<table>
<thead>
<tr>
<th>Title</th>
<th>Borcherds Lifts, Symmetry Relations, and Applications (Automorphic forms, automorphic representations and related topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Heim, Bernhard; Murase, Atsushi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2010), 1715: 1-10</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/170307">http://hdl.handle.net/2433/170307</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Borcherds Lifts, Symmetry Relations, and Applications

Bernhard Heim and Atsushi Murase

Abstract. This paper is related to the authors’ talk at the RIMS conference 2010 on: Automorphic forms, automorphic representations and related topics in Tokyo. We mainly study holomorphic Siegel modular forms on $\text{Sp}_2(\mathbb{Z})$ obtained as Borcherds lifts and the connection with the Witt and Siegel $\Phi$-operator. As a direct consequence we obtain for example that Siegel Eisenstein series are not Borcherds lifts.

Mathematics Subject Classification (2000): 11F41

Keywords: Siegel modular forms, Borcherds products, modular polynomials.

1 Introduction and the main results

1.1 Introduction

In this note we mainly summarize the results presented at the RIMS conference 2010 on: Automorphic forms, automorphic representations and related topics in Tokyo. A Borcherds lift ([Bo1],[Bo2],[Bo3]) on $\Gamma_2 = \text{Sp}_2(\mathbb{Z})$ is a meromorphic automorphic form $F$ on $\Gamma_2$ (with a multiplier system of finite order) whose divisor is of the form $\sum d A(d) H_d$, where $d$ runs over the positive integers congruent to 0 or 1 modulo 4, $A(d) \in \mathbb{Z}$ ($A(d) = 0$ except for a finite number of $d$) and $H_d$ is the Humbert surface of discriminant $d$. Since every Borcherds lift is a quotient of holomorphic Borcherds lifts, we mainly consider the holomorphic case in this paper.

We employ our previous result on the multiplicative symmetries for Borcherds lifts ([HM]; see Theorem 3.1). We obtain that the image of a holomorphic Borcherds lift on $\Gamma_2$ under the Siegel operator is proportional to a power of $\Delta$, the Ramanujan discriminant function. This implies that the Siegel Eisenstein series is never a Borcherds lift. Then we show that a holomorphic Borcherds lift on $\Gamma_2$ with trivial character is proportional to $\chi_{10}^a \chi_{35}^b F'$, where $\chi_{10}$ and $\chi_{35}$ are Borcherds lifts of weight 10 and 35, respectively, $a \in \mathbb{Z}_{\geq 0}, b \in \{0,1\}$ and $F'$ is a Borcherds lift of weight divisible by 12 such that the image of $F'$ under the Witt operator is nonzero (Corollary 1.5).

1.2 Siegel modular forms

To explain our results more precisely, let

$$\Gamma_n := \left\{ \gamma \in \text{GL}_n(\mathbb{Z}) \mid \Gamma \gamma \begin{pmatrix} 0 \ & \ 1_n \\ -1_n & 0_n \end{pmatrix} \gamma = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \right\}$$

The first author was partially supported by a grant of Prof. T. Ibukiyama (Grants-in-Aids from JSPS (21244001)). Part of the notes had been written at his stay in the summer of 2010 at the Max-Planck-Institut für Mathematik in Bonn. The second author was partially supported by Grants-in-Aids from JSPS (20540031).
be the Siegel modular group of degree $n$ and $\mathcal{H}_n := \{ Z \in M_n(\mathbb{C}) \mid \text{Im}(Z) > 0 \}$ be the upper half space of degree $n$, where $0_n$ (respectively $1_n$) is the zero (respectively identity) matrix of degree $n$.

Let $M_k(\Gamma_n)$ denote the space of holomorphic automorphic forms of weight $k$ on $\Gamma_n$ and $S_k(\Gamma_n)$ be the subspace of cuspforms.

In the case $n = 2$ which we are mainly interested in we often write $(\tau_1, z, \tau_2)$ for a point

\[
\begin{pmatrix}
\tau_1 & z \\
z & \tau_2
\end{pmatrix} \in \mathcal{H}_2.
\]

For $F \in M_k(\Gamma_2)$, we put

\[
\Phi(F)(\tau) := \lim_{y \to \infty} F(\tau, 0, iy) \quad (\tau \in \mathfrak{H}_1),
\]

\[
\mathcal{W}(F)(\tau_1, \tau_2) := F(\tau_1, 0, \tau_2) \quad (\tau_1, \tau_2 \in \mathfrak{H}_1).
\]

Then $\Phi(F) \in M_k(\Gamma_1)$ and $\mathcal{W}(F) \in \text{Sym}^2(M_k(\Gamma_1))$. The operator $\Phi$ (respectively $\mathcal{W}$) is called the Siegel (respectively Witt) operator. Then $S_k(\Gamma_2) = \{ F \in M_k(\Gamma_2) \mid \Phi(F) = 0 \}$ is the space of cusp forms.

For $F \in M_k(\Gamma_2)$ admits the Fourier expansion

\[
F(\tau_1, z, \tau_2) = \sum_{n,r,m \in \mathbb{Z}} A_F(n, r, m) e(n\tau_1 + rz + m\tau_2),
\]

where we put $e(z) = \exp(2\pi iz)$ for $z \in \mathbb{C}$. Note that $A_F(n, r, m) = 0$ unless $n, m, 4nm - r^2 \geq 0$.

For $k \geq 4$ let $E_k(Z)$ denote the Siegel Eisenstein series on $\Gamma_2$ of weight $k$. Due to Igusa ([Ig]), the graded ring $\bigoplus_{k \geq 0} M_k(\Gamma_2)$ is generated by $E_4, E_6, \chi_{10}, \chi_{12}$ and $\chi_{35}$, where

\[
\chi_{10} := -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} (E_4 E_6 - E_{10}) \in S_{10}(\Gamma_2),
\]

\[
\chi_{12} := 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} (3^2 \cdot 7^2 E_4^3 + 2 \cdot 5^3 E_6^2 - 691 E_{12}) \in S_{12}(\Gamma_2)
\]

and $\chi_{35}$ is a unique element of $S_{35}(\Gamma_2)$ up to constant multiples. Note that we follow Igusa's normalizations of $\chi_{10}$ and $\chi_{12}$ so that

\[
A_{\chi_{10}}(1,1,1) = -1/4,
\]

\[
A_{\chi_{12}}(1,1,1) = 1/12.
\]

We also recall that van der Geer ([Ge1]) defined a Siegel modular form

\[
G_{24} := (\chi_{12} - 2^{-12} \cdot 3^{-6}(E_6^2 + E_4^3))^2 - E_4 (2 \cdot 3^{-1} \chi_{10} - 2^{-11} \cdot 3^{-6} E_4 E_6)^2 \in M_{24}(\Gamma_2),
\]

whose divisor is the Humbert surface of discriminant 5 (for the definition of Humbert surfaces, see 2.2). It is known that $\chi_{10}, \chi_{35}$ and $G_{24}$ are Borcherds lifts (see [GN1] and [GN2]), but $\chi_{12}$ is not a Borcherds lift (see [HM]).
1.3 Main results

Employing our previous result on the multiplicative symmetries for Borcherds lifts ([HM]; see Theorem 3.1), we give several necessary conditions for \( F \in M_k(\Gamma_2) \) to be a Borcherds lift.

**Theorem 1.1.** Assume that \( F \in M_k(\Gamma_2) \) is a Borcherds lift. Then \( \Phi(F) \) is proportional to a power \( \Delta^r \) of the modular discriminant \( \Delta \) with \( r \geq 0 \).

**Corollary 1.2.** If \( F \in M_k(\Gamma_2) \setminus S_k(\Gamma_2) \) is a Borcherds lift, then the weight \( k \) is divisible by 12.

We note that \( \chi_{10} \in S_{10}(\Gamma_2) \) is a Borcherds lift, and hence that the assumption of noncuspidality is necessary.

**Corollary 1.3.** The Siegel Eisenstein series \( E_k \) is not a Borcherds lift.

Moreover we have the following result:

**Theorem 1.4.** If \( F \in M_k(\Gamma_2) \) is a Borcherds lift and \( \mathcal{W}(F) \neq 0 \), then the weight \( k \) is divisible by 12 and greater than 12.

**Corollary 1.5.** Let \( F \in M_k(\Gamma_2) \) be a Borcherds lift. We let \( b = 0 \) if \( k \) is even and \( b = 1 \) otherwise. Define \( a \in \mathbb{Z}_{\geq 0} \) such that the coefficient of \( H_{1} \) in the divisor of \( F \) is equal to \( 2a + b \).

Then there exists a Borcherds lift \( F' \in M_{k'}(\Gamma_2) \) with \( \mathcal{W}(F') \neq 0 \) such that \( F \) is proportional to \( \chi_{10}^a \chi_{35}^b F' \). In particular, the weight \( k \) of \( F \) is of the form

\[
10a + 35b + 12c \quad (a \in \mathbb{Z}_{\geq 0}, b \in \{0, 1\}, c \in \mathbb{Z}_{\geq 0}, c \neq 1).
\]

2 Borcherds lifts

2.1 Jacobi forms

For \( k \in \mathbb{Z} \), let \( J_{k,1}^{wh} \) denote the space of holomorphic functions on \( \mathfrak{H} \times \mathbb{C} \) satisfying the following conditions:

(i) \( \phi\left(\begin{array}{c}
\tau + b \\
\frac{c\tau + d}{c\tau + d}
\end{array}\right) = (c\tau + d)^k e\left(\frac{cz^2}{c\tau + d}\right) \phi(\tau, z) \quad \left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \Gamma_1, \tau \in \mathfrak{H}, z \in \mathbb{C}\right). \)

(ii) \( \phi(\tau, z + \lambda \tau + \mu) = e(-\lambda^2 \tau - 2\lambda z) \phi(\tau, z) \quad (\lambda, \mu \in \mathbb{Z})\).

(iii) Let \( \phi(\tau, z) = \sum_{n,r \in \mathbb{Z}} a_{\phi}(n, r)e(n\tau + rz) \) be the Fourier expansion of \( \phi \). Then \( a_{\phi}(n, r) = 0 \) if \( 4n - r^2 \) is sufficiently small.

We call \( J_{k,1}^{wh} \) the space of weakly holomorphic Jacobi forms of weight \( k \) and index 1. The Fourier coefficient \( a_{\phi}(n, r) \) depends only on \( N = 4n - r^2 \) and is often denoted by \( a_{\phi}(N) \). We put \( a_{\phi}(N) = 0 \) if \( N \equiv 1 \) or 2 (mod 4). We then have

\[
\phi(\tau, z) = \sum_{N \in \mathbb{Z}} a_{\phi}(N) \sum_{r \in \mathbb{Z}, r^2 \equiv -N \mod 4} e\left(\frac{N + r^2}{4} \tau + rz\right).
\]

For \( \phi \in J_{0,1}^{wh} \), we call \( \{a_{\phi}(N) \mid N < 0\} \) the principal part of \( \phi \), which determines \( \phi \) since the space of holomorphic Jacobi forms of weight 0 and index 1 vanishes.
2.2 Humbert surfaces

Let

\[ Q := \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 0 \end{pmatrix} \]

Put \( Q(X, Y) := {}^tXQY \) and \( Q[X] := {}^tXQX \) for \( X, Y \in \mathbb{C}^5 \). For \( Z = (\tau_1, z, \tau_2) \in \mathfrak{H}_2 \) put \( \tilde{Z} := \tau_1\tau_2 + z^2, \tau_1, z, \tau_2, 1 \) \( \in \mathbb{C}^5 \). Note that \( Q[\tilde{Z}] = 0 \) and \( Q(\tilde{Z}, \overline{\tilde{Z}}) = 4 \det(\text{Im}(Z)) > 0 \).

There exists a homomorphism \( \iota : \text{Sp}_2(\mathbb{R}) \to O(Q)_{\mathbb{R}} \) such that \( g\langle Z \rangle = j(g, Z)^{-1}\iota(g)\tilde{Z} \) for \( g \in \text{Sp}_2(\mathbb{R}) \) and \( Z \in \mathfrak{H}_2 \).

Let \( \mathcal{H}_d := \sum_{X \in \mathcal{L}_d} \{ Z \in \mathfrak{H}_2 | Q(X, \tilde{Z}) = 0 \} \), where \( \mathcal{L}_d := \{ X \in \mathfrak{H}_2^* | Q[X] = -d/2 \} \). Note that \( \mathcal{H}_d = 0 \) unless \( d > 0 \) and \( d \equiv 0 \) or 1 (mod 4). Since \( \mathfrak{H}_d^* \) is \( \iota(\Gamma_2) \)-invariant, \( \mathcal{H}_d \) is \( \Gamma_2 \)-invariant. Denote by \( \mathcal{H}_d \) the image of \( \mathcal{H}_d \) in \( \Gamma_2 \backslash \mathfrak{H}_2 \) by the natural projection \( \mathfrak{H}_2 \to \Gamma_2 \backslash \mathfrak{H}_2 \). The divisor \( \mathcal{H}_d \) of \( \Gamma_2 \backslash \mathfrak{H}_2 \) is called the Humbert surface of discriminant \( d \). It is known that \( \mathcal{H}_d \) is nonzero and irreducible if \( d \equiv 0 \) or 1 (mod 4) (see [Ge2], page 212, Theorem 2.4; see also [GH], Section 3). Note that

\[ \mathcal{H}_1 = \bigcup_{\gamma \in \Gamma_2} \gamma \{ (\tau_1, 0, \tau_2) | \tau_1, \tau_2 \in \mathfrak{H} \} \]

\[ \mathcal{H}_4 = \bigcup_{\gamma \in \Gamma_2} \gamma \{ (\tau, z, \tau) | \tau \in \mathfrak{H}, z \in \mathbb{C} \} . \]

Let \( v \) be the unique nontrivial quadratic character of \( \Gamma_2 \) and \( M_k(\Gamma_2, v) \) the space of Siegel modular forms on \( \Gamma_2 \) of weight \( k \) with character \( v \). The following result of Igusa is quite useful (see [GN1], Corollary 1.4).

**Lemma 2.1.** Let \( F \in M_k(\Gamma_2, v) \). If \( k \) is odd, \( \chi_5^{-1}F \in M_{k-5}(\Gamma_2) \). If \( k \) is even, \( \chi_3^{-1}F \in M_{k-30}(\Gamma_2) \).

2.3 Borcherds lifts on \( \Gamma_2 \)

As a special case of Borcherds theory ([Bo1] and [Bo2]; see also [GN3], §2.1), we have the following result:
Theorem 2.2. Let $\phi \in J_{0,1}^{wh}$ and write $a(N)$ for $a_\phi(N)$. Assume that $a(N) \in \mathbb{Z}$ if $N < 0$.

(i) Set

$$\delta := \sum_{r \in \mathbb{Z}} a(-r^2),$$
$$\rho := \frac{1}{2} \sum_{r \in \mathbb{Z}, r > 0} a(-r^2) r,$$
$$\nu := \frac{1}{4} \sum_{r \in \mathbb{Z}} a(-r^2) r^2$$

and

$$\Lambda := \{(m, r, n) \in \mathbb{Z}^3 \mid m > 0 \text{ or } m = 0, n > 0 \text{ or } m = n = 0, r > 0\}.$$

Then

$$\Psi_\phi(\tau_1, z, \tau_2) := e\left(\frac{\delta}{24} \tau_2 - \rho z + \nu \tau_1\right) \prod_{(m, r, n) \in \Lambda} (1 - e(m\tau_1 + rz + n\tau_2))^{a(4mn - r^2)}$$

converges absolutely if $\det(\text{Im}(Z))$ is sufficiently large, and is meromorphically continued to $\mathfrak{H}_2$.

(ii) The function $\Psi_\phi$ is a meromorphic modular form on $\Gamma_2$ of weight $k_\phi = a(0)/2$ and character $\nu^\alpha$ ($\alpha \in \{0, 1\}$).

(iii) The divisor of $\Psi_\phi$ is

$$\sum_d a(-d) H_d^*,$$

where $d$ runs over the positive integers congruent to 0 or 1 modulo 4 and

$$H_d^* := \sum_{f > 0, f^2 | d} H_{f^{-2}d}.$$

The meromorphic modular form $\Psi_\phi$ is called the Borcherds lift of $\phi$.

Remark 2.3. It is well-known that the weight of Borcherds lifts is related to the Cohen numbers $H(N) = H(2, N)$. These are the coefficients of the Cohen Eisenstein series

$$\sum_{N \geq 0} H(2, N) e(N\tau),$$

of weight $5/2$. For convenience we put $h(N) = \sum_{f^2 | N} \mu(f) H(f^{-2}N)$, where $\mu$ is the Möbius function. Moreover put $\hat{H}(N) = -60H(N)$ and $\hat{h}(N) = -60h(N)$. Then we have
Theorem 2.4.

(i) For each positive integer \(d\) with \(d \equiv 0 \text{ or } 1 \pmod{4}\), there exists an \(F_d \in M_{k_d}(\Gamma_2, v^{\alpha_d})\) with \(\alpha_d \in \{0,1\}\) satisfying \(\text{div}(F_d) = H_d\).

(ii) We have \(k_d = \hat{h}(d)\).

(iii) We have \(F_1 \in M_5(\Gamma_2, v), F_4 \in M_{30}(\Gamma_2, v)\) and \(F_d \in M_{k_d}(\Gamma_2)\) if \(d > 4\).

(iv) A Borcherds lift \(F \in M_k(\Gamma_2, v^\alpha)\) \((\alpha \in \{0,1\})\) is a constant multiple of \(\prod_d F_d^{A(d)}\), where \(d\) runs over the positive integers with \(d \equiv 0 \text{ or } 1 \pmod{4}\), and \(A(d)\) is a nonnegative integer \((A(d) = 0 \text{ except for a finite number of } d)\) satisfying \(A(1) + A(4) \equiv \alpha \pmod{2}\). Furthermore we have

\[
k = \sum_{d>0} A(d) \hat{h}(d).
\]

Moreover we have

Theorem 2.5. The weight \(k_d\) of \(F_d\) is divisible by 24 if and only if \(d > 4\) and \(d \neq 8\).

Remark 2.6. The Borcherds lifts in \(M_k(\Gamma_2)\) with \(k \leq 60\) are listed as follows:

<table>
<thead>
<tr>
<th>Borcherds lift</th>
<th>weight</th>
<th>divisor</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_1^{2a}) (1 (\leq a \leq 6))</td>
<td>10a</td>
<td>(2aH_1)</td>
</tr>
<tr>
<td>(F_1^{2a+1}F_4) (1 (\leq a \leq 2))</td>
<td>10a + 35</td>
<td>((2a + 1)H_1 + H_4)</td>
</tr>
<tr>
<td>(F_1^{2a}F_5) (1 (\leq a \leq 3))</td>
<td>10a + 24</td>
<td>(2aH_1 + H_5)</td>
</tr>
<tr>
<td>(F_4^2)</td>
<td>60</td>
<td>(2H_4)</td>
</tr>
<tr>
<td>(F_5^2)</td>
<td>48</td>
<td>(2H_5)</td>
</tr>
<tr>
<td>(F_8)</td>
<td>60</td>
<td>(H_8)</td>
</tr>
</tbody>
</table>

The table shows that every Borcherds lift of weight less than or equal to 60 is a monomial of \(F_1, F_4, F_5\) and \(F_8\). We also see that there is no holomorphic Borcherds lift of weight 12. This gives another proof of the fact that \(\chi_{12}\) is not a Borcherds lift, which was proved in [HM] in a different way.

2.4 The image of \(\Psi_\phi\) under the Witt operator

For \(m \in \mathbb{Z}_{>0}\), let \(\mathcal{M}_m\) be the set of matrices in \(M_2(\mathbb{Z})\) of determinant \(m\). As is well-known, there exists a polynomial \(\Phi_m\) in \(\mathbb{Z}[X,Y]\), called the modular polynomial of degree \(m\), such that

\[
\prod_{M \in \text{SL}_2(\mathbb{Z}) \backslash \mathcal{M}_m} (X - j(M(\tau))) = \Phi_m(X, j(\tau)).
\]
The degree of $\Phi_m(X,Y)$ in $X$ is equal to $\sigma_1(m) = \sum_{0<d|m} d$. Let
\[
\eta(\tau) := e(\tau/24) \prod_{n=1}^{\infty} (1 - e(n\tau)) \quad (\tau \in \mathfrak{H})
\]
be the Dedekind's eta function.

**Theorem 2.7.** Let $\phi \in J_{0,1}^{wh}$ and suppose that $a(N) := a_\phi(N) \in \mathbb{Z}$ if $N < 0$. Assume that the Borcherds lift $\Psi_\phi$ of $\phi$ is holomorphic.

(i) We have $W(\Psi_\phi) = 0$ if and only if $\sum_{r>0} a(-r^2) > 0$.

(ii) Assume that $\sum_{r>0} a(-r^2) = 0$. Then
\[
W(\Psi_\phi) = c (\eta(\tau_1)\eta(\tau_2))^{b(0)} \prod_{n>0} \Phi_n(j(\tau_1), j(\tau_2))^{b(-n)},
\]
where $c \in \mathbb{C}^\times$ and
\[
b(n) := \sum_{r \in \mathbb{Z}} a(4n - r^2).
\]

(iii) Assume that $\sum_{r>0} a(-r^2) = 0$. The automorphic form $W(\Psi_\phi)$ belongs to $\text{Sym}^2(S_{b(0)/2}(\Gamma_1))$ if and only if $\sum_{r \in \mathbb{Z}} a(-r^2)r^2 > 0$.

**Remark 2.8.** The degree of $W(\Psi_\phi)$ in $q_1 = e[\tau_1]$ is equal to
\[
b(0)/24 - \sum_{n>0} \sigma_1(n)b(-n).
\]

**Corollary 2.9.** Let $d > 4$. Then $F_d \in S_{k_d}(\Gamma_2)$ if and only if $d = \square$.

# 3 Multiplicative symmetries and the main theorems

## 3.1 The multiplicative symmetries

For $F \in M_k(\Gamma_2)$ and a prime number $p$, we put
\[
F|\mathcal{T}_p^\uparrow(\tau_1, z, \tau_2) = F(p\tau_1, pz, \tau_2) \prod_{a=0}^{p-1} F(\frac{\tau_1+a}{p}, z, \tau_2),
\]
\[
F|\mathcal{T}_p^\downarrow(\tau_1, z, \tau_2) = F(\tau_1, pz, p\tau_2) \prod_{a=0}^{p-1} F(\tau_1, z, \frac{\tau_2+a}{p}).
\]

We say that $F$ satisfies the multiplicative symmetries if the condition
\[
(\text{MS})_p
\]
holds with $\epsilon_p(F) \in \mathbb{C}^\times$, $|\epsilon_p(F)| = 1$ for any prime number $p$. Denote by $A_{F,p}^\dagger(n, r, m)$ (respectively $A_{F,p}(n, r, m)$) the coefficient of $e(n\tau_1 + rz + m\tau_2)$ in the Fourier expansion of $F|\mathcal{T}_p^\dagger(\tau_1, z, \tau_2)$ (respectively $F|\mathcal{T}_p(\tau_1, z, \tau_2)$). If $F$ satisfies (MS)$_p$, then we have
\[
A_{F,p}^\dagger(m, r, n) = \epsilon_p(F) A_{F,p}(m, r, n)
\]
for any $(m, n, r)$. As a special case of [HM], we have the following result.
Theorem 3.1. Suppose that $F \in M_k(\Gamma_2)$ is a Borcherds lift. Then $F$ satisfies the multiplicative symmetries.

3.2 A characterization of powers of the modular discriminant

Let $k$ be a positive integer greater than or equal to 4. Denote by $M_k(\Gamma_1)$ (respectively $S_k(\Gamma_1)$) the space of holomorphic automorphic (respectively cusp) forms on $\Gamma_1 = \text{SL}_2(\mathbb{Z})$ of weight $k$. Recall that $S_{12}(\Gamma_1) = \mathbb{C} \cdot \Delta$ and that $\Delta$ has no zeros in $\mathfrak{H}$.

For $f \in M_k(\Gamma_1)$ and a prime number $p$, we define the multiplicative Hecke operator by

$$(f|\mathcal{T}_p)(\tau) = f(p\tau) \prod_{c=0}^{p-1} f\left(\frac{\tau+c}{p}\right).$$

Then $f|\mathcal{T}_p \in M_{(p+1)k}(\Gamma_1)$. The following property plays a crucial role in the proof of Theorem 1.1.

Proposition 3.2. Let $f$ be a nonzero element of $M_k(\Gamma_1)$. Then $f$ satisfies

$$(*)_p \quad f|\mathcal{T}_p(\tau) = \epsilon_p(f) f(\tau)^{p+1} \quad (\tau \in \mathfrak{H})$$

for any prime number $p$ with $\epsilon_p(f) \in \mathbb{C}^\times, |\epsilon_p(f)| = 1$ if and only if $f$ is a constant multiple of $\Delta^r$ ($r \in \mathbb{Z}_{\geq 0}$).

Remark 3.3. If $f \in M_k(\Gamma_1)$ satisfies $(*)_2$, $f$ is a constant multiple of $\Delta^r$.

3.3 Multiplicative symmetries for $\text{Sym}^2(M_k(\Gamma_1))$

For $\varphi \in \text{Sym}^2(M_k(\Gamma_1))$ and a prime number $p$, we define the multiplicative Hecke operators by

$$(\varphi|\mathcal{T}_p^\uparrow)(\tau_1, \tau_2) = \varphi(p\tau_1, \tau_2) \prod_{c=0}^{p-1} \varphi\left(\frac{\tau_1+c}{p}, \tau_2\right),$$

$$(\varphi|\mathcal{T}_p^\downarrow)(\tau_1, \tau_2) = \varphi(\tau_1, p\tau_2) \prod_{c=0}^{p-1} \varphi\left(\tau_1, \frac{\tau_2+c}{p}\right).$$

We say that $\varphi$ satisfies the multiplicative symmetry for $p$ if there exists an $\epsilon_p(\varphi) \in \mathbb{C}^\times, |\epsilon_p(\varphi)| = 1$ such that

$$(\text{ms})_p \quad \varphi|\mathcal{T}_p^\uparrow = \epsilon_p(\varphi) \varphi|\mathcal{T}_p^\downarrow$$

holds. For $\varphi \in \text{Sym}^2(M_k(\Gamma_1))$, put $\Phi'(\varphi)(\tau) = \lim_{y \to \infty} \varphi(\tau, iy)$. Then $\Phi'(\varphi) \in M_k(\Gamma_1)$. The following facts can be verified.

Lemma 3.4. If $\varphi \in \text{Sym}^2(M_k(\Gamma_1))$ satisfies $(\text{ms})_p$ and $f = \Phi'(\varphi) \neq 0$, then $f$ satisfies $(*)_p$. In particular, $f$ is a constant multiple of $\Delta^r$ and $k$ is divisible by 12.

Proposition 3.5. If $\varphi \in \text{Sym}^2(M_k(\Gamma_1)) \setminus \{0\}$ satisfies $(\text{ms})_2$, $k$ is divisible by 12.

Proposition 3.6. Suppose that $F \in M_k(\Gamma_2)$ satisfies $(\text{MS})_p$ for a prime $p$. Put $f = \Phi(F)$ and $\varphi = \mathcal{W}(F)$. Then, for any prime number $p$, $f$ (respectively $\varphi$) satisfies $(*)_p$ (respectively $(\text{ms})_p$) and $\epsilon_p(F) = \epsilon_p(f) = \epsilon_p(\varphi)$. 
3.4 Proof of Theorem 1.1

By Proposition 3.6 and Proposition 3.3, we obtain the following result, from which Theorem 1.1 follows.

**Proposition 3.7.** Assume that $F \in M_k(\Gamma_2)$ satisfies (MS)$_2$ and $f = \Phi(F) \neq 0$. Then $f = c\Delta^r$ ($c \in \mathbb{C}^\times, r \in \mathbb{Z}_{\geq 0}$). In particular, the weight $k$ is divisible by 12.

3.5 Proof of Theorem 1.4

Theorem 1.4 is a direct consequence of Theorem 3.1 and the following result.

**Proposition 3.8.** If $F \in M_k(\Gamma_2)$ satisfies (MS)$_2$ and $\mathcal{W}(F) \neq 0$, then $k$ is divisible by 12.

**Proof.** Let $\varphi = \mathcal{W}(F)$. Then $\varphi \neq 0$ and $\varphi$ satisfies (ms)$_2$. The proposition now follows from Proposition 3.5. $\square$

References


Bernhard Heim
German University of Technology in Oman, Way No. 36, Building No. 331, North Ghubrah, Muscat, Sultanate of Oman
e-mail: bernhard.heim@gutech.edu.om

Atsushi Murase
Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan
e-mail: murase@cc.kyoto-su.ac.jp