E/\mathbb{Q}$, i.e.

$$r_{p} := \begin{cases} 1 \quad (p: \text{non-ramified}) \\ 2 \quad (p: \text{ramified}) \end{cases}$$

We further put $A(\chi) := \prod_{p < \infty} p^{i_{p}(\chi)}$.

We set

$$\langle f', f' \rangle := \int_{Z'(\mathbb{A}_{\mathbb{Q}})B^{x} \backslash B_{\mathbb{A}_{\mathbb{Q}}}^{x}} (f'(b), f'(b))_{\kappa} db,$$

where $Z'$ denotes the center of $B^{x}$ and $(\ast, \ast)_{\kappa}$ stands for an inner product of $V_{\kappa}$.

By $\langle \ast, \ast \rangle_{\infty}$ we denote an inner product of $\pi_{\infty}''$. As this inner product we can take

$$\langle h_{\infty}, h_{\infty}' \rangle_{\infty} := \int_{\mathbb{H}^{1}} (h_{\infty}(u), h_{\infty}'(u))_{\kappa} du \quad (h_{\infty}, h_{\infty}' \in \pi_{\infty}'' \simeq V_{\kappa}),$$

where $du$ denotes the invariant measure of $\mathbb{H}^{1}$ normalized so that $\text{vol}(\mathbb{H}^{1}) = 1$. Let $v_{\kappa, \xi}$ be a highest weight vector of $V_{\kappa}$ with respect to $\sigma_{\kappa}(\mathbb{R}(\xi)^{\times})$-action, and let $v_{\kappa, \xi}' \in V_{\kappa}$ be the dual of $v_{\kappa, \xi}$ with respect to $(\ast, \ast)_{\infty}$. We set $f'_{\infty, \kappa}(b_{\infty}) := (f'_{\infty}(b_{\infty}), v_{\kappa, \xi}')_{\kappa} v_{\kappa, \xi}$ for $b_{\infty} \in B_{\infty}^{x}$. The notation $\zeta(s) = \prod_{v \leq \infty} \zeta_{v}(s)$ stands for the Riemann zeta function with a local factor $\zeta_{v}(s)$ at each place $v$. With the assumption on $\chi$ in Proposition 1.1 we then have the following:

**Proposition 2.5.**

$$\frac{|P_{X}(f'; \gamma_{0}')|^{2}}{\langle f', f' \rangle} = \begin{cases} C(f', \chi)L(\Pi', \chi^{-1}, \frac{1}{2}) \quad (\pi_{p}''|_{E_{p}^{\times}} = \chi_{p} \text{ when } p \text{ divides } d_{B} \text{ and is ramified in } E), \\ 0 \quad \text{(otherwise)}, \end{cases}$$

where

$$C(f', \chi) := \frac{\sqrt{|d|} (\kappa + 1)}{2^{4\pi A(\chi)}} \prod_{p|A(\chi)} L_{p}(\eta_{p}, 1)^{2} \prod_{p|d_{B}} r_{p}(1-p^{-1}) (f'_{\infty, \kappa}, f'_{\infty, \kappa})_{\infty} \frac{\zeta(2)}{(f'_{\infty}, f'_{\infty})_{\infty}} L(\pi(\text{JL}(f')), Ad, 1).$$
2.5 Main result (first form)

Theorem 2.4 and Proposition 2.5 imply the theorem as follows:

**Theorem 2.6.** Under the assumption in Theorem 1.2 we have

\[
\frac{||\mathcal{L}(f, f')_{\xi}^\chi(g_0)||^2}{(f, f)(f', f')} = C(f, f', \xi, \chi)L(\Pi, \chi^{-1}, \frac{1}{2})L(\Pi', \chi^{-1}, \frac{1}{2}),
\]

where, if \(\pi_p''|_{E_p^\cross} = \chi_p\) for \(p|d_B\) ramified in \(E\) and \(S_1 = S_2^+ = \emptyset\), \(C(f, f', \xi, \chi) = \frac{2^{2\kappa+|\delta(D)|-9}(\kappa+1)n(\xi)^{\frac{\kappa}{2}}|d_\xi|^\frac{3}{2}w(E)^2}{\pi h(E)^2A(\chi)^2D^{\frac{3}{2}}\prod_{p|A(\chi)}L_{p}(\eta_{p}, 1)^{4}\prod_{p|d_B}r_{p}(1-p^{-1})\prod_{p<\infty}C_{p}(f,\xi, \chi)^{2}}\cdot e^{-8\pi\sqrt{n(\xi)}}\cdot \frac{\langle f_{\infty,\kappa}',f_{\infty,\kappa}'\rangle_{\infty}}{\langle f_{\infty}',f_{\infty}'\rangle_{\infty}}\)

\[
\frac{\zeta(2)}{L(\pi(f), Ad, 1)L(\pi(JL(f')), Ad, 1)}\]

and \(C(f, f', \xi, \chi) = 0\) otherwise.

2.6 Main result (second form)

We introduce the degree eight \(L\)-function attached to \(\mathcal{L}(f, f')\) and \(\chi\), and relate its central value to the square norm \(||\mathcal{L}(f, f')_{\xi}^\chi(g_0)||^2\) when \(D=1\).

We now recall that \(\mathcal{L}(f, f')\) belongs to \(S_\kappa\) (cf. §1.1). Before introducing the degree eight \(L\)-function, we define the spinor \(L\)-function for a Hecke eigenform \(F \in S_\kappa\). In [M-N-1, §5.1] we introduced three Hecke operators \(\mathcal{T}_{p}^i\) with \(0 \leq i \leq 2\) for \(p \not| d_B\). Let \(\Lambda_p^i\) be the Hecke eigenvalue of \(\mathcal{T}_{p}^i\) for \(F\) with \(0 \leq i \leq 2\). For \(p \not| d_B\) we put

\[
Q_{F,p}(t) := 1 - p^{-\frac{3}{2}}\Lambda_p^1t + p^{-2}(\Lambda_p^2 + p^2 + 1)t^2 - p^{-\frac{3}{2}}\Lambda_p^1t^3 + t^4.
\]

For this we note that \(Q_{F,p}(p^{-s})^{-1}\) coincides with the local spinor \(L\)-function for a unramified principal series of the group of \(\mathbb{Q}_p\)-rational points for the split symplectic \(\mathbb{Q}\)-group \(GSp(2)\) of degree two with similitudes. Here recall that \(GSp(2)\) is defined by

\[
GSp(2)_{\mathbb{Q}} := \left\{ g \in GL(4)_{\mathbb{Q}} \mid t g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}, \nu(g) \in \mathbb{Q}^\cross \right\}.
\]

On the other hand, in [M-N-1, §5.2], we introduced two Hecke operators \(T_{p}^i\) with \(0 \leq i \leq 1\) for \(p|d_B\). Let \(\Lambda_p^i\) be the Hecke eigenvalue of \(T_{p}^i\) for \(F\) with \(0 \leq i \leq 1\). For \(p|d_B\) we put

\[
Q_{F,p}(t) := (1 - p^{-\frac{3}{2}}(\Lambda_p^1 - (p^{A_p} - 1)\Lambda_p^0)t + t^2)(1 - \Lambda_p^0p^{-\frac{3}{2}}t^2),
\]

where \(A_p := \begin{cases} 1 & (p \not| D) \\ 2 & (p|D) \end{cases}\). This is due to Sugano [Su, (3.4)].

We define the spinor \(L\)-function \(L(F, \text{spin}, s)\) of \(F\) by

\[
L(F, \text{spin}, s) := \prod_{v \leq \infty} L_v(F, \text{spin}, s),
\]
where
\[
L_v(F, \text{spin}, s) := \begin{cases} 
Q_{F,p}(p^{-s})^{-1} & (v = p < \infty), \\
\Gamma_C(s + \kappa - \frac{1}{2})\Gamma_C(s + \kappa + \frac{1}{2}) & (v = \infty). 
\end{cases}
\]

This is a modification of the definition in [M-N-1, §5.3]. We can then reformulate [M-N-1, Corollary 5.3] when \(D = 1\).

**Proposition 2.7.** The spinor \(L\)-function for \(\mathcal{L}(f, f')\) decomposes into
\[
L(\mathcal{L}(f, f'), \text{spin}, s) = L(\pi(f), s)L(\pi(JL(f')), s).
\]

Of course, when \(D = 1\), we see that \(L(\mathcal{L}(f, f'), \text{spin}, s)\) has the meromorphic continuation and satisfies the functional equation between \(s\) and \(1 - s\) since so do \(L(\pi(f), s)\) and \(L(\pi(JL(f')), s)\).

We now introduce the \(L\)-function
\[
L(F, \chi^{-1}, s) := \prod_{v \leq \infty} L_v(F, \chi^{-1}, s)
\]
of degree eight for a Hecke eigenform \(F \in S_{\kappa}\) and \(\chi\). Here the local factors \(L_v(F, \chi^{-1}, s)\) are given as
\[
L_v(F, \chi^{-1}, s) := \begin{cases} 
Q_{F,p}(\alpha_p^{\chi}p^{-s})^{-1}Q_{F,p}(\beta_p^{\chi}p^{-s})^{-1} & (\chi \text{ is unramified at } v = p < \infty), \\
1 & (\chi \text{ is ramified at } v = p < \infty), \\
\Gamma_C(s + \kappa - \frac{1}{2})\Gamma_C(s + \kappa + \frac{1}{2}) & (v = \infty),
\end{cases}
\]
where
\[
(\alpha_p^{\chi}, \beta_p^{\chi}) := \begin{cases} 
(\chi_p(\varpi_{p,1})^{-1}, \chi_p(\varpi_{p,2})^{-1}) & (v = p: \text{split}), \\
(\chi_p(p)^{-1}, -\chi_p(p)^{-1}) = (1, -1) & (v = p: \text{inert}), \\
(\chi_p(\varpi_p)^{-1}, 0) & (v = p: \text{ramified})
\end{cases}
\]
with prime elements \(\varpi_{p,1}, \varpi_{p,2}\) and \(\varpi_p\) introduced in Lemma 2.1.

**Proposition 2.8.** Let \(D = 1\). We have
\[
L(\mathcal{L}(f, f'), \chi^{-1}, s) = L(\Pi, \chi^{-1}, s)L(\Pi', \chi^{-1}, s).
\]
This has meromorphic continuation to \(\mathbb{C}\) and satisfies the functional equation
\[
L(\mathcal{L}(f, f'), \chi^{-1}, s) = \epsilon(\Pi, \chi^{-1})\epsilon(\Pi', \chi^{-1})L(\mathcal{L}(f, f'), \chi^{-1}, 1 - s),
\]
where \(\epsilon(\Pi, \chi^{-1})\) (resp. \(\epsilon(\Pi', \chi^{-1})\)) denotes the \(\epsilon\)-factor of \(L(\Pi, \chi^{-1}, s)\) (resp. \(L(\Pi', \chi^{-1}, s)\)).

In view of Proposition 2.4, Proposition 2.5 and Proposition 2.8, the central value \(L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2})\) is meaningful when \(D = 1\). We are now able to reformulate Theorem 2.6 as follows:

**Theorem 2.9.** Let the assumption be as in Theorem 1.2 and let \(D = 1\). We have
\[
\frac{||\mathcal{L}(f, f')\xi(\mathfrak{g}_0)||^2}{\langle f, f \rangle \langle f', f' \rangle} = C(f, f', \xi, \chi)L(\mathcal{L}(f, f'), \chi^{-1}, \frac{1}{2}).
\]
2.7 Main result (third form)

Note that the group $G = GSp(1,1)$ is an inner form of the symplectic $\mathbb{Q}$-group $GSp(2)$ of degree two with similitudes. As is well-known, the Langlands principle of functoriality [Lg-1] suggests that the $L$-function for a Hecke eigenform $F \in S_\kappa$ should be an $L$-function for some automorphic form (or automorphic representation) of $GSp(2)_{A_\mathbb{Q}}$.

T. Okazaki [O] has recently constructed cusp forms on $GSp(2)_{A_\mathbb{Q}}$, which contain forms whose theta lifting from $GL(2)_{A_\mathbb{Q}} \times GL(2)_{A_\mathbb{Q}}$ to $GSp(2)_{A_\mathbb{Q}}$ (or from $GO(2,2)_{A_\mathbb{Q}}$ to $GSp(2)_{A_\mathbb{Q}}$) and follows the formulation of Harris-Kudla [H-K]. He specifies an appropriate Schwartz-Bruhat function on $M_2(A_{\mathbb{Q}})^\oplus 2$ to construct the theta kernel for the lifting. We denote this lifting by $\mathcal{L}_{GSp(2)}$. His result is then stated as follows:

**Theorem 2.10 (Okazaki).** For two non-zero primitive cusp forms $(f_1, f_2) \in S_{\kappa_1}(N_1) \times S_{\kappa_2}(N_2)$, $\mathcal{L}_{GSp(2)}(f_1, f_2)$ is a non-zero generic cusp form on $GSp(2)_{A_\mathbb{Q}}$ with the following properties:

1. it is a paramodular form of level $N_1N_2$, namely, at a prime $p \mid N := N_1N_2$, it is right invariant by a paramodular group

$$\left(\begin{array}{cccc}
Z_p & Z_p & N^{-1}Z_p & Z_p \\
NZ_p & Z_p & Z_p & Z_p \\
NZ_p & Z_p & Z_p & Z_p \\
NZ_p & Z_p & Z_p & Z_p \\
\end{array}\right) \cap GSp(2)_{\mathbb{Q}_p},$$

2. at the Archimedean place, the highest weight of $\mathcal{L}_{GSp(2)}(f_1, f_2)$ is

$$\left(\frac{\kappa_1 + \kappa_2}{2}, -\frac{\kappa_1 - \kappa_2}{2}\right).$$

The global spinor $L$-function of $\mathcal{L}_{GSp(2)}(f_1, f_2)$ (in the sense of Novodvorsky [No]) decomposes into

$$L(\pi(f_1), s)L(\pi(f_2), s).$$

Also for $\mathcal{L}_{GSp(2)}(f_1, f_2)$ and a Hecke character $\chi$ we can similarly define the degree eight $L$-function $L(\mathcal{L}_{GSp(2)}(f_1, f_2), \chi^{-1}, s)$. For a Hecke eigenform $f' \in A_\kappa$ we have put $\text{JL}(f')$ to be the Jacquet-Langlands lift of $f'$ (cf. §2.1) and note that $\text{JL}(f')$ is primitive (cf. [E-1], [E-2], [Sh, §6]). From the theorem just mentioned and Proposition 2.8 we deduce the following:

**Proposition 2.11.** Let $(f, f') \in S_\kappa(1) \times A_\kappa$ be Hecke eigenforms with $L(f, f') \neq 0$. Then we have

$$L(\mathcal{L}(f, f'), \text{spin}, s) = L(\mathcal{L}_{GSp(2)}(f, \text{JL}(f')), \text{spin}, s),$$

$$L(\mathcal{L}(f, f'), \chi^{-1}, s) = L(\mathcal{L}_{GSp(2)}(f, \text{JL}(f')), \chi^{-1}, s).$$
We thus know that Theorem 2.9 is restated as follows:

**Theorem 2.12.** Let the assumption be as in Theorem 1.2 and let $D = 1$. We have

$$\frac{||L(f, f')_{\xi}(g_0)||^2}{(f, f')(f', f')} = C(f, f', \xi, \chi)L(L_{GSp(2)}(f, JL(f'))), \chi^{-1}, \frac{1}{2}).$$

**Remark 2.13.** (1) This theorem is compatible with the conjecture of Furusawa-Martín [F-M] and Furusawa-Shalika [F-S], which are inspired by Böcherer [B].

(2) The example of the functorial correspondence above is essentially predicted by T. Ibukiyama in his study of automorphic forms on the compact inner form of $GSp(2)$ (cf. [I]). We can expect that automorphic forms in $S_{\kappa}$ (cf. §1.1) with $D = 1$ should functorially correspond to paramodular forms on $GSp(2)_{AQ}$ of level $d_B$.

(3) At the Archimedean place $L(f, f')$ generates the quaternionic discrete series with minimal $K_{\infty}$-type $\tau_{\kappa}$, while $L_{GSp(2)}(f, JL(f'))$ generates the large discrete series with the Blattner parameter $(\kappa + 1, -1)$. The latter is due to Przebinda [P, Chap.III, §3]. These two discrete series representations have the same infinitesimal character. That is, the functorial correspondence of these two global theta lifts is compatible with the Archimedean local functorial correspondence established by Langlands [Lg-2].

### 2.8 Strictly positive central L-values

As an application of our main results we show the existence of strictly positive central values for the $L$-functions in our concern. In this subsection we fix the quaternion algebra $B$ and a maximal order $\mathcal{O}$ of $B$ as

$$B = \mathbb{Q} + \mathbb{Q} \cdot i + \mathbb{Q} \cdot j + \mathbb{Q} \cdot k \quad (i^2 = j^2 = -1, \; ij = -ji = k),$$

$$\mathcal{O} = \mathbb{Z} \cdot \frac{1 + i + j + k}{2} + \mathbb{Z} \cdot i + \mathbb{Z} \cdot j + \mathbb{Z} \cdot k.$$

We note that $d_B = 2$ for this $B$. In [M-N-2, §14] we have shown the following:

**Proposition 2.14.** Suppose that $\xi = i/2$ and $\chi$ is unramified at every finite prime, i.e. $i_p(\chi) = 0$ for any finite prime $p$. Let $D \in \{1, 2\}$ and $\kappa \geq 12$ (resp. $\kappa \geq 8$) be divisible by 4 (resp. by 8) when $D = 1$ (resp. $D = 2$). Then there exist Hecke eigenforms $(f, f') \in S_{\kappa}(D) \times A_{\kappa}$ such that $p_{\kappa}(f, \gamma_0)^{\kappa}p_{\kappa}(f', \gamma_0) \neq 0$ (which implies $L(f, f')_{\xi}^{\kappa} \neq 0$).

We now assume that $D = 1$ and $\chi$ is unramified at every finite prime. From Proposition 2.4, Proposition 2.5 and this proposition we deduce the following:

**Theorem 2.15.** Under the assumption in Proposition 2.14 there exist Hecke eigenforms $(f, f') \in S_{\kappa}(1) \times A_{\kappa}$ such that

$$L(\Pi, \chi^{-1}, \frac{1}{2}) > 0, \; L(\Pi', \chi^{-1}, \frac{1}{2}) > 0.$$
Theorem 2.16. Keeping the assumption in Proposition 2.14, there are Hecke eigenforms $(f, f') \in S_\kappa(1) \times A_\kappa$ such that

$$L(L(f, f'), \chi^{-1}, \frac{1}{2}) = L(L_{GSp(2)}(f, JL(f')), \chi^{-1}, \frac{1}{2}) > 0.$$  

Remark 2.17. We remark that there are several results on the non-negativity of the central-values of the $L$-functions in Theorem 2.15 and Theorem 2.16 (cf. [Gu], [J-C], [Lp] etc.).

3 Outline of the proof of Proposition 2.5

In order to verify Theorem 2.6 and Theorem 2.9 it remains to show Proposition 2.5. In this section we choose the measure of $A^\times_{\mathbb{Q}}$ so that

$$\text{vol}(\mathbb{Z}_{p}^\times) = 1$$

for any $p < \infty$.

For the proof of the proposition we need two formulas. The first formula is due to Waldspurger [Wa, Proposition 7]. In order to review it we need several remarks. We first remark that every $\pi''_{v}$ is unitary and thus equipped with a unitary inner product. When $v = p \not| d_{B}$ we embed $E_{p}^\times$ into $B_{p}^\times = GL_{2}(\mathbb{Q}_{p})$ by $b_{\xi}$ (cf. §1.4). Recall that, for the quadratic character $\eta$ attached to the quadratic extension $E/\mathbb{Q}$, we have let $L(\eta, s)$ be the $L$-function defined by $\eta$ and $L_{v}(\eta_{v}, s)$ the local factor of $L(\eta, s)$ at a place $v$ (cf. §2.3).

Proposition 3.1 (Waldspurger). For $b = (b_{v})_{v \leq \infty} \in B_{A_{\mathbb{Q}}}^\times$,

$$\frac{||P_{\chi}(f'^{f}; b)||^{2}}{\langle f, f \rangle} = \frac{\sqrt{|d_{\xi}|}}{8\pi} \frac{\zeta(2)L(\Pi', \chi^{-1}, \frac{1}{2})}{L(\pi(JL(f')), \text{Ad}, 1)} \prod_{v \leq \infty} \alpha_{v}(f_{v}', \chi_{v}, b_{v})$$

where

$$\alpha_{v}(f_{v}', \chi_{v}, b_{v}) := \frac{L_{v}(\eta_{v}, 1)L_{v}(\pi'_{v}, \text{Ad}, 1)}{\z_{v}(2)L_{v}(\Pi', \chi_{v}^{-1}, \frac{1}{2})} \int_{Q_{v}^\times \backslash E_{v}^\times} \frac{\langle \pi''_{v}(tb_{v})f'_{v}, \pi''_{v}(b_{v})f_{v}' \rangle_{v}}{\langle f'_{v}, f_{v}' \rangle_{v}} \chi_{v}(t)^{-1} dt$$

with an inner product $\langle *, * \rangle_{v}$ of $\pi''_{v}$ for each place $v$.

Here we note that the normalization of our measure $dt$ differs from that of [Wa] by $\frac{\sqrt{|d_{\xi}|}}{4\pi}$-multiple, and furthermore note that the toral integral in the sense of [Wa] is replaced by ours under the normalization of the measure of $A_{\mathbb{Q}}^\times$ mentioned above.

The second formula is the well-known explicit formula for zonal spherical functions by Macdonald [Ma, Chap.V, §3, (3.4)]. We put

$$\phi_{v}(g) := \frac{\langle \pi''_{v}(g)f'_{v}, f''_{v} \rangle_{v}}{\langle f'_{v}, f''_{v} \rangle_{v}} \quad (g \in B_{v}^\times)$$

for each place $v$. We note that $\pi''_{p}$ at $p \not| d_{B}$, which is isomorphic to $\pi'_{p}$, is a unitary unramified principal series representation of $GL_{2}(\mathbb{Q}_{p})$ with a spherical vector $f'_{p}$. The Satake parameter of $\pi''_{p}$ is of the form $(\alpha_{p}, \alpha_{p}^{-1}) \in (\mathbb{C}^\times)^{2}$ with $|\alpha_{p}| = 1$. We see that $\phi_{p}(g)$ is a zonal spherical function on $GL_{2}(\mathbb{Q}_{p})$ for $v = p \not| d_{B}$. 


Proposition 3.2 (Macdonald). Let \( p \) be a finite prime not dividing \( d_B \). For \( a_m := \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} \) with \( m \geq 0 \) we have
\[
\phi_p(a_m) = \frac{p^{-\frac{m}{2}}}{1+p^{-1}} \left( \alpha_p^m \frac{1-p^{-1}\alpha_p^{-2}}{1-\alpha_p^{-2}} + \alpha_p^{-m} \frac{1-p^{-1}\alpha_p^2}{1-\alpha_p^2} \right).
\]

This formula is useful to evaluate the local integrals involved in \( \alpha_v(f_v', \chi_v, b_v) \) for \( v = p \) not dividing \( d_B \). The evaluation of such local integrals at other places is settled by a direct calculation. We furthermore note that \( \alpha_v(f_v', \chi_v, b_v) \)’s have factors contributed by ratios of local \( L \)-functions, which are computed by Lemma 2.1, Lemma 2.2 and Lemma 2.3. We now state the following formula.

Proposition 3.3. (1) For \( p \nmid d_B \),
\[
\alpha_p(f_p', \chi_p, \gamma_{0,p}') = \begin{cases} 1 & (i_p(\chi) = 0), \\ p^{-i_p(\chi)} L_p(\eta_p, 1)^2 & (i_p(\chi) > 0). \end{cases}
\]

(2) When \( p \mid d_B \) we have
\[
\alpha_p(f_p', \chi_p, \gamma_{0,p}') = \begin{cases} r_p(1-p^{-1}) & (p \text{ is iner}t \text{ or } p \text{ is ramified and } \omega_p = \delta_p), \\ 0 & (p \text{ is ramified and } \omega_p \neq \delta_p), \end{cases}
\]
where \( r_p \) is the ramification index of \( p \) for the quadratic extension \( E/Q \) (cf. \S 2.4).

(3) When \( v = \infty \) we have
\[
\alpha_\infty(f_\infty', \chi_\infty^{-1}, \gamma_{0,\infty}') = \frac{\kappa + 1}{4} \langle f_\infty'^{\kappa}, f_\infty'^{\kappa} \rangle_\infty,
\]
where see \S 2.4 for \( f_\infty'^{\kappa} \).

As a result of Proposition 3.1 and Proposition 3.3 we have proved Proposition 2.5.

References


Atsushi Murase: Department of Mathematical Science, Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan
E-mail address: murase@cc.kyoto-su.ac.jp

Hiro-aki Narita: Department of Mathematics, Kumamoto University, Kurokami, Kumamoto 860-8555, Japan
E-mail address: narita@sci.kumamoto-u.ac.jp