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On P-adic families of modular forms

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Abstract. We describe a new approach to the theory of p-adic families of modular forms, which is based on a comparison of trace formulas. We apply it to give new proofs for the existence of p-adic continuous families of modular forms in the finite slope case and for the existence of p-adic analytic families of modular forms in the slope 0, i.e. in the ordinary case.

Introduction. The theory of p-adic families of automorphic forms originates with the work of H. Hida who showed that any ordinary modular eigenform f (i.e. f has slope 0) fits in a p-adic analytic family of eigenforms $(f_k)_k$ of varying weight k (see section 1 below for a precise statement of the notion of analytic family). He used two different approaches to this result, one based on use of group cohomology and, more recently, he described an approach based on use of algebro-geometric methods, which is less elementary (cf. [Hi 1,2]). Later, Mazur and Gouvea conjectured that an analogous statement holds for modular forms of arbitrary finite slope α (cf. [M-G]) and their conjecture has been proven by Coleman and Wan (cf. [C], [W]). Coleman's proof is not elementary and relies on methods from rigid analytic geometry. We note that Buzzard has given a proof of the boundedness of the dimension of the slope α-subspace of the space of modular foms as the weight k varies, which only uses group cohomology and is elementary (cf. [Bu]). Meanwhile, the work of Hida and Coleman has been generalized to groups of higher rank by Hida, Tilouine, Ash-Stevens, Emerton, Buzzard, ... Moreover, there is very recent work by G. Harder.

In this note we want to describe a new approach to the construction of p-adic families of eigenforms, different from the existing ones, which is based on the trace formula. We will not give full proofs, which can be found in [M]; rather we want to describe the essential content of our approach. We were guided by the analogy with the functoriality principle. Given an automorphic form f on a group G the functoriality principle yields the existence of a form f' on a group G', whose existence then follows by comparing trace formulas on G and G'. Somewhat similar, the theory of p-adic families is an existence statement for automorphic forms: given a modular form f in weight k_0 the theory of p-adic families predicts the existence of modular forms f_k in weights k. In contrast to the functoriality principle, there are now infinitely many forms f_k, which are explicitly determined by f only modulo some power of p and which are related to each other (the family $(f_k)_k$ depends analytically on the weight k). We want to deduce the existence of the p-adic family $(f_k)_k$ passing through a given form f of weight k_0 by comparing trace formulas at weight k_0 and at weights k.

We would like to explain our motivation. 1.) Such an approach would confirm the idea of the trace formula as a universal, unifying principle in the theory of automorphic forms: the trace formula is a common source, which can yield the existence of those automorphic forms, which are predicted by the functoriality principle, but it can also yield the existence of those modular forms, which are predicted by the theory of p-adic families. 2.) The use of the trace formula gives another perspective on the theory of p-adic families. Similar to the functoriality principle our proof of the existence of p-adic families relies on certain (simple) trace identities (cf. equations (3) and (4) in section 2). In case of the functoriality principle these identities hold due to the Fundamental Lemma; in our case these identities hold essentially because ℓ^k, $\ell \in \mathbb{Z}_p^*$, is a p-adically continuous and analytic function of k. We note that these look like abelian conditions; they also are the same conditions which essentially yield the existence of p-adic families of Eisenstein series. 3.) The approach seems to carry over directly to any reductive group G, which has discrete series representations and it yields the existence of families of true cusp forms passing through a given cusp form. For other groups one has to replace the use of the topological trace formula by the more difficult Arthur-Selberg trace formula. 4.) The approach is elementary: besides the topological trace formula, which has an elementary proof (cf. [Be]), we need Buzzard's Theorem and only very basic facts from algebra and number theory. In particular, we hope to obtain an elementary proof of the full Mazur-Gouvea Conjecture in this way (one has to generalize equation (4) in section 2 to arbitrary slope spaces).

We mention related work. The idea of applying the trace formula to the theory of p-adic families of modular forms is mentioned in [C]. Buzzard and Calegari used an explicit trace formula and computer calculations to find an explicit counterexample to the Mazur-Gouvea Conjecture in its strong form (cf. [B-C]). More related to ours is the work of Koike who used the trace formula to examine p-adic properties of the Hecke operator T_p (cf. [K 1,2]).

In section 1 we introduce the notation and we formulate our main results. In section 2 we describe our approach. In section 3 we give the application to the construction of p-adically continuous families of eigenforms in the finite slope case; in section 4 we will show that any continuous family of slope 0 already must be analytic; this yields the existence of R-families of eigenforms passing through a given eigenform of slope 0. Finally, in section 5 we show that if f is a cusp form, then the R-family, in which f fits, consists of true cusp forms.

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0.1 Description of main results.

We fix a prime $p \in \mathbb{N}$, an integer $N \in \mathbb{N}$ such that $(p, N) = 1$ and a Dirichlet character $\chi : \mathbb{Z}/(Np)^* \to \bar{\mathbb{Q}}^*$. We denote by $\omega : \mathbb{Z}/(p)^* \to \mu_{p-1} \subset \mathbb{C}^*$ the Teichmuller character; thus, ω is determined by the condition $\omega(z) \equiv z \pmod p$ for all z, which are relatively prime to p. We denote by $\Gamma = \Gamma_1(Np)$ the Hecke subgroup of level Np. We define the Hecke algebra $\mathcal{H} = \Gamma\backslash\Delta/\Gamma$, where

$$\Delta = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \pmod{Np},\ (a, Np) = 1\}$$

and we denote by $\mathcal{H}_1 = \langle T_\ell, \ell\,\text{prime}\rangle \leq \mathcal{H}$ the subalgebra generated by the Hecke operators $T_\ell = \Gamma \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} \Gamma$. We further denote by $\mathcal{M}_k = \mathcal{M}_k(\Gamma, \chi\omega^{-k})$ the space of all (complex) modular forms of level Γ, nebentype $\chi\omega^{-k}$ and weight k. For any $\gamma \in \bar{\mathbb{Q}}$ we denote by $\mathcal{M}_k(\gamma)$ the generalized eigenspace attached to T_p and the eigenvalue γ. We fix a p-adic valuation v_p on $\bar{\mathbb{Q}}_p$; the slope α-subspace $\mathcal{M}_k^\alpha$ of $\mathcal{M}_k$ then is defined as

$$\mathcal{M}_k^\alpha = \bigoplus_{\gamma,\, v_p(\gamma) = \alpha} \mathcal{M}_k(\gamma).$$

Instead of eigenforms $f \in \mathcal{M}_k^\alpha$ we will work with the corresponding system of Hecke eigenvalues. We denote by Φ_k^α the set of all sequences $\lambda = (\lambda_\ell)_\ell$, where ℓ runs over all primes, such that there is an eigenform $f \in \mathcal{M}_k^\alpha$ satisfying $T_\ell f = \lambda_\ell f$ for all primes ℓ (i.e. λ is the eigenvalue corresponding to f).

Our first result asserts that the dimension of the slope α subspace is locally constant as a function of the weight.

Corollary 1. *There are $K(\alpha)$, $B(\alpha) \in \mathbb{N}$ only depending on p, N, χ and α such that for all $k, k' \geq K(\alpha)$ satisfying $k \equiv k' \pmod{p^{B(\alpha)}}$ we have*

$$\dim \mathcal{M}_k^\alpha = \dim \mathcal{M}_{k'}^\alpha.$$

In the ordinary case we obtain $\dim \mathcal{M}_k^0 = \dim \mathcal{M}_k^0$ *for all $k > 2$.*

We call a family $(\lambda_k)_k$, $\lambda_k \in \Phi_k^\alpha$, continuous or a Lipschitz family of exponent (a, b) if $k \equiv k' \pmod{p^m}$ implies $\lambda_k \equiv \lambda_{k'} \pmod{p^{\mathsf{a}m+\mathsf{b}}}$ (this is defined as $v_p(\lambda_{k,\ell} - \lambda_{k',\ell}) \geq \mathsf{a}m + \mathsf{b}$ for all primes ℓ).

Theorem 2. *There are $\mathsf{a} \in \mathbb{Q}_{>0}$ and $\mathsf{b} \in \mathbb{Q}$ only depending on N, p, χ and α such that any $\lambda \in \Phi_{k_0}^\alpha$ fits in a Lipschitz family $(\lambda_k)_k$ of exponent (a, b), i.e. there are $\lambda_k \in \Phi_k^\alpha$, $k \in k_0 + p^{K(\alpha)}\mathbb{Z}$, such that $\lambda_{k_0} = \lambda$ and $k \equiv k' \pmod{p^m}$ implies that $\lambda_k \equiv \lambda_{k'} \pmod{p^{\mathsf{a}m+\mathsf{b}}}$. Moreover,*

$$0 < \mathsf{a} < \frac{1}{2 \dim \mathcal{M}_{k_0}^\alpha}.$$

In the ordinary case the family $(\lambda_k)_k$ exists for all $k > 2$.

A family $(\lambda_k)_k$ is p-adically analytic if there are power series $F_\ell \in \mathbb{Z}_p[[X]]$ such that $F_\ell((1+p)^k-1) = \lambda_{k,\ell}$ for all ℓ and all k. We will need a more general notion of analyticity. Let R be a finite free $\mathbb{Z}_p[[X]]$-algebra and let $\varphi_k : R \to \bar{\mathbb{Q}}_p$ be a family of morphisms. $(\lambda_k)_k$ is a R-family if there are $\Omega_\ell \in R$ such that $\varphi_k(\Omega_\ell) = \lambda_{k,\ell}$ for all k and all primes ℓ.

Theorem 3 a. *There are a finite, free $\mathbb{Z}_p[[X]]$-algebra R of rank less than or equal to $\dim \mathcal{M}_k^0$, a family of morphisms $\varphi_k : R \to \bar{\mathbb{Q}}_p$ and a finite set $S \subset \mathbb{N}$ such that any Lipschitz family $(\lambda_k)_k$, $\lambda_k \in \Phi_k^0$, is locally an R-family, i.e. for all $k_0 \notin S$ there is $\epsilon > 0$ and $\Omega_\ell \in R$ such that $\lambda_{k,\ell} = \varphi_k(\Omega_\ell)$ for all ℓ and all $k \in U_\epsilon(k_0)$.*

Here, $U_\epsilon(k_0) = \{k,\ v_p(k-k_0) < \epsilon)\}$. Essentially the same methods as in the proof of Theorem 3a yield that any $\lambda \in \Phi_{k_0}^0$, $k_0 \notin S$, fits in a R-family:

Theorem 3 b. *For any $\lambda \in \Phi_{k_0}^0$, $k_0 \notin S$ there are $\Omega_\ell \in R$ such that $(\varphi_k(\Omega_\ell))_\ell \in \Phi_k^0$ for all $k \notin S$ and $(\varphi_{k_0}(\Omega_\ell))_\ell = \lambda$.*

0.2 A trace formula approach to the construction of p-adic families of modular forms.

We describe our approach based on the trace formula. In a first step we will show that any λ fits in a Lipschitz family and in a second step we will show that any Lipschitz family of slope 0 is an R-family.

We look at the first step. We denote by $\mathcal{X}$ the set of all characters $\lambda : \mathcal{H}_1 \to \bar{\mathbb{Q}}$. Since $\mathcal{H}_1$ is generated by the Hecke operators T_ℓ, λ can be identified with the sequence $(\lambda_\ell)_\ell$, where $\lambda_\ell = \lambda(T_\ell)$. We say that two characters $\lambda, \mu \in \mathcal{X}$ are congruent mod p^c, if $\lambda(T) \equiv \mu(T) \pmod{p^c}$ for all $T \in \mathcal{H}_1$; this is equivalent to $\lambda_\ell \equiv \mu_\ell \pmod{p^c}$ for all primes ℓ. For any character $\lambda = (\lambda_\ell)_\ell$ we denote by $\mathcal{M}_k^\alpha(\lambda)$ the generalized eigenspace attached to λ, i.e. $\mathcal{M}_k^\alpha(\lambda)$ consists of all $f \in \mathcal{M}_k^\alpha$ such that $(T_\ell - \lambda_\ell)^n f = 0$ for some $n = n_\ell$. We obtain a decomposition as $\mathcal{H}$-modules

$$\mathcal{M}_k^\alpha = \bigoplus_{\lambda \in \Phi_k^\alpha} \mathcal{M}_k^\alpha(\lambda).$$

If now any $\lambda \in \Phi_{k_0}^\alpha$ fits in a Lipschitz family $(\lambda_k)_k$ then for any k, $k \equiv k_0 \pmod{p^m}$ there is a map

$$\psi_k : \Phi_{k_0}^\alpha \to \Phi_k^\alpha$$

such that $\psi_k(\lambda) \equiv \lambda \pmod{p^{\mathsf{a}m+\mathsf{b}}}$ for all $\lambda \in \Phi_{k_0}^\alpha$. We will see that it is sufficient to establish the existence of the maps ψ_k. This in turn relies on a reformulation in terms of certain reduced multiplicities; for any $\lambda \in \mathcal{X}$ we define its $\pmod{p^c}$-multiplicity as

$$\mathbf{m}_k^\alpha(\lambda, c) = \sum_{\substack{\mu \in \Phi_k^\alpha \\ \mu \equiv \lambda \pmod{p^c}}} \dim \mathcal{M}_k^\alpha(\mu).$$

Thus, $\mathbf{m}_k^\alpha(\lambda, c)$ is the multiplicity of λ in the $\pmod{p^c}$-reduction of $\mathcal{M}_k^\alpha$. ψ_k then exists if we can show for all $\lambda \in \mathcal{X}$ that $\mathbf{m}_{k_0}^\alpha(\lambda, \mathsf{a}m+\mathsf{b}) \neq 0$ implies $\mathbf{m}_k^\alpha(\lambda, \mathsf{a}m+\mathsf{b}) \neq 0$. This kind of statement does not seem to be related to a simple trace identity. We therefore assume stronger that even equality of multiplicities holds:

$$\mathbf{m}_{k_0}^\alpha(\lambda, \mathsf{a}m+\mathsf{b}) = \mathbf{m}_k^\alpha(\lambda, \mathsf{a}m+\mathsf{b}) \tag{1}$$

for all $\lambda \in \mathcal{X}$. This implies that the $\pmod{p^{\mathsf{a}m+\mathsf{b}}}$-reductions of $\mathcal{M}_{k_0}^\alpha$ and $\mathcal{M}_k^\alpha$ are isomorphic as Hecke modules

$$\mathcal{M}_{k_0}^\alpha[p^{\mathsf{a}m+\mathsf{b}}] \to \mathcal{M}_k^\alpha[p^{\mathsf{a}m+\mathsf{b}}] = \mathcal{M}_{k_0}^\alpha/p^{\mathsf{a}m+\mathsf{b}}\mathcal{M}_{k_0}^\alpha, \tag{2}$$

hence, the following simple trace identity holds:

$$\operatorname{tr} T|_{\mathcal{M}_{k_0}^\alpha} \equiv \operatorname{tr} T|_{\mathcal{M}_k^\alpha} \pmod{p^{\mathsf{a}m+\mathsf{b}}}. \tag{3}$$

for all $T \in \mathcal{H}$. Using the topological trace formula, we prove an identity of this kind in section 3. On the other hand, using it we are only able to prove a local version of the isomorphism (2): (2) is equivalent to equality (1); using (3) we will show that for any $\lambda \in \mathcal{X}$ there is a $c = c(\lambda) \geq \mathsf{a}m+\mathsf{b}$ such that

$$\mathbf{m}_{k_0}^\alpha(\lambda, c) = \mathbf{m}_k^\alpha(\lambda, c).$$

Still, this is strong enough to deduce the existence of continuous families passing through a given eigenvalue λ as in Theorem 2.

In a second step again using the trace formula, we show that any Lipschitz family of slope 0 is (locally) an R-family. We will show that the trace functional on the slope 0 subspace depends analytically on the weight k, i.e. there is a power series F with p-adic coefficients such that

$$\operatorname{tr} T|_{\mathcal{M}_{k_0}^0} = F((1+p)^k - 1) \tag{4}$$

for all all Hecke operators T. As a consequence, we obtain that the characteristic polynomial $\mathbf{Ch}_{T,k_0} \in K[Y]$ of T acting on $\mathcal{M}_{k_0}^\alpha$ fits into a analytic family, i.e. there is a polynomial $\mathbf{Ch}_T = \sum_i A_i Y^i \in K[[X]][Y]$ such that $\mathbf{Ch}_T((1+p)^k - 1) = \mathbf{Ch}_{T,k}$. We let $\lambda_{T,i}$, $i = 1, \ldots, s$ be the roots of $\mathbf{Ch}_T$ in a splitting field $\mathbb{E}$. The specializations of $\lambda_{T,i}$ at weight k are precisely the roots of $\mathbf{Ch}_{T,k}$, hence, any eigenvalue of T acting on $\mathcal{M}_k^0$ fits into a p-adic analytic family (given by some of the $\lambda_{T,i}$). We have to find out how to collect the $\lambda_{T_\ell,i}$ as ℓ runs over the primes into systems of eigenvalues, i.e. we have to show that we can choose for any ℓ an index $i(\ell)$ such that $\lambda = (\lambda_{T_\ell,i(\ell)})_\ell$ specializes under any φ_k to an element in Φ_k^0. To this end we use the result of the first step. This will finally yield Theorem 3a and 3b.

0.3 Continuous families of modular forms

We set

$$\mathcal{M}_k^{\le\alpha} = \bigoplus_{\beta\le\alpha} \mathcal{M}_k^{\beta} \quad \text{and} \quad \mathcal{M}_k^{>\alpha} = \bigoplus_{\beta>\alpha} \mathcal{M}_k^{\beta}.$$

We will use the following reformulation of a Theorem of Buzzard (cf. [Bu]).

Theorem (Buzzard). *There are numbers $M(\alpha)$ only depending on α (and N and p) such that*

$$\sum_{0\le\beta\le\alpha} \dim \mathcal{M}_k(\Gamma)^{\beta} \le M(\alpha)$$

for all $k \ge 2$. Moreover the $M(\alpha)$ can be chosen such that $M(\alpha)$ grows linearly in α.

We denote by $\Phi_{p,k}$ resp. $\Phi_{p,k}^{\le\alpha}$ the set of all roots of the characteristic polynomial of T_p acting on $\mathcal{M}_k$ resp. on $\mathcal{M}_k \le \alpha$. For a polynomial $p = \sum_{i\ge 0} a_i X^i \in \bar{\mathbb{Q}}_p[[X]]$ we define the slope $\mathbf{S}(p)$ of p as

$$\mathbf{S}(p) = \sup\{s \in \mathbb{Q} : v_p(a_i) \ge si \text{ for all } i \ge 0\}.$$

We select two weights k, k'. Using Lagrange interpolation we construct an element $\mathbf{e}_{k,k'}^{\le\alpha} = p_{k,k'}^{\le\alpha}(T_p) \in \bar{\mathbb{Q}}[T_p]$ ($p_{k,k'}^{\le\alpha} \in \bar{\mathbb{Q}}[X]$) such that the following holds.

Lemma 1. *1.) For any $\gamma \in \Phi_{p,k} - \Phi_{p,k}^{\le\alpha}$ we have*

$$\mathcal{D}_{\mathcal{B}}(\mathbf{e}_{k,k'}^{\le\alpha}|_{\mathcal{M}_k(\gamma)}) = \begin{pmatrix} \zeta & & * \\ & \ddots & \\ & & \zeta \end{pmatrix}$$

where $\zeta \in \mathcal{O}_{\bar{\mathbb{Q}}}$ and $v_p(\zeta) \ge 1/(2M(\alpha))$. An analogous statement holds for $\gamma \in \Phi_{p,k'} - \Phi_{p,k'}^{\le\alpha}$.

2.) For any $\gamma \in \Phi_{p,k}^{\le\alpha}$ we have

$$\mathcal{D}_{\mathcal{B}}(\mathbf{e}_{k,k'}^{\le\alpha}|_{\mathcal{M}_k(\gamma)}) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Again, an analogous statement holds for $\gamma \in \Phi_{p,k'}^{\le\alpha}$

3.)

$$\mathbf{S}(p_{k,k'}^{\le\alpha}) \ge -\alpha.$$

4.)

$$\deg p_{k,k'}^{\le\alpha} \le 2M(\alpha).$$

Remark. The Lemma implies that

$$\lim_{L\to\infty} \operatorname{tr} \mathbf{e}_{k,k'}^{\leq\alpha\,L}|_{\mathcal{M}_k^{\leq\alpha}} = \dim \mathcal{M}_k^{\alpha}$$

$$\lim_{L\to\infty} \operatorname{tr} \mathbf{e}_{k,k'}^{\leq\alpha\,L}|_{\mathcal{M}_k^{>\alpha}} = 0$$

An analogous statement holds if we replace k by k'. Thus, $\mathbf{e}_{k,k'}^{\leq\alpha}$ is an approximate idempotent attached to the slope $\leq \alpha$-subspace in weights k and k'.

We denote by L_k the irreducible representation of GL_2 of dimension $k+1$ and central character $x \mapsto x^{k-2}$. We set $e_{\chi\omega^{-k}} = \frac{1}{\varphi(Np)} \sum_{\epsilon\in(\mathbb{Z}/Np\mathbb{Z})^*} \chi\omega^{-k}(\epsilon)\,\langle\epsilon\rangle$ ($\langle\epsilon\rangle$ is the diamond operator). $e_{\chi\omega^{-k}}$ is a projector onto the $\chi\omega^{-k}$-nebentype and using the Eichler-Shimura isomorphism we obtain

$$\begin{aligned} \operatorname{tr} T|_{\mathcal{M}_k^{\leq\alpha}} &= \lim_{L\to\infty} \operatorname{tr} T\mathbf{e}_{k,k'}^{\leq\alpha\,L} e_{\chi\omega^{-k}}|_{\mathcal{M}_k(\Gamma)} \\ &= \lim_{L\to\infty} \operatorname{tr} T\mathbf{e}_{k,k'}^{\leq\alpha\,L} e_{\chi\omega^{-k}}|_{H^1(\Gamma,L_k)} \\ &= \lim_{L\to\infty} \operatorname{Lef}(T\mathbf{e}_{k,k'}^{\leq\alpha\,L} e_{\chi\omega^{-k}}|H^\bullet(\Gamma,L_k)). \end{aligned}$$

Of course, the same equation holds for weight k. On the other hand, the Lefschetz number

$$\operatorname{Lef}(T\mathbf{e}_{k,k'}^{\leq\alpha\,L} e_{\chi\omega^{-k}}|H^\bullet(\Gamma,L_k)) = \sum_i \operatorname{tr} T(\mathbf{e}_{k,k'}^{\leq\alpha\,L} e_{\chi\omega^{-k}}|H^i(\Gamma,L_k))$$

can be computed using the topological trace formula. We formulate the result. We define the functions $f_s : C \mapsto \frac{1}{2M(\alpha)}[\frac{m}{C}]$ and $f_g : C \mapsto (1 - \frac{2\alpha M(\alpha)}{C})m - v_p(\varphi(N))$, which map $\mathbb{R}$ to $\mathbb{R}$.

Proposition. *Fix $\alpha \in \mathbb{Q}_{\geq 0}$ and let $C \in \mathbb{Q}_{>0}$. Assume that $k,k' \in \mathbb{N}$ satisfy $k,k' \geq (C+1)^2+2$ and $k \equiv k' \pmod{p^m}$ with $m \geq C+1$. Then for all Hecke operators $T \in \mathcal{H}_1$ the following congruence holds true:*

$$\operatorname{tr} T|_{\mathcal{M}_k^{\leq\alpha}} \equiv \operatorname{tr} T|_{\mathcal{M}_{k'}^{\leq\alpha}} \pmod{p^{\Box}},$$

where

$$\Box = \min\{f_s(C), f_g(C)\}.$$

We want to choose C such that $\Box$ becomes maximal. Since f_s is monoton decreasing in C and f_g is monoton increasing we obtain a maximum for $\Box$ if we choose C such that $f_s(C) = f_g(C)$. We slightly simplify and choose C such that

$$\frac{m}{2M(\alpha)C} = (1 - \frac{2\alpha M(\alpha)}{C})m,$$

i.e. we choose for C the value

$$K(\alpha) = 2\alpha M(\alpha) + \frac{1}{2M(\alpha)} \ (\in \mathbb{Q}_{>0}).$$

This implies

$$\square \geq \frac{m}{1 + 4\alpha M(\alpha)^2} - \frac{1}{2M(\alpha)} - v_p(\varphi(N)).$$

We abbreviate

$$\Delta = v_p(\varphi(N)) + 1$$

and note that $\Delta \geq \frac{1}{2M(\alpha)} + v_p(\varphi(N))$ and Δ only depends on N and p. Under the assumptions of the above Proposition we then obtain the congruence

Corollary 1. *For all Hecke operators $T \in \mathcal{H}_1$ the following congruence holds*

$$\operatorname{tr} T|_{\mathcal{M}_k^\alpha} \equiv \operatorname{tr} T|_{\mathcal{M}_{k'}^\alpha} \pmod{p^{\frac{m}{1+4\alpha M(\alpha)^2} - \Delta}}.$$

In the ordinary case we obtain a somewhat stronger result.

Corollary 1^{ord}. *For all Hecke operators $T \in \mathcal{H}_1$ the following congruence holds*

$$\operatorname{tr} T|_{\mathcal{M}_k^0} \equiv \operatorname{tr} T|_{\mathcal{M}_{k'}^0} \pmod{p^{m - v_p(\varphi(N))}}.$$

As an immediate consequence of the trace identity we obtain the local constance of the dimension of the slope subspaces. We set

$$B(\alpha) = (1 + 4\alpha M(\alpha)^2)(M(\alpha) + \Delta)$$

We note that Buzzard's Theorem implies that $B(\alpha)$ grows like α^4.

Corollary 2. *Fix an arbitrary slope $\alpha \in \mathbb{Q}_{\geq 0}$. For all pairs of integers $k, k' \in \mathbb{N}$ satisfying $k, k' \geq (K(\alpha) + 1)^2 + 2$ and $k \equiv k' \pmod{p^m}$ with $m > B(\alpha)$ it holds that*

$$\dim \mathcal{M}_k^\alpha = \dim \mathcal{M}_{k'}^\alpha.$$

Proof. The above Theorem in particular applies to the Hecke operator T_1, which acts as the identity. The Corollary implies that

$$\operatorname{tr} T_1|_{\mathcal{M}_k^\alpha} \equiv \operatorname{tr} T_1|_{\mathcal{M}_{k'}^\alpha} \pmod{p^{\frac{m}{1+4\alpha M(\alpha)^2} - \Delta}}.$$

Since $T_1 = \mathrm{id}$ and $m > B(\alpha)$ implies $\frac{m}{1+4\alpha M(\alpha)^2} - \Delta > M(\alpha)$ this yields

$$\dim \mathcal{M}_k^\alpha \equiv \dim \mathcal{M}_{k'}^\alpha \pmod{p^{M(\alpha)}}.$$

Since $\dim \mathcal{M}_k^\alpha$ and $\dim \mathcal{M}_{k'}^\alpha$ are smaller than $M(\alpha)$ by Buzzard's Theorem we deduce that $\dim \mathcal{M}_k^\alpha = \dim \mathcal{M}_{k'}^\alpha$. Thus the proof of the Corollary is finished.

We explain how to deduce the existence of Lipschitz families from the above trace identity. Let

$$\lambda : \mathcal{H}_1 \to \bar{\mathbb{Q}}$$

be a character of $\mathcal{H}_1$. We recall that we have set

$$\Phi_k^\alpha = \{\lambda = (\lambda_\ell)_\ell : \mathcal{M}_k(\lambda) \neq 0 \text{ and } v_p(\lambda_p) = \alpha\}.$$

and the space of modular forms decomposes

$$\mathcal{M}_k^\alpha = \bigoplus_{\lambda \in \Phi_k^\alpha} \mathcal{M}_k(\lambda).$$

Moreover, we defined the reduced multiplicity

$$\mathbf{m}_k^\alpha(\lambda, c) = \sum_{\substack{\gamma \in \Phi_k^\alpha \\ \gamma \equiv \lambda \pmod{p^c}}} \dim \mathcal{M}_k(\gamma).$$

In addition we define the following rational numbers

$$\mathsf{a} = \mathsf{a}(\alpha) = \frac{1}{2M(\alpha) + 8\alpha M(\alpha)^3} \ (\in \mathbb{Q}_{>0})$$

and

$$\mathsf{b} = \mathsf{b}(\alpha) = -\frac{\Delta + l}{2M(\alpha)} - (2M(\alpha) + 2)l,$$

where we have set $l = [\log_p M(\alpha)] + 1$ ($\log_p$ is the complex logarithm with base p). Note that a is strictly positive.

Theorem. *Fix an arbitrary $\alpha \in \mathbb{Q}_{\geq 0}$ and assume that $k, k' > (K(\alpha)+1)^2 + 2$ and $k \equiv k' \pmod{p^m}$ with $m > K(\alpha) + 1$. Then, for any character $\lambda = (\lambda_\ell)_\ell$ there is $c \in \mathbb{Q}$ with $c \geq \mathsf{a}m + \mathsf{b}$ such that*

$$\mathbf{m}_k^\alpha(\lambda, c) = \mathbf{m}_{k'}^\alpha(\lambda, c).$$

The proof rests on the existence of certain elements in the Hecke algebra.

Lemma 2. *There are an integer $c \in \mathbb{N}$, $c \geq \mathsf{a}m + \mathsf{b}$ and an element $e(\lambda) \in \mathcal{H}_1 \otimes \bar{\mathbb{Q}}$ such that*

- $e(\lambda) \in \frac{1}{2cM(\alpha)}\mathcal{H}_1$, *i.e. $e(\lambda)$ has bounded denominators*
- $\operatorname{tr} e(\lambda)|_{\mathcal{M}_k^\alpha} \equiv \mathbf{m}_k^\alpha(\lambda, c) \pmod{p^l}$
- $\operatorname{tr} e(\lambda)|_{\mathcal{M}_{k'}^\alpha} \equiv \mathbf{m}_{k'}^\alpha(\lambda, c) \pmod{p^l}$.

Applying Corollary 1 to the element $e(\lambda)$ we obtain

$$\mathbf{m}_k^\alpha(\lambda, c) \equiv \mathbf{m}_{k'}^\alpha(\lambda, c)$$

modulo a power of p, which is bigger than $M(\alpha)$. Since $\dim \mathcal{M}_k^\alpha$ and $\dim \mathcal{M}_{k'}^\alpha$ are smaller than $M(\alpha)$ this implies $\mathbf{m}_k^\alpha(\lambda, c) = \mathbf{m}_{k'}^\alpha(\lambda, c)$.

As a Corollary the above Theorem yields the existence of p-adic Lipschitz families of finte slope modular forms. First we immediately obtain the following kind of transfer for modular forms from weight k to weight k':

Corollary 3. *Let the assumptions be as in the above Theorem. Then for any $\lambda \in \Phi_k^\alpha$ there is a $\lambda \in \Phi_{k'}^\alpha$ such that*

$$\lambda \equiv \lambda' \pmod{p^{\mathsf{a}m+\mathsf{b}}}.$$

Proof. If $\lambda \in \Phi_k^\alpha$ then $\mathbf{m}_k^\alpha(\lambda, c) \neq 0$, where c is as in the above Theorem. Hence, we obtain $\mathbf{m}_{k'}^\alpha(\lambda, c) \neq 0$, i.e. there is $\lambda' \in \Phi_{k'}^\alpha$ such that $\lambda \equiv \lambda' \pmod{p^c}$. Since $c \geq \mathsf{a}m + \mathsf{b}$ this yields the claim and the Corollary is proven.

Using Corollary 1 we obtain

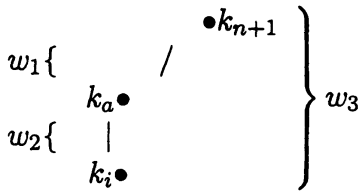
Corollary 4. *Fix an arbitrary slope $\alpha \in \mathbb{Q}_{\geq 0}$. Assume that $k_0 > (K(\alpha)+1)^2 + 2)$ and let $\lambda \in \Phi_{k_0}^\alpha$. Then there is a family $(\lambda_k)_k$, where $\lambda_k \in \Phi_k^\alpha$ and k runs over all weights satisfying $k > (K(\alpha)+1)^2 + 2$ and $k \equiv k_0 \pmod{p^{K(\alpha)+1}}$ such that the following holds: $\lambda_{k_0} = \lambda$ and $k \equiv k' \pmod{p^m}$ implies $\lambda_k \equiv \lambda_{k'} \pmod{p^{\mathsf{a}m+\mathsf{b}}}$.*

Proof. We enumerate the set of all weights k satisfying $k > (K(\alpha)+1)^2 + 2$ and $k \equiv k_0 \pmod{p^{K(\alpha)+1}}$ in a sequence $k_0, k_1, k_2, k_3, \ldots$. We inductively construct elements $\lambda_{k_i} \in \Phi_{k_i}^\alpha$, $i = 0, 1, 2, 3, \ldots$ such that $\lambda_{k_0} = \lambda$ and $k_i \equiv k_j \pmod{p^m}$ implies $\lambda_{k_i} \equiv \lambda_{k_j} \pmod{p^{\mathsf{a}m+\mathsf{b}}}$. Clearly, we set $\lambda_{k_0} = \lambda$. Assume that $\lambda_{k_0}, \ldots, \lambda_{k_n}$ have been defined such that $k_i \equiv k_j \pmod{p^m}$ implies that $\lambda_{k_i} \equiv \lambda_{k_j} \pmod{p^{\mathsf{a}m+\mathsf{b}}}$ for all $i, j = 0, \ldots, n$. To define $\lambda_{k_{n+1}}$ we select $a \in \{0, 1, 2, \ldots, n\}$ such that

$$v_p(k_{n+1} - k_a) \geq v_p(k_{n+1} - k_i) \quad \text{for all } i = 0, \ldots, n.$$

By Corollary 1 there is $\lambda \in \Phi_{k_{n+1}}^\alpha$ such that $\lambda \equiv \lambda_{k_a} \pmod{p^{\mathsf{a}w_1+\mathsf{b}}}$, where $w_1 = v_p(k_{n+1} - k_a)$. We then set $\lambda_{k_{n+1}}$ equal to this λ.

Let $i \in \{0, \ldots, n\}$ be arbitrary and set $w_3 = v_p(k_{n+1} - k_i)$. We have to show that $\lambda_{k_{n+1}} \equiv \lambda_{k_i} \pmod{p^{\mathsf{a}w_3+\mathsf{b}}}$. To this end we set $w_2 = v_p(k_a - k_i)$.



We know that $\lambda_{k_{n+1}} \equiv \lambda_{k_a} \pmod{p^{\mathsf{a}w_1+\mathsf{b}}}$ by definition of $\lambda_{k_{n+1}}$ and that $\lambda_{k_a} \equiv \lambda_{k_i} \pmod{p^{\mathsf{a}w_2+\mathsf{b}}}$ by our induction hypotheses, hence,

(1) $$\lambda_{k_{n+1}} \equiv \lambda_{k_i} \pmod{p^{\mathsf{a}\min\{w_1,w_2\}+\mathsf{b}}}.$$

We distinguish cases.

Case A $w_2 > w_1$. In this case $\min\{w_1, w_2\} = w_1$ and $w_3 = w_1$ by the p-adic triangle inequality. Hence, equation (1) implies that $\lambda_{k_{n+1}} \equiv \lambda_{k_i} \pmod{p^{\mathsf{a}w_3+\mathsf{b}}}$.

Case B $w_2 < w_1$. In this case $\min\{w_1, w_2\} = w_2$ and $w_3 = w_2$. Hence, equation (1) implies that $\lambda_{k_{n+1}} \equiv \lambda_{k_i} \pmod{p^{\mathsf{a}w_3+\mathsf{b}}}$.

Case C $w_2 = w_1$. In this case $\min\{w_1, w_2\} = w_1$. On the other hand, by the choice of a we know that $w_1 \geq w_3$; thus equation (1) yields $\lambda_{k_{n+1}} \equiv \lambda_{k_i} \pmod{p^{\mathsf{a}w_3+\mathsf{b}}}$.

This completes the proof of the Corollary.

0.4 Analytic families of ordinary modular forms.

From now on we restrict to the ordinary case. We denote by $\mathfrak{o}$ the ring of integers in the field $\mathbb{Q}(Np)$, which is obtained from $\mathbb{Q}$ by adjoining all $\varphi(pN)$-th roots of unity. Using the (topological) trace formula one can show the following

Theorem. *Let $T = \Gamma\alpha\Gamma$, $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ be any Hecke operator. There is $F_T \in \frac{1}{\varphi(N)}\mathfrak{o}[[X]]$ such that*

$$\operatorname{tr} T|_{\mathcal{M}_k^0} = F_T((1+p)^k - 1)$$

for all $k \geq 2$.

We set d_k equal to the dimension of $\dim \mathcal{M}_k^0$ and we denote by

$$\mathbf{Ch}_{T,k}(Y) = \sum_{j=0}^{d_k} (-1)^j a_{j,k} Y^{d-j}$$

the characteristic polynomial of $T|_{\mathcal{M}_k^0}$. The coefficients of $\mathbf{Ch}_{T,k}$ are given by the recursive formula $a_{0,k} = 1$ and $a_{j,k} = \frac{1}{j}\sum_{h=1}^{j} (-1)^{h+1}\, a_{j-h,k} \operatorname{tr} T^h|_{\mathcal{M}_k^0}$, $j = 1, 2, 3, \ldots, d_k$; moreover, if $j > d_k$ we know that $a_{j,k}$ as defined above equals 0 (cf. [Koe], 3.4.6 Satz, p. 117). A straightforward induction using the Theorem and these recursive formulas shows then that there are

$$A_j(X) = A_{T,j}(X) \in \frac{1}{j!M^j\varphi(N)^j}\mathfrak{o}[[X]]$$

such that $A_j(u^k - 1) = a_{j,k}$ for all $j = 0, 1, 2, \ldots$ and all $k \geq 2$. Since $d_k = \dim \mathcal{M}_k^0 \leq M(0)$ we deduce that $a_{j,k} = 0$ for all k if $j > M(0)$, hence, $A_j(X) = 0$ for all $j > M(0)$. We set

$$\mathbf{Ch}_T(Y) = \sum_{j=0}^{M(0)} (-1)^j A_j(X) Y^{d-j}$$

and obtain

Proposition 1. *For all weights $k \geq 2$ we have*

$$\mathbf{Ch}_T(u^k - 1)(Y) = \mathbf{Ch}_{T,k}(Y),$$

i.e. the characteristic polynomials of the Hecke operators $T|_{\mathcal{M}_k^0}$, $k \geq 2$, form a p-adic analytic family. Moreover, the j-th coefficient $A_j = A_{T,j}$ of $\mathbf{Ch}_T$ is contained in $\frac{1}{j!M^j\varphi(N)^j}\mathcal{O}[[X]]$ and $A_0 = 1$.

We denote by $\mathbb{K} = \{f/g,\ f, g \in \mathfrak{o}[[X]]\}$ the quotient field of $\mathfrak{o}[[X]]$. $\mathbb{K}$ is a subfield of the field of all formal Laurent series in X. In particular, $\mathbf{Ch}_T$ is contained in $\mathbb{K}[Y]$. We denote by $\mathbb{E}/\mathbb{K}$ a splitting field for $\mathbf{Ch}_T$. Hence, in $\mathbb{E}[Y]$ the polynomial $\mathbf{Ch}_T$ splits completely

$$\mathbf{Ch}_T = \prod_{i=1}^{r}(Y - \lambda_{T,i})^{\mathbf{m}(\lambda_{T,i})},$$

where $\lambda_{T,i} \in \mathbb{E}$ and $r = r_T$ depends on T. We denote by $R = R(T)$ the integral closure of $\mathfrak{o}[[X]]$ in $\mathbb{E}$. Since $\mathfrak{o}[[X]]$ is a unique factorization domain, it is integrally closed. Since $\mathbb{E}/\mathbb{K}$ is a finite separable extension we thus know that R is a finite $\mathfrak{o}[[X]]$-module.

$$\begin{array}{ccc} & & \mathbb{E} \\ & / & | \\ R & & \mathbb{K}. \\ | & / & \\ \mathfrak{o}[[X]] & & \end{array} \tag{11}$$

For any k we choose an extension $\varphi_k : R \to \bar{\mathbb{Q}}$ of the evaluation morphism $\mathfrak{o}[[X]] \to \bar{\mathbb{Q}}$, $F \mapsto F((1+p)^k - 1)$. Using Proposition 1 it is not difficult to see that the following holds.

Proposition 2. *Let $T \in \mathcal{H}_1$. Let $\lambda_{T,1}, \dots, \lambda_{T,r}$, $r = r_T$ be the roots of $\mathbf{Ch}_T$ appearing with multiplicities $\mathbf{m}(\lambda_{T,1}), \dots, \mathbf{m}(\lambda_{T,r})$. Then, $\lambda_{T,i} \in \frac{1}{E}R$, where $E = p\varphi(N)$, and for all weights $k \geq 2$ the eigenvalues of T acting on $\mathcal{M}_k^0$ (counted with multiplicities) are given by the sequence*

$$\underbrace{\varphi_k(\lambda_{T,1}), \dots, \varphi_k(\lambda_{T,1})}_{\mathbf{m}(\lambda_{T,1})}, \dots, \underbrace{\varphi_k(\lambda_{T,r}), \dots, \varphi_k(\lambda_{T,r})}_{\mathbf{m}(\lambda_{T,r})}.$$

Thus, any eigenvalue λ of T acting on $\mathcal{M}_k^0$ fits in a R-family given by some $\lambda_{T,i}$. We have to find out how to choose for any ℓ an index $i(\ell)$ such that $(\lambda_{T_\ell,i(\ell)})_\ell$ specializes under φ_k to an element in Φ_k^0 for all k, i.e. $(\lambda_{T_\ell,i(\ell)})_\ell$ corresponds to an R-family of modular eigenform. To this end we choose an element $e \in \mathcal{H}_1$ such that for almost all k (i.e. for all $k \notin S$) the values $\lambda(e)$, $\lambda \in \Phi_k^0$ are pairwise different. We apply the preceeding with $T = e$, i.e. we set $r = r_e$ and $R = R(e)$ is the integral closure of

$\mathfrak{o}[[X]]$ in a splitting field $\mathbb{E}$ of $\mathbf{Ch}_e$. In particular, $|\Phi_k^0| = r$ for all $k \notin S$ and we write $\Phi_k^0 = \{\lambda_{1,k}, \ldots, \lambda_{r,k}\}$. Let $k_0 \notin S$. We have already seen that any λ_{i,k_0} fits in a Lipschitz family $(\lambda_{i,k})_k$. On the other hand, Proposition 2 implies (after eventually reordering the $\lambda_{e,i}$) that $\lambda_{i,k_0}(e) = \varphi_{k_0}(\lambda_{e,i})$ for all $i = 1, \ldots, r$. Since the $\lambda_{i,k_0}(e)$ are pairwise different and since the $\lambda_{i,k}(e)$ as well as the $\varphi_k(\lambda_{e,i})$ are continuous functions of k (in the p-adic sense) we deduce that

$$\varphi_k(\lambda_{e,i}) = \lambda_{i,k}(e)$$

for all k contained in some neighbourhood $U_\epsilon(k_0)$ of k_0. Let $T \in \mathcal{H}_1$. We define the matrix

$$\mathbf{A} = (\lambda_{e,i}^j)_{i,j=1,\ldots,r},$$

the vector

$$\mathbf{b}(T) = (F_{Te^j})_{j=1,\ldots,r}$$

(cf. the above Theorem for the definition of F_{Te^j}) and we denote by

$$\mathcal{D} = \prod_{i<j}(\lambda_{e,i} - \lambda_{e,j})$$

the discriminant of $\mathbf{Ch}_e$. The Theorem and Proposition 2 imply that

$$\varphi_k(\mathbf{A}) = (\varphi_k(\lambda_{e,i}^j))_{i,j} = (\lambda_{i,k}^j(e))_{i,j} \tag{1}$$

and

$$\varphi_k(\mathbf{b}) = (\varphi_k(F_{Te^j}))_j = (\operatorname{tr} Te^j|_{\mathcal{M}_k^0})_j. \tag{2}$$

and

$$\varphi(\mathcal{D}) = \prod_{i>j}(\lambda_{i,k}(e) - \lambda_{j,k}(e)). \tag{3}$$

Proposition 3. *Let $T \in \mathcal{H}_1$ be any Hecke operator. Then, for all $k \in U_\epsilon(k_0)$, $\lambda_{i,k}(T)$ equals the i-th coefficient of the vector*

$$\frac{1}{\mathbf{m}(\lambda_{e,i})}\,\frac{\varphi_k(\operatorname{ad}\mathbf{A})\,\varphi_k(\mathbf{b})}{\varphi_k(\mathcal{D})};$$

here $\operatorname{ad}\mathbf{A}$ *is the adjoint matrix of* $\mathbf{A}$ *and* ϵ *is defined in Lemma 1.*

In matrix form Proposition 3 may be rewritten as

$$\begin{pmatrix} \lambda_{1,k}(T) \\ \vdots \\ \lambda_{r,k}(T) \end{pmatrix} = \frac{1}{\mathbf{m}(\lambda_{e,i})}\,\frac{\varphi_k(\operatorname{ad}\mathbf{A})\,\varphi_k(\mathbf{b}(T))}{\varphi_k(\mathcal{D})} \tag{4}$$

for all $k \in U_\epsilon(k_0)$. We note that ϵ does not depend on T. Equation (4) in particular holds for all Hecke operators T_ℓ and we obtain that for any i the family $(\lambda_{i,k})_{k\in U_\epsilon(k_0)}$ is an R-family, which proves Theorem 3 a. The proof of Theorem 3 b essentially is a variant of the above proof.

The *Proof* of Proposition 3 rests on the following system of linear equations. We set $\mathbf{m}_i = \mathbf{m}(\lambda_{e,i}) = \mathbf{m}_k^0(\lambda_{i,k})$ for all $k \in U_\epsilon(k_0)$; Proposition 2 implies that for all $k \in U_\epsilon(k_0)$ and all $1 \le j \le r$

$$\operatorname{tr} Te^j|_{\mathcal{M}_k^0} = \sum_{i=1}^{r} \mathbf{m}_i \lambda_{i,k}(e^j)\lambda_{i,k}(T).$$

We set $A = (\lambda_{i,k}(e^j))_{i,j}$ and $b = (\operatorname{tr} Te^j|_{\mathcal{M}_k^0})_j$; the above equation may be rewritten as

$$A \begin{pmatrix} \mathbf{m}_1\lambda_{1,k}(T) \\ \vdots \\ \mathbf{m}_r\lambda_{s,k}(T) \end{pmatrix} = b$$

for all $k \in U_\epsilon(k_0)$. Since A is a matrix of Vandermonde type we know $A^{-1} = \frac{1}{\prod_{i<j}\lambda_{i,k}(e)-\lambda_{j,k}(e)} \operatorname{ad} A$ (the $\lambda_{i,k}(e)$ are pairwise different) and the above equation is equivalent to

$$\begin{pmatrix} \mathbf{m}_1\lambda_{1,k}(T) \\ \vdots \\ \mathbf{m}_r\lambda_{s,k}(T) \end{pmatrix} = \frac{\operatorname{ad} A\, b}{\prod_{i<j}\lambda_{i,k}(e)-\lambda_{j,k}(e)}.$$

Using equations (1,2,3) we obtain the claim and the Proposition therefore is proven.

0.5 Cuspidality of analytic families of ordinary families of modular forms.

In this last section we show that our trace identities expressed in Corollary 1 and Corollary 1^{ord} in section 3 and in the Theorem in section 4 also hold on the slope α subspace $\mathcal{S}_k^\alpha$ of the space $\mathcal{S}_k = \mathcal{S}_k(\Gamma, \chi\omega^{-k})$ of cusp of level Γ, weight k and nebentype $\chi\omega^{-k}$. To this end we show that they hold on the orthogonal complement $\mathcal{E}_k$ of $\mathcal{S}_k$ in $\mathcal{M}_k$. As Hecke module, $\mathcal{E}_k$ is a direct sum of induced representations

$$\mathcal{E}_k \cong \bigoplus_{\Theta} (\operatorname{Ind}_{B(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)} \Theta_f)^K, \tag{2}$$

where $B \le \mathrm{GL}_2$ is the Borel subgroup consisting of all upper triangular matrices and

$$\Theta = (\Theta_1, \Theta_2) : T_2(\mathbb{Q})\backslash T_2(\mathbb{A}) \to \mathbb{C}^*$$

runs over all characters satisfying the following conditions:

(3.a) $\Theta_{1,\infty}|_{\mathbb{R}_+} = |\cdot|_\infty^{k-3/2}$, $\Theta_{2,\infty}|_{\mathbb{R}_+} = |\cdot|_\infty^{-1/2}$ with $\Theta_{1,\infty}\Theta_{2,\infty}^{-1}(-1) = (-1)^k$

(3.b) $\Theta_1\Theta_2 = |\cdot|^{k-2}\tilde{\chi}\tilde{\omega}^{-k}$

(3.c) $(\mathrm{Ind}_{B(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)}\Theta_f)^K \neq 0$

($K = K_1(Np) \leq \mathrm{GL}_2(\hat{\mathbb{Z}})$ is the Hecke subgroup correspondiong to $\Gamma = \Gamma_1(Np)$). We denote by $\mathbf{T}_\ell = K_{1,\ell}(Np)\begin{pmatrix}\ell & \\ & 1\end{pmatrix}K_{1,\ell}(Np)$ the local Hecke operator and we determine the slope decomposition of a constituent of $\mathcal{E}_k$.

Proposition 1. *Let Π be any automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ such that Π_f occurs in $\mathcal{E}_k$.*

- If $\mathrm{cond}\,\Theta_p = (1,1)$, i.e. Θ_p is unramified, then with respect to some basis of $\Pi_p^{K_p}$ the Hecke operator $\mathbf{T}_p$ on $\Pi_p^{K_p}$ is represented by the matrix

$$\begin{pmatrix}p^{1/2}\Theta_{1,p}(p) & \\ & p^{1/2}\Theta_{2,p}(p)\end{pmatrix}.$$

- If $\mathrm{cond}\,\Theta_p = (p,1)$ then $\mathbf{T}_p$ acts on $\Pi_p^{K_p}$ as multiplication with $\Theta_{1,p}(p)p^{1/2}$.

- If $\mathrm{cond}\,\Theta_p = (1,p)$ then $\mathbf{T}_p$ acts on $\Pi_p^{K_p}$ as multiplication with $\Theta_{2,p}(p)p^{1/2}$.

Since the classical Hecke operator T_p corresponds to the local Hecke operator $p^{k-2}\tilde{\chi}_p\tilde{\omega}_p^{-k}(p^{-1})\,\mathbf{T}_p$ we obtain that the nontrivial slopes of Π_f^K with respect to T_p are $0, k-1$ resp. 0 resp. $k-1$ in the first resp. second resp. third case of Proposition 1. Since we are interested in families of constant slope we have to restrict to the slope 0-subspace of $\mathcal{E}_k$ with respect to T_p, which is the slope $2-k$ subspace with respect to $\mathbf{T}_p$. We fix a weight k_0 and we let $\Pi_f^{K,2-k_0} = (\mathrm{Ind}_{B(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)}\Theta_f)^{K,2-k_0}$ be a constituent of $\mathcal{E}_{k_0}^0$, i.e. $\Theta = (\Theta_1, \Theta_2)$ satisfies (3 a,b,c) (with k replaced by k_0) and $\mathrm{cond}\,\Theta = (1,1)$ or $= (p,1)$. We define a character $\Theta_k = (\Theta_{1,k}, \Theta_2)$ by setting

$$\Theta_{1,k} = \Theta_1|\cdot|^{k-k_0}\omega^{k_0-k}.$$

Θ_k again satisfies (3 a,b,c) and the condition on the conductor. Hence, $\Pi_{k,f}^{K,0} = (\mathrm{Ind}_{B(\mathbb{A}_f)}^{\mathrm{GL}_2(\mathbb{A}_f)}\Theta_{k,f})^{K,2-k}$ is a nontrivial constituent of $\mathcal{E}_k^0$.

Proposition 2. *1.) For all primes ℓ the following holds:*

1.) $\mathrm{tr}\,\mathbf{T}_\ell|_{\Pi_{k,f}^{K,2-k}}$ depends analytically on k, i.e. there is $F_{^a\Theta_\ell} \in \mathfrak{o}[[X]]$ such that

$$\mathrm{tr}\,\mathbf{T}_\ell|_{\Pi_{k,f}^{K,2-k}} = F_{^a\Theta_\ell}((1+p)^k - 1)$$

for all k. Here, $^a\Theta = \delta_{B,f}^{1/2}\Theta$, i.e. $\Pi_{k,f}$ is algebraically induced from $^a\Theta$.

2.) $\sigma(F_{^a\Theta_\ell}) = F_{\sigma(^a\Theta)_\ell}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}_p/\mathbb{Q}_p)$.

As a consequence of Proposition 2 we obtain

Theorem. *For all primes ℓ there is a power series $F_\ell \in \mathfrak{o}[[X]]$ such that*

$$\mathrm{tr}\, \mathbf{T}_\ell|_{\mathcal{E}_k^0} = F_\ell((1+p)^k - 1)$$

for all weights k.

The above trace identity in particular implies that $\mathrm{tr}\, T_\ell|_{\mathcal{E}_k^0} \equiv \mathrm{tr}\, T_\ell|_{\mathcal{E}_{k'}^0} \pmod{p^m}$ if $k \equiv k' \pmod{p^m}$. Thus, our trace identitites also hold on $\mathcal{E}_k^0$ and, hence, on $\mathcal{S}_k^0$. Corollary 1 and Theorems 2, 3a, 3b of section 1 therefore also hold in the cuspidal case. In particular, since cuspidal eigenforms forms are determined by their corresponding system of Hecke eigenvalues we obtain

Corollary. *Any cuspidal eigenform $f \in \mathcal{S}_{k_0}^0$ fits in an R-family of true cuspidal eigenforms $(f_k)_k$.*

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